

CHAPTER 1

First order PDE (Evans PDE 3.1 3.2)

The PDE we consider here is

$$F(Du, u, x) = 0, \quad F : \mathbb{R}^d \times \mathbb{R} \times U \rightarrow \mathbb{R}, \quad (0.1)$$

with $U \subset \mathbb{R}^d$ open, and we assume that F is smooth.

The initial/boundary condition is that we require

$$u|_{\Gamma} = g|_{\Gamma}, \quad \Gamma \subset U, g \text{ given}, \quad (0.2)$$

being Γ a smooth $d - 1$ -dimensional surface.

1. Construction of solutions

DEFINITION 1.1. A C^2 function $U \times A \ni (x, a) \mapsto u(x, a)$, $U, A \subset \mathbb{R}^d$ open, is a *complete integral* in $U \times A$ if:

- $u(\cdot, a)$ solves the PDE (0.1) for all $a \in A$;
- $\text{rank}(D_a u, D_{x,a}^2 u) = d$.

REMARK 1.2. The last condition requires that the solution u depends on the full parameters a .

EXAMPLE 1.3. For

$$x \cdot Du + f(Du) = u$$

take

$$u = a \cdot x + f(a).$$

EXAMPLE 1.4. For the HJ equation

$$H(Du) = 0$$

take

$$u = a \cdot x + b, \quad H(a) = 0,$$

assuming the non degeneracy condition that $\{H = 0\}$ is a smooth $d - 1$ surface. For example

$$|Du| - 1 = 0$$

gives

$$a \in \mathbb{S}^{d-1}, \quad b \in \mathbb{R}.$$

EXAMPLE 1.5. For the HJ equation

$$u_t + H(Du) = 0$$

a complete family of solutions is

$$u(t, x; a, b) = a \cdot x - tH(a) + b.$$

1.1. Envelopes. The *envelope* v of the family of functions $u(x, a)$ is given by inverting (whenever possible)

$$D_a u(x, a) = 0 \quad \Rightarrow \quad a = \phi(x),$$

and then defining

$$v(x) = u(x, \phi(x)).$$

Sometimes, it is called *singular integral*.

LEMMA 1.6. *If the envelope exists and is C^1 , then v solves (0.1).*

PROOF. Just compute

$$\begin{aligned} F(D_x v, v, x) &= F(D_x u(x, \phi) + D_a u(x, \phi) D_x \phi, u(x, \phi(x)), x) \\ &= F(D_x u(x, \phi(x)), u(x, \phi(x)), x) = 0. \end{aligned}$$

□

EXAMPLE 1.7. Consider

$$u^2(1 + |Du|^2) - 1 = 0,$$

whose complete integral is

$$u(x, a) = \pm \sqrt{1 - |x - a|^2}, \quad |x - a| < 1.$$

Then

$$D_a u = \mp \frac{x - a}{\sqrt{1 - |x - a|^2}} = 0, \quad a = x,$$

so the singular integral is

$$u = \pm 1.$$

A second construction is by considering for a smooth function $h : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ and setting

$$v(t, a') = u(t, (a', h(a'))).$$

Assume that v is C^1 : inverting when possible

$$D_{a'} v(t, a') = 0 = D_a u \cdot (\text{id}, h') = 0, \quad a' = a'(x),$$

we obtain again

$$D_x v(t, a'(x)) = D_x u + D_a u(\text{id}, h') D_x a' = D_x u,$$

so that being the PDE of first order we still have a solution.

EXAMPLE 1.8. Consider $H(p) = \frac{p^2}{2}$, $u_t + H(p) = 0$, and the complete integral

$$u(t, x, a) = a \cdot x - \frac{a^2}{2} t + b.$$

We have

$$D_{a,b} u = (x - ta, 1)$$

which is never 0 so that we cannot find an the envelope solution

If instead we fix b , then

$$D_a u = x - ta = 0, \quad a = \frac{x}{t},$$

and the envelope is

$$u(t, x) = \frac{x^2}{t} - \frac{x^2}{2t} + b = \frac{x^2}{2t} + b.$$

2. Characteristics

We want to convert (0.1) into a system of ODEs. Let

$$\mathbb{R} \ni s \mapsto x(s) \in \mathbb{R}^d$$

be a smooth curve, and let

$$z(s) = u(x(s)), \quad p(s) = Du(x(s)). \quad (2.1)$$

By differentiating, one obtains

$$\dot{z} = Du \cdot \dot{x} = p \cdot \dot{x}, \quad \dot{p} = D^2u \dot{x}.$$

Differentiating the PDE

$$D_p F \cdot D^2u + F_u Du + D_x F = 0,$$

so that if now

$$\dot{x} = D_p F, \quad (2.2)$$

then we obtain the following:

$$\dot{z} = p \cdot D_p F, \quad \dot{p} = -F_u p - D_x F,$$

DEFINITION 2.1. The *characteristic equations* are

$$\begin{cases} \dot{x} = D_p F(p, z, x), \\ \dot{z} = D_p F(p, z, x) \cdot p, \\ \dot{p} = -D_x F(p, z, x) - F_z(p, z, x)p, \end{cases} \quad (2.3)$$

with the requirement that the initial data (x_0, z_0, p_0) satisfy $F(p_0, z_0, x_0) = 0$.

LEMMA 2.2. If $F(p_0, z_0, x_0) = 0$, then the solution $(x, z, p)(s)$ to (2.3) satisfies $F(p, z, x) = 0$.

PROOF. From the definition of (x, z, p) , it follows that $\frac{d}{ds} F(p(s), z(s), x(s)) = 0$ by substitution. $\square$

It is also immediate to check that

PROPOSITION 2.3. If u is a C^2 solution to (0.1), then the variables z, p given by (2.1) satisfy (2.3) (where $x(s)$ is defined there).

REMARK 2.4. Note that we require C^2 because we take the derivative of the PDE to compute the ODE for p . In fact, to define a classical solution u , it is required only to be C^1 . When we construct a local solution, we will see that $s \mapsto u(x(s))$ is of regularity C^2 , that is, it has two directional derivatives along the flow, even if u may only be C^1 .

2.1. Examples.

2.1.1. F linear. Assume that

$$F(p, z, x) = a(x) \cdot p + b(x)z + c(x).$$

Then the equation for characteristics is

$$\begin{cases} \dot{p} = -Dc(x) - b(x)p, \\ \dot{z} = a(x) \cdot p = -b(x)z - c(x), \\ \dot{x} = a(x). \end{cases}$$

The *projected characteristic* $x(s)$ is thus autonomous, and the ODE for z does not depend on p . Clearly such a solution is well defined as long as the coefficients a, b, c are regular: moreover, the map

$$u(x(s)) = z(s)$$

is well defined since $x(s)$ is invertible.

The equation for p is not needed: if $X(s, y)$ is the flow generated by $\dot{x} = a(x)$, then

$$u(X(s, y)) = z(s, y),$$

and differentiating w.r.t. s

$$Du \cdot a(x) = \dot{z} = -b(x)z - c(x),$$

which is the PDE. Observe also that to give a distributional solution it is enough to have $s \mapsto u(x(s))$ be differentiable, i.e. we could allow discontinuous functions to the "solutions" (in this extended sense).

EXAMPLE 2.5. The PDE

$$x_1 u_{x_2} - x_2 u_{x_1} = u$$

gives the ODE

$$\dot{x}_1 = -x_2, \quad \dot{x}_2 = x_1, \quad \dot{z} = z,$$

so that the solution is (i imaginary unit, identifying $\mathbb{R}^2$ with the complex plane)

$$x(s) = x_0 e^{is}, \quad z(s) = z_0 e^s.$$

With this formula one can construct the general solution if the initial data is given on a single point of every circle.

2.2. F quasilinear (semilinear). The function F is now

$$F(p, z, x) = a(x, z) \cdot p + b(x, u).$$

The ODE are now

$$\dot{x} = a(x, z), \quad \dot{z} = a(x, z) \cdot p = -b(x, u).$$

In this case, the solution of the ODE exists locally, but it may blow up or the map

$$u(x(s)) = z(s)$$

may not be invertible because we only know that $(x_0, z_0) \mapsto (x(s; x_0, z_0), z(s; x_0, z_0))$ is.

If F is semilinear, i.e. the coefficients of p depend only on x , the function $(s, y) \mapsto x(s, y)$ is always invertible, so that the local existence may occur because of the blowup of $i(x(s))$.

EXAMPLE 2.6. If

$$u_{x_1} + u_{x_2} = u^2,$$

then

$$\dot{x} = (1, 1), \quad \dot{z} = z^2,$$

so that

$$u(x_0 + (s, s)) = \frac{1}{\frac{1}{z_0} - s},$$

which blows up in finite time.

EXAMPLE 2.7. If

$$u_{x_1} + uu_{x_2} = 0,$$

then the characteristics are

$$\dot{x} = (1, z), \quad \dot{z} = 0,$$

so that

$$u(x_0 + (s, z_0 s)) = z_0.$$

However the map $x_0 + (s, z_0 s)$ may not be invertible.

EXAMPLE 2.8. Consider

$$u_{x_1} u_{x_2} = u.$$

The ODE for characteristics is

$$\dot{x} = (p_2, p_1), \quad \dot{z} = 2p_1 p_2, \quad \dot{p} = p.$$

Hence

$$p = p_0 e^s, \quad x = x_0 + p_0^\perp (e^s - 1), \quad z = z_0 + p_{0,1} p_{0,2} (e^{2s} - 1).$$

If the data is given on

$$\Gamma = \{x_1 = 0\}, \quad u|_\Gamma = x_2^2,$$

then we compute from the PDE $F(p, z, x) = 0$

$$p_2 = u_{x_2} = 2x_2, \quad p_1 = u_{x_1} = \frac{u}{u_{x_2}} = \frac{x_2}{2},$$

so that the solution is along the characteristic starting from $(0, y)$

$$\begin{aligned} p(s; y) &= (2y, y/2) e^s, \\ z(s, y) &= y^2 + y^2 (e^{2s} - 1) = y^2 e^{2s}, \\ x(s; y) &= (0, y) + (2y, y/2) (e^s - 1). \end{aligned}$$

The last equation gives

$$\frac{x_2 - y}{x_1} = \frac{1}{4}, \quad y = x_2 - \frac{x_1}{4}, \quad e^s = 1 + \frac{x_1}{2x_2 - \frac{x_1}{2}} = \frac{4x_2 + x_1}{4x_2 - x_1},$$

$$u(x) = \left(x_2 - \frac{x_1}{4}\right)^2 \left(\frac{4x_2 + x_1}{4x_2 - x_1}\right)^2 = \frac{(x_1 + 4x_2)^2}{16}.$$

Indeed

$$\frac{(x_1 + 4x_2)}{8} \frac{(x_1 + 4x_2)4}{8} = \frac{(x_1 + 4x_2)^2}{16}.$$

Note that in this case, even if no curves passes through $\{x_2 = \frac{x_1}{4}\}$, the final formula is valid also there.

3. Initial/Boundary conditions

Aim here is to construct a local solution to

$$F(Du, u, x) = 0, \quad u|_{\Gamma} = g|_{\Gamma},$$

As a first step, assuming Γ smooth, we make a change of variable such that $\Gamma = \{x_d = 0\}$ close to a point $x_0 \in \Gamma$.

Let $\Phi, \Psi = \Phi^{-1}$ be such a smooth mappings: then

$$y = \Phi(x), \quad \Phi(\Gamma) = \{y_d = 0\}, \quad v(y) = u(\Psi(y)).$$

The PDE becomes

$$Du(x) = Dv(\Phi(x))D\Phi(x),$$

$$0 = F(Du, u, x) = F(Dv(\Phi(x))D\Phi(x), v(\Phi(x)), x) = F(Dv(y)D\Phi(\Psi(y)), v(y), \Psi(y)) = G(Dv, v, y).$$

The latter is the expression of the PDE in the new coordinates, having as solutions the image of the solutions of F . The boundary data become

$$h(y) = g(\Psi(y)), \quad y \in \Phi(\Gamma).$$

We will thus consider the boundary problem

$$\begin{cases} F(Du, u, x) = 0, \\ u|_{\{x_d=0\}} = h. \end{cases} \quad (3.1)$$

3.1. Compatibility of the initial data. In order to use the characteristic ODE to construct a solution, one has to find the appropriate initial data. Now, given h , we know the quantities

$$x_0 = (x'_0, 0), \quad z_0 = u(x_0), \quad u_{x_i}(x_0) = h_{x_i}(x_0) \text{ for } i \neq d.$$

The last identity is the PDE by Proposition 2.3, i.e.

$$F(h_{x_1}(x_0), \dots, h_{x_{d-1}}(x_0), u_{x_d}, u(x_0), x_0) = 0.$$

In general, it may happen that u_{x_d} does not exists or it is not unique.

3.1.1. *Noncharacteristic initial data.* In order to construct a solution in a neighborhood of x_0 , we need

- there exists a unique (local) solution $u_d(x_0)$ to

$$F(h_{x_1}(x_0), \dots, h_{x_{d-1}}(x_0), u_{x_d}, u(x_0), x_0) = 0; \quad (3.2)$$

- the flow

$$\mathbb{R} \times \mathbb{R}^{d-1} \ni (s, y) \mapsto x(s, y)$$

generated by

$$\dot{x} = D_p F(p, z, x), \quad y \in \Gamma,$$

is invertible in a neighborhood of $(0, x_0)$.

Assume that there is a local solution to (3.2), i.e. a function

$$y \mapsto p_0(y) = (h_{x_1}(y), \dots, h_{x_{d-1}}(y), p_d(y)), \quad F(p_0(y), u_0(y), (y, 0)) = 0,$$

and let $(p_0(y, 0), z_0(y, 0), (y, 0))$ be the triple corresponding to this solution.

LEMMA 3.1. *If*

$$\partial_{p_d} F(p_0(x_0), z_0(x_0), x_0) \neq 0 \quad \text{at } (y, 0) = x_0,$$

then there is a local solution to the characteristic ODE.

PROOF. By the Implicit Function Theorem, there is a unique smooth function

$$p_{d,0}(y), \quad F(p', p_d, z, x) = 0.$$

Then, since we now know the initial data for the ODEs, the solution is defined locally for every initial point on $\Gamma = \{x_d = 0\}$. $\square$

REMARK 3.2. In the original coordinates, the non characteristic condition is

$$D_p F(p_0, z_0, x_0) D\Phi(x_0) \neq 0 \quad \Leftrightarrow \quad D_p F(p_0, z_0, x_0) \cdot \mathbf{n}(x_0) \neq 0,$$

where $\mathbf{n}$ is the unit normal to Γ .

3.2. Local solution. Still it is not know if the flow is invertible. Assume that the non characteristic condition holds, and let

$$x(s, y), \quad z(s, y), \quad p(s, y)$$

be the solution of the ODE starting from the point $y \in \Gamma$: s is the time-parametrization of the trajectory.

LEMMA 3.3. *The equation*

$$\mathbb{R} \times \mathbb{R}^{d-1} \ni (s, y) \mapsto x(s, y) \in \mathbb{R}^d$$

is locally invertible.

PROOF. The map is smooth, and the derivative at $(0, y_0)$ is

$$x(s, y) = (y, 0) + (D_{p'} F, D_p F) s + o(y, s), \quad D_{s,y} x(0, y_0) = \begin{bmatrix} D_{p'} F & \text{id}_{d-1} \\ D_{p_d} F & 0 \end{bmatrix}$$

which by the non characteristic condition is non singular. $\square$

REMARK 3.4. Another proof is that since $D_{p_d} F \neq 0$, we can use x_d as the time coordinates, obtaining

$$\frac{dx}{dx_d} = \frac{D_p F}{D_{p_d} F}.$$

In particular the characteristics are flowing transversally through Γ .

Hence we can write

$$s = s(x), \quad y = y(x),$$

and the solution to the PDE as a function of x :

$$u(x) = u(s(x), y(x)), \quad p(x) = p(s(x), y(x)). \quad (3.3)$$

THEOREM 3.5. *The function given by (3.3) is a solution of the PDE problem (3.1).*

PROOF. We have to prove two things: the PDE is satisfied and p is the gradient of u .

First of all, by Lemma 2.2 we have that

$$F(p(x), u(x), x) = 0$$

in a neighborhood of x_0 .

It remains to prove that

$$p(x(s)) = D_x u(x).$$

Set

$$r = D_{s,y} z - p \cdot D_{s,y} x = (D_x u - p) D_{s,y} x.$$

For the s direction we have that $r_s = 0$, because it corresponds to the ODE for z ,

$$z_s = \dot{z} = p \cdot D_p F = p \cdot \dot{x} = p \cdot x_s.$$

For the other directions, taking a derivative in s

$$\dot{r}_y = D_y \dot{z} - \dot{p} D_y x - p D_y \dot{x}.$$

Since by the ODE

$$D_y \dot{z} = D_y p \dot{x} + p D_y \dot{x}, \quad \dot{p} = -F_z p - D_x F$$

we obtain

$$\dot{r}_y = D_y p \dot{x} - \dot{p} D_y x = D_y p D_p F - (-F_z p - D_x F) D_y x.$$

Differentiating $F(p, z, x) = 0$ by y

$$D_p F D_y p + F_z D_y z + D_x F D_y x = 0,$$

so that

$$\dot{r} = F_z (p D_y x - D_y z) = -F_z r.$$

On Γ this quantity is 0 by the choice of the initial data, so we deduce that $r = 0$, i.e. $p = Du$. $\square$

4. Exercises

1) Find a solution to

$$u_x + u_y = u, \quad u(0, y) = \sin(y).$$

2) For which initial data there exists a global solution to

$$u_x + u_y = u^2, \quad u(x_1, 0) = v(x_1)?$$

3) For the HJ equation

$$u_t + H(Du) = 0, \quad u(0, x) = u_0(x)$$

find the existence time of the smooth solution.

4) Find the existence time of a smooth solution for the quasilinear equation

$$u_t + \lambda(u) u_x = 0, \quad u(0, x) = u_0(x).$$

5) Consider the PDE

$$\begin{cases} u_{x_1} = 0, & |(x_1, x_2)| < 1, \\ u(\cos(\theta), \sin(\theta)) = u_0(\theta). \end{cases}$$

(In this case the data are more like boundary data.) Determine the conditions on u_0 for there is a solution. Verify that the local existence theorem holds.