

# Game theory

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# 4

## 4.1 Basic definitions

There are many situations in which several decision makers compete with one another to arrive at the best outcome. These competitive decision-making situations are the subject of **game theory**. The study of game theory dates back to 1944, the year in which John von Neumann and Oskar Morgenstern published their classic book *Theory of Games and Economic Behaviour*.

If you have played card games or board games you will be familiar with situations in which competitors develop plans of action in order to win. Game theory deals with similar situations in which competing decision makers develop plans of action in order to win. In addition, game theory develops several mathematical techniques to aid the decision maker in selecting the plan of action that will result in the best outcome.

## 4.2 Types of game situations

Competitive games can be subdivided into several categories. One classification is based on the number of competitive decision makers, called **players**, involved in the game.

- A game situation consisting of two players is called a two person game.

We will confine our attention to such games.

Games can also be classified according to their outcomes in terms of each player's gains and losses.

- If the sum of the players' gains and losses is zero then the game is called a zero-sum game.

In this chapter we will be concerned only with such games. In a **two person zero-sum game** one player's gain is the other player's loss. For example, if one player wins £100 then the other player loses £100, and so the sum of the values is zero.



## 4.3 Two person zero-sum games

By way of illustration consider the following situation.

A camera company X is planning to introduce a new instant camera and wants to capture as large an increase in its market share as possible. The other dominant camera company is Y. Any gain in market share for X will result in an identical loss in market share for Y. Company Y therefore wishes to minimise X's market share increase. Each company has three plans of action they can follow. The table below shows the percentage increase or decrease in market share for X.

|   |       | Y     |       |       |
|---|-------|-------|-------|-------|
|   |       | $Y_1$ | $Y_2$ | $Y_3$ |
| X | $X_1$ | 9     | 7     | 2     |
|   | $X_2$ | 11    | 8     | 4     |
|   | $X_3$ | 4     | 1     | -7    |

This is called the **payoff matrix** or **table**. The payoff table is organised so that the player who is trying to **maximise** the outcome of the game (the offensive player) is down the left-hand side and the player who is trying to minimise the outcome (the defensive player) is across the top. By convention, payoffs are shown for only one of the game players, player X in this case.

In game theory it is assumed that the payoff table is known to both the offensive and defensive players.

A **strategy** is a plan of action to be followed by each player. Each player in a game has two or more strategies, only one of which is selected for each playing of the game. The values in the table are the payoffs or outcomes associated with each player's strategies.

Let us consider two examples.

- (i) If X chooses strategy  $X_2$  and Y chooses strategy  $Y_1$  the outcome is an 11% **gain** for X and an 11% **loss** for Y. The number 11 is called the **value** of the game.
- (ii) If X chooses strategy  $X_3$  and Y chooses strategy  $Y_3$  the outcome is a **loss** of 7% for X and a 7% **gain** for Y. The **value** of this game is -7.

The purpose of the game is for each player to select the strategy that will result in the best possible payoff or outcome **regardless of what the opponent does**. The best strategy for each player is known as the **optimal strategy**.

When each player in the game adopts a **single strategy** as an optimal strategy then the game is a **pure strategy** game. The value of a pure strategy game is the same for both offensive and defensive players. In contrast, in a **mixed strategy** game the players adopt **different strategies** each time the game is played.



## 4.4 Pure strategy games

### Example 1

Consider the following payoff table in which the row player, X, has three possible strategies,  $X_1$ ,  $X_2$  and  $X_3$ , and the column player, Y, has three possible strategies,  $Y_1$ ,  $Y_2$  and  $Y_3$ . The numbers in the table give points scored when the strategies are adopted.

|       | $Y_1$ | $Y_2$ | $Y_3$ |
|-------|-------|-------|-------|
| $X_1$ | 8     | 12    | 9     |
| $X_2$ | 13    | 9     | 8     |
| $X_3$ | 11    | 14    | 10    |

In a pure strategy game each player follows a single strategy regardless of the other player's strategy. Each player chooses the strategy that enables him to do the best he can, given that his opponent knows the strategy he is following.

Player X's objective is to obtain as many points as possible.

If player X chooses strategy  $X_1$  (row 1) his minimum points score will be  $\min(8, 12, 9) = 8$ .

If player X chooses strategy  $X_2$  (row 2) his minimum points score will be  $\min(13, 9, 8) = 8$ .

If player X chooses strategy  $X_3$  (row 3) his minimum points score will be  $\min(11, 14, 10) = 10$ .

We may record this by adding an extra column labelled 'row minimum' to the table:

|       | $Y_1$ | $Y_2$ | $Y_3$ | Row minimum |
|-------|-------|-------|-------|-------------|
| $X_1$ | 8     | 12    | 9     | 8           |
| $X_2$ | 13    | 9     | 8     | 8           |
| $X_3$ | 11    | 14    | 10    | 10          |

Since player X wishes to score as many points as possible he will choose the strategy that guarantees the maximum gain, that is player X should choose  $X_3$  since:

$$\max(8, 8, 10) = 10 \quad (1)$$

This is called a **maximin strategy** and  $10 = \text{maximum over all rows (row minimum)}$ .

To find the **maximin strategy**:

- Find the smallest number in each row. (Remember that  $-6$  is smaller than  $-1$  and  $-2$  is smaller than  $1$ .)
- Select from the numbers found in (i) the largest. The row in which this occurs gives the required strategy.



Player Y's objective is to lose as few points as possible.

If player Y chooses strategy  $Y_1$  (column 1) his maximum loss of points will be  $\max(8, 13, 11) = 13$ .

If he chooses strategy  $Y_2$  (column 2) his maximum loss of points will be  $\max(12, 9, 14) = 14$ .

If he chooses strategy  $Y_3$  (column 3) his maximum loss of points will be  $\max(9, 8, 10) = 10$ .

We may record this by adding an extra row labelled 'column maximum':

|                | $Y_1$ | $Y_2$ | $Y_3$ |
|----------------|-------|-------|-------|
| $X_1$          | 8     | 12    | 9     |
| $X_2$          | 13    | 9     | 8     |
| $X_3$          | 11    | 14    | 10    |
| Column maximum | 13    | 14    | 10    |

Since player Y wishes to keep his loss of points to a minimum, he will choose the strategy that guarantees this, that is he should choose  $Y_3$  since:

$$\min(13, 14, 10) = 10 \quad (2)$$

This is called a **minimax strategy** and  
 $10 = \text{minimum over all columns (column maximum)}$ .

To find the **minimax strategy**:

- (i) Find the largest number in each column. (Remember that 2 is greater than -6 and -1 is greater than -4.)
- (ii) Select from the numbers found in (i) the smallest. The column in which this occurs gives the required strategy.

We may include all the above in a single table:

|                | $Y_1$ | $Y_2$ | $Y_3$ | Row minimum |
|----------------|-------|-------|-------|-------------|
| $X_1$          | 8     | 12    | 9     | 8           |
| $X_2$          | 13    | 9     | 8     | 8           |
| $X_3$          | 11    | 14    | 10    | 10          |
| Column maximum | 13    | 14    | 10    |             |

}  $\max = 10$

$\underbrace{\hspace{10em}}_{\min = 10}$

Notice that in this case, from equations (1) and (2):

$$\begin{array}{ccc} \text{max over all rows} & = & \text{min over all columns} \\ \text{(row minimum)} & & \text{(column maximum)} \end{array} \quad (3)$$



- Any two person zero-sum game satisfying equation (3) is said to have a *saddle point*.
- If a game has a saddle point then we call the common value of both sides of equation (3) the *value  $v$*  of the game to the row player X.

In this example a saddle point occurs when the row player chooses row 3 and the column player chooses column 3. The numerical value of the saddle point, 10 in this case, is the value of the game. Notice that 10 is **the smallest number in its row** and **the largest number in its column**. Thus, like the centre point of a horse's saddle, a saddle point for a two person zero-sum game is a local minimum in one direction (looking across the row) and a local maximum in another direction (looking up and down the column).

A saddle point can also be thought of as an **equilibrium point** in that neither player can benefit from a unilateral change in strategy. If X, the row player, were to change from his optimal strategy  $X_3$  to either  $X_1$  or  $X_2$  his score would decrease.

If Y, the column player, were to change from his optimal strategy  $Y_3$  to either  $Y_1$  or  $Y_2$  his loss would increase. A saddle point is therefore **stable** in that neither player has an incentive to move away from it.

Many two person zero-sum games do not have a saddle point, as the example below shows. Later in this chapter we will see how to find the value and optimal strategies for two person zero-sum games that do not have saddle points.

### Example 2

Show that the two person zero-sum game given by the following payoff table does not have a saddle point.

|   |   |    |
|---|---|----|
| 7 | 1 | 4  |
| 4 | 8 | 11 |
| 2 | 7 | 9  |

Calculating the row minimum and the column maximum we obtain:

|                |   |    |   |
|----------------|---|----|---|
| Row minimum    |   |    |   |
| 7              | 1 | 4  | 1 |
| 4              | 8 | 11 | 4 |
| 2              | 7 | 9  | 2 |
| Column maximum |   |    |   |
| 7              | 8 | 11 |   |

Maximum of row minimum = 4

Minimum of column maximum = 7

Since these are not equal there is no saddle point.



**Example 3**

Obtain the saddle point for the two person zero-sum game given by the following payoff table.

|    |    |
|----|----|
| -1 | 2  |
| -6 | -4 |

The minimum in row 1 is  $\min(-1, 2) = -1$ .

The minimum in row 2 is  $\min(-6, -4) = -6$ .

(Note that  $-6 < -4$ .)

The maximum of these is  $-1$ , since  $-1 > -6$ .

The maximum in column 1 is  $\max(-1, -6) = -1$ .

The maximum in column 2 is  $\max(2, -4) = 2$ .

The minimum of these is  $-1$ .

Hence the saddle point condition is satisfied and  $(-1)$  is the saddle point and the value of the game.

**Exercise 4A**

- 1 Show that the following two person zero-sum games do not have saddle points. *PUNTO DI EQUILIBRIO*

(a)

|   |   |   |
|---|---|---|
| 3 | 5 | 2 |
| 4 | 1 | 3 |
| 1 | 4 | 6 |

(b)

|   |   |
|---|---|
| 4 | 5 |
| 5 | 3 |

(c)

|   |   |   |   |
|---|---|---|---|
| 8 | 6 | 2 | 1 |
| 7 | 0 | 1 | 3 |

(d)

|   |   |
|---|---|
| 6 | 5 |
| 4 | 8 |
| 2 | 7 |

- 2 For the payoff tables below:

- (a) show that there is a saddle point *= punto di equilibrio*  
 (b) obtain the strategies for the players  
 (c) give the value of the game.

(i)

|   |   |
|---|---|
| 5 | 4 |
| 3 | 2 |

(ii)

|    |   |
|----|---|
| -2 | 3 |
| -1 | 1 |

(iii)

|    |    |
|----|----|
| -2 | -3 |
| 4  | -1 |
| 1  | 0  |

(iv)

|    |    |    |
|----|----|----|
| 5  | 2  | 4  |
| -2 | 1  | -1 |
| 2  | -1 | 3  |

(v)

|   |   |   |   |
|---|---|---|---|
| 3 | 6 | 5 | 4 |
| 4 | 7 | 6 | 5 |



## 4.5 Mixed strategies

When there is no saddle point, no pure strategy exists. Players will then play each strategy for a certain proportion (fraction) of the time. This is called a **mixed strategy** game. Let us begin by looking at  $2 \times 2$  games.

### Mixed strategies for $2 \times 2$ games

Consider the  $2 \times 2$  payoff table:

|       | $Y_1$ | $Y_2$ |
|-------|-------|-------|
| $X_1$ | 4     | 2     |
| $X_2$ | 3     | 10    |

For this game: maximum of row minimum = 3  
minimum of column maximum = 4

so there is **no saddle point**.

Suppose X plays strategy  $X_1$  a fraction  $p$  ( $\geq 0$ ) of the time, then she will play strategy  $X_2$  a fraction  $(1 - p)$  of the time.

The gain for X if Y plays strategy  $Y_1$  is:

$$4p + 3(1 - p) = p + 3$$

The gain for X if Y plays strategy  $Y_2$  is:

$$2p + 10(1 - p) = -8p + 10$$

Let  $v$  be the value of the game to X.

That means  $v$  is the amount X will win per game if she plays her best strategy consistently and Y plays the best possible counter strategy against her.

Then our problem is to find  $p$  so as to maximise  $v$  where:

$$v \leq p + 3$$

and 
$$v \leq -8p + 10$$

These inequalities hold since the right-hand sides are what X will win per game against Y's pure strategies  $Y_1$  and  $Y_2$ .

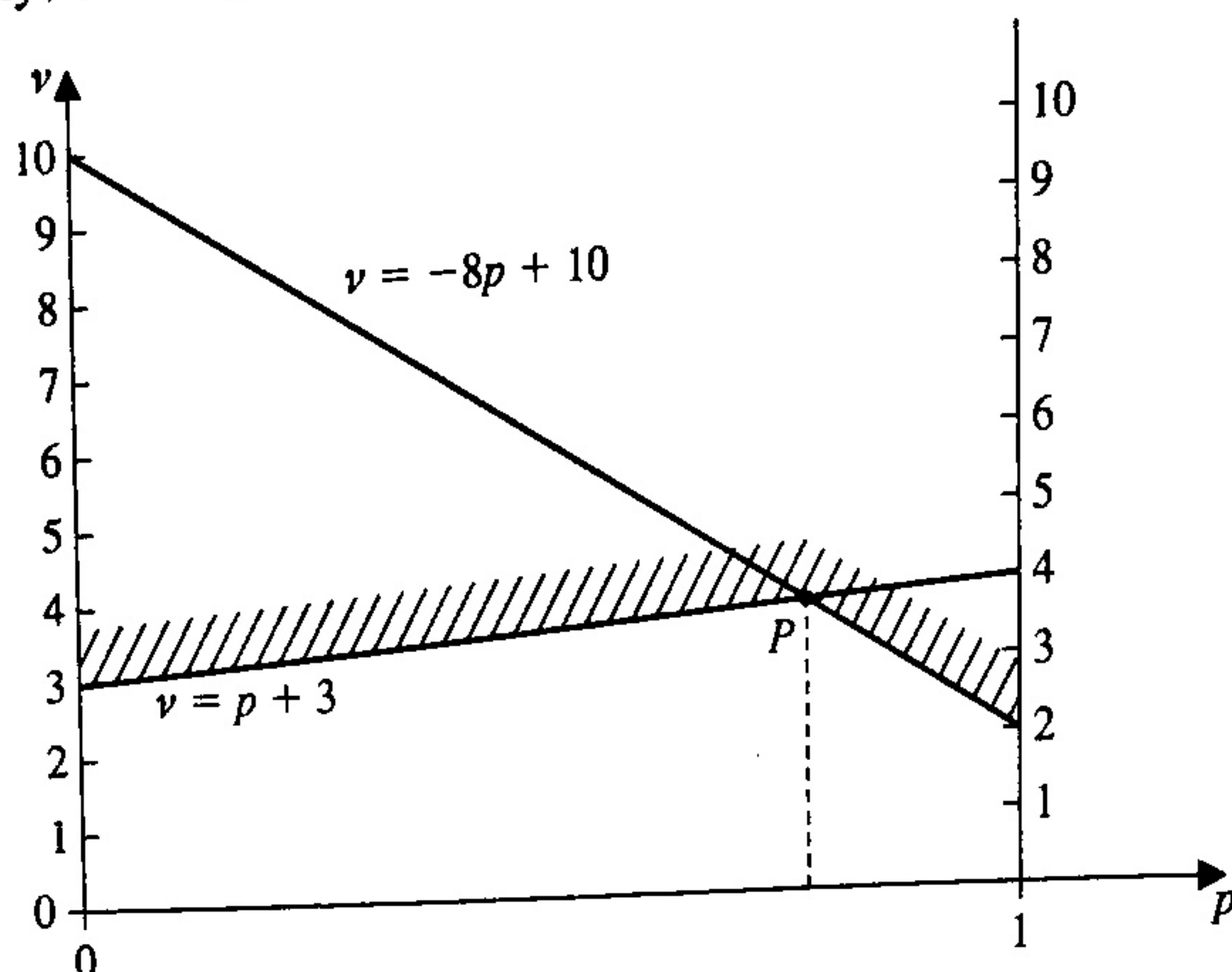
Thus we want to find, for each value of  $p$ , the smaller of the two functions  $(p + 3)$  and  $(-8p + 10)$ , and then choose the value of  $p$  for which the smaller function is as large as possible.

Consider the graphs of the functions  $v = p + 3$  and  $v = -8p + 10$ , for values of  $p$  between 0 and 1 ( $p$  is a fraction and so  $0 \leq p \leq 1$ ).

When  $p = 0$ ,  $p + 3 = 3$   
 $p = 1$ ,  $p + 3 = 4$



So  $v = p + 3$  goes through  $(0, 3)$  and  $(1, 4)$ .  
 Similarly,  $v = -8p + 10$  goes through  $(0, 10)$  and  $(1, 2)$ .



We see that the unshaded region bounded by two line segments, has its highest point at the point  $P$ , where the two lines intersect. This must be the case in every such problem.

Hence the required value of  $p$  is obtained when:

$$p + 3 = -8p + 10$$

or

$$9p = 7$$

so

$$p = \frac{7}{9} \text{ and } (1 - p) = \frac{2}{9}$$

$X$  should therefore use strategy  $X_1$  a fraction  $\frac{7}{9}$  of the time and  $X_2$  a fraction  $\frac{2}{9}$  of the time. The value  $v$  of the game is given by:

$$(p + 3) \quad \text{when} \quad p = \frac{7}{9}$$

or

$$-8p + 10 \quad \text{when} \quad p = \frac{7}{9}$$

So

$$v = \cancel{\frac{37}{9}} \quad 3 + \frac{7}{9}$$

The point  $P$  has coordinates  $(\frac{7}{9}, \frac{37}{9})$ .  $3 + \frac{7}{9}$

We can deal with  $Y$ 's strategies in the same way. Suppose she uses strategy  $Y_1$  for a fraction  $q$  of the time and strategy  $Y_2$  for a fraction  $(1 - q)$  of the time.

The gain for  $Y$  is then:

$$-[4q + 2(1 - q)] \text{ if } X \text{ plays strategy } X_1$$

and

$$-[3q + 10(1 - q)] \text{ if } X \text{ plays strategy } X_2$$

(Notice the minus signs here.)



In the light of the above,  $q$  is determined from:

$$4q + 2(1 - q) = 3q + 10(1 - q)$$

$$2q + 2 = -7q + 10$$

$$9q = 8$$

or

So  $q = \frac{8}{9}$  and  $(1 - q) = \frac{1}{9}$

Y should therefore use strategy  $Y_1$  for  $\frac{8}{9}$  of the time and strategy  $Y_2$  for  $\frac{1}{9}$  of the time.

Y's loss is given by  $-(2q + 2)$  when  $q = \frac{8}{9}$  and so is  $-3\frac{7}{9}$ .

#### Example 4

The payoff table for a two person zero-sum game is:

|       | $Y_1$ | $Y_2$ |
|-------|-------|-------|
| $X_1$ | 2     | 10    |
| $X_2$ | 6     | 1     |

Find the best combination of strategies for each player and the value of the game.

Suppose X uses strategy  $X_1$  for a fraction  $p$  of the time and strategy  $X_2$  for a fraction  $(1 - p)$  of the time. Equating the gain for X when Y plays  $Y_1$ , and the gain for X when Y plays  $Y_2$  we obtain:

$$2p + 6(1 - p) = 10p + (1 - p)$$

$$-4p + 6 = 9p + 1$$

or

$$5 = 13p$$

So

$$p = \frac{5}{13}$$

and

$$(1 - p) = \frac{8}{13}$$

X should therefore use  $X_1$  for a fraction  $\frac{5}{13}$  of the time and  $X_2$  for a fraction  $\frac{8}{13}$  of the time.

Her gain will be  $(9p + 1)$  when  $p = \frac{5}{13}$ , that is  $4\frac{6}{13}$ . This is the value of the game.

Suppose Y uses strategy  $Y_1$  for a fraction  $q$  of the time and  $Y_2$  for a fraction  $(1 - q)$  of the time.

Then:

$$2q + 10(1 - q) = 6q + (1 - q)$$

$$-8q + 10 = 5q + 1$$

or

$$13q = 9$$

So

$$q = \frac{9}{13}$$

and

$$(1 - q) = \frac{4}{13}$$

The loss of Y will be  $-(5q + 1)$  when  $q = \frac{9}{13}$ , that is  $-4\frac{6}{13}$ .



**Exercise 4B**

- 1 In each of the following  $2 \times 2$  games find:
- the optimal mixed strategies for both players
  - the value of the game.

(a)

|   |   |
|---|---|
| 7 | 1 |
| 4 | 8 |

(b)

|    |    |
|----|----|
| 5  | 15 |
| 11 | 6  |

(c)

|    |    |
|----|----|
| 5  | -2 |
| -8 | 6  |

**Mixed strategies for  $2 \times 3$  and  $3 \times 2$  games**

Games without saddle points where one player has only two possible strategies can be solved by a straightforward extension of the graphical approach used above.

**Example 5**

Consider the two person zero-sum game for which the payoff matrix is:

|   |                |                |                |                |
|---|----------------|----------------|----------------|----------------|
|   |                | Y              |                |                |
|   |                | Y <sub>1</sub> | Y <sub>2</sub> | Y <sub>3</sub> |
| X | X <sub>1</sub> | 4              | 2              | 6              |
|   | X <sub>2</sub> | 2              | 10             | 1              |

The game does not have a saddle point.

Since X has only two possible strategies suppose he uses strategy X<sub>1</sub> for a fraction  $p$  of the time and strategy X<sub>2</sub> for a fraction  $(1 - p)$  of the time. Then his expected gain if Y plays Y<sub>1</sub> is:

$$4p + 2(1 - p) = 2 + 2p$$

His expected gain if Y plays Y<sub>2</sub> is:

$$2p + 10(1 - p) = 10 - 8p$$

His expected gain if Y plays Y<sub>3</sub> is:

$$6p + (1 - p) = 1 + 5p$$

So X's problem is to maximise his game value  $v$  subject to:

$$v \leq 2 + 2p, v \leq 10 - 8p, v \leq 1 + 5p$$

by an appropriate choice of  $p$ .

We can find the value of  $p$  that achieves this by plotting the lines with equations:

$$v = 2 + 2p \quad (l_1)$$

$$v = 10 - 8p \quad (l_2)$$

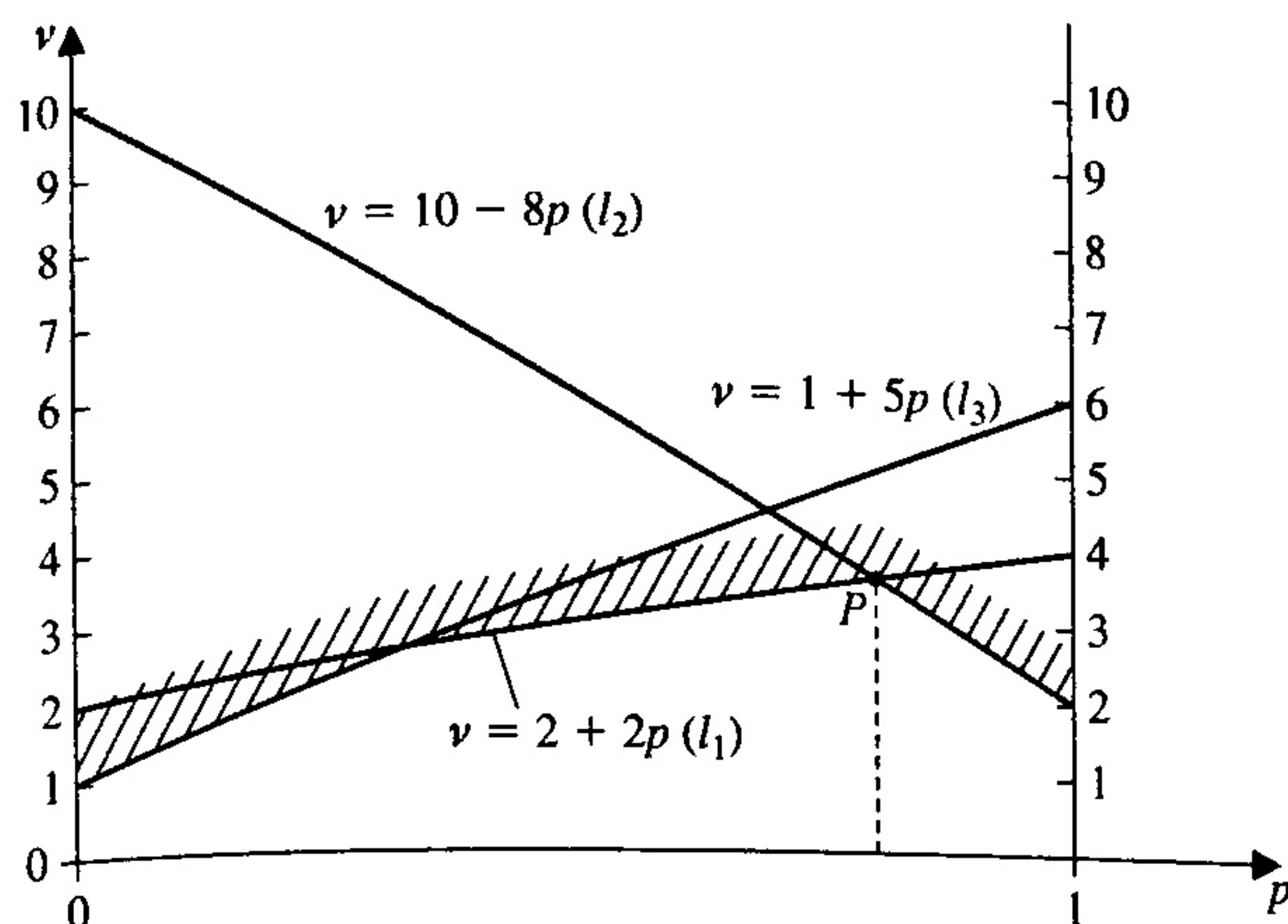
$$v = 1 + 5p \quad (l_3)$$

$$0 \leq p \leq 1$$

and  
for



These are shown below.



Hence X can maximise the value of the game by choosing the value of  $p$  corresponding to the highest point  $P$  of the unshaded region. From the diagram,  $P$  is at the intersection of  $l_1$  and  $l_2$ , so the value of  $p$  is given by:

$$2 + 2p = 10 - 8p$$

or 
$$10p = 8$$

So 
$$p = \frac{4}{5} \quad \text{and} \quad (1 - p) = \frac{1}{5}$$

and the value of the game is  $(2 + 2p)$  when  $p = \frac{4}{5}$ , that is  $v = 3\frac{3}{5}$ .

X should therefore use strategy  $X_1$  for  $\frac{4}{5}$  of the time and strategy  $X_2$  for  $\frac{1}{5}$  of the time.

We also note from the above that Y must only use strategies  $Y_1$  and  $Y_2$ , corresponding to  $l_1$  and  $l_2$ , and never strategy  $Y_3$  if he is to minimise X's gain, when the latter plays the best combination of strategies. This illustrates a general feature of  $2 \times n$  games, which is that the player with  $n$  strategies will, in fact, be able only to use two of them in his best mix. This follows from the fact that the combination the opponent will use is determined by the intersection of lines corresponding to two strategies. Introducing any strategies other than these two will give the opponent a *greater* gain than necessary.

Having discovered Y will only use strategies  $Y_1$  and  $Y_2$ , suppose he uses  $Y_1$  for a fraction  $q$  of the time and hence  $Y_2$  for a fraction  $(1 - q)$  of the time.

Then using:

$$(\text{gain for Y if X uses } X_1) = (\text{gain for Y if X uses } X_2)$$

we obtain:

$$4q + 2(1 - q) = 2q + 10(1 - q)$$



$$\begin{aligned} \text{or} \quad 2q + 2 &= -8q + 10 \\ 10q &= 8 \end{aligned}$$

$$\text{So} \quad q = \frac{8}{10} = \frac{4}{5} \quad \text{and} \quad (1 - q) = \frac{1}{5}$$

Hence Y should use strategy  $Y_1$  for  $\frac{4}{5}$  of the time and strategy  $Y_2$  for  $\frac{1}{5}$  of the time and never use  $Y_3$ .

### Example 6

Consider the two person zero-sum game for which the payoff matrix is:

|       | $Y_1$ | $Y_2$ |
|-------|-------|-------|
| $X_1$ | 9     | 1     |
| $X_2$ | 6     | 5     |
| $X_3$ | 3     | 7     |

The game does not have a saddle point.

Since Y has only two possible strategies, suppose she uses strategy  $Y_1$  for a fraction  $q$  of the time and strategy  $Y_2$  for a fraction  $(1 - q)$  of the time. Then her expected gain if X plays  $X_1$  is  $-X$ 's gain:

$$\begin{aligned} &= -[9q + (1 - q)] \\ &= -[8q + 1] \end{aligned}$$

Her expected gain if X plays  $X_2$  is:

$$\begin{aligned} &= -[6q + 5(1 - q)] \\ &= -[q + 5] \end{aligned}$$

Her expected gain if X plays  $X_3$  is:

$$\begin{aligned} &= -[3q + 7(1 - q)] \\ &= -[-4q + 7] \end{aligned}$$

If  $v$  is the value of the game to X, then Y wishes her gain, in each case, to be  $\geq (-v)$ .

So we have

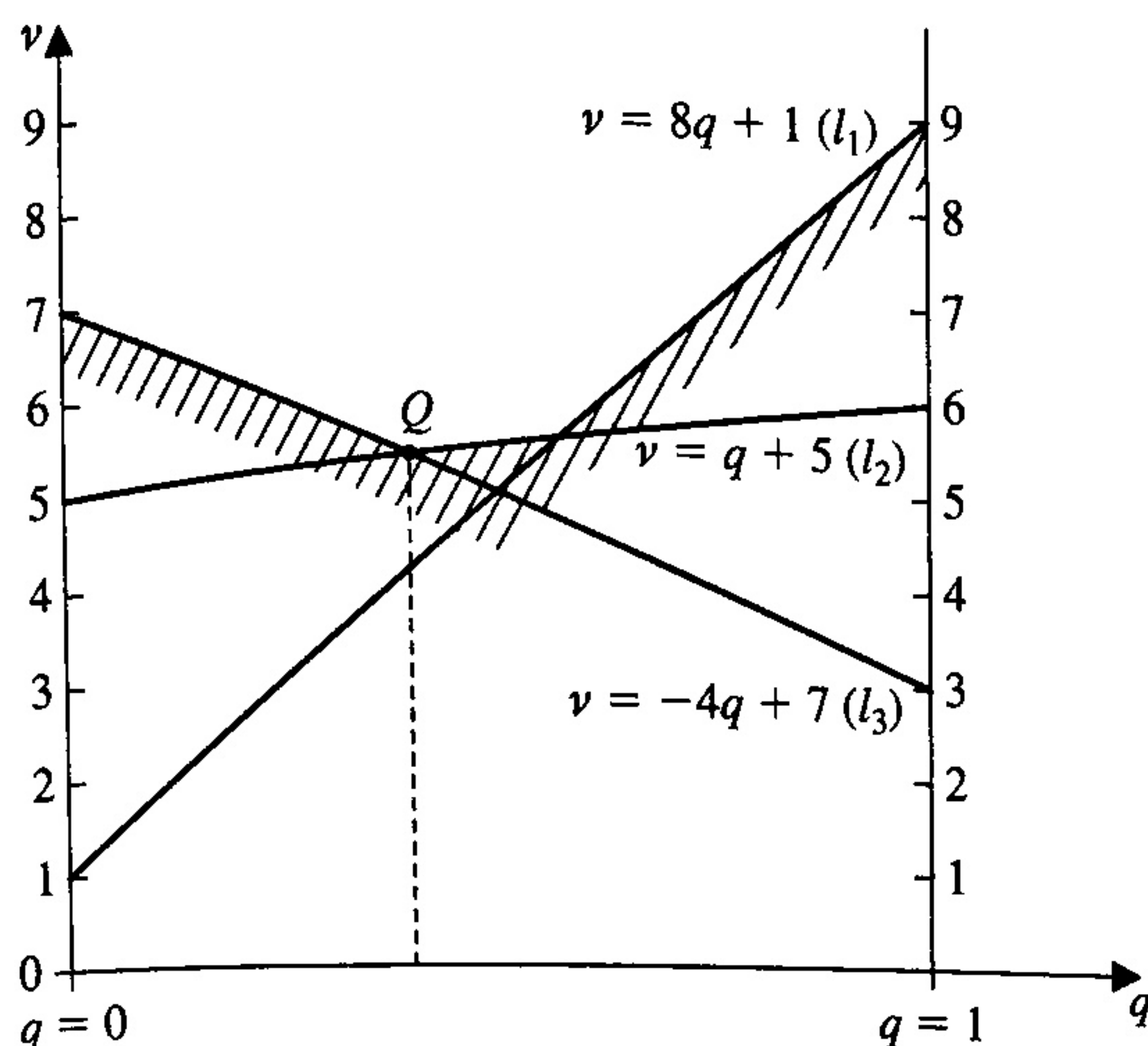
$$\begin{aligned} \text{or} \quad &-(8q + 1) \geq -v \\ &8q + 1 \leq v \end{aligned} \tag{1}$$

$$\begin{aligned} \text{or} \quad &-(q + 5) \geq -v \\ &q + 5 \leq v \end{aligned} \tag{2}$$

$$\begin{aligned} \text{and} \quad &-(-4q + 7) \geq -v \\ \text{or} \quad &-4q + 7 \leq v \end{aligned} \tag{3}$$

Since Y wishes to keep X's gain as small as possible we choose the value of  $q$  that satisfies all these constraints and for which  $v$  is as small as possible.





The graph shows the lines:

$$\begin{aligned} l_1: v &= 8q + 1 \\ l_2: v &= q + 5 \\ l_3: v &= -4q + 7 \end{aligned}$$

The inequalities (1), (2) and (3) are satisfied in the unshaded region and the point  $Q$  gives the required value of  $q$ . This is the intersection of lines  $l_2$  and  $l_3$ .

So 
$$q + 5 = -4q + 7$$

or 
$$5q = 2$$

So 
$$q = \frac{2}{5} \quad \text{and} \quad 1 - q = \frac{3}{5}$$

Y therefore plays  $Y_1$  for  $\frac{2}{5}$  of the time and  $Y_2$  for  $\frac{3}{5}$  of the time. The value of the game is  $(q + 5)$  when  $q = \frac{2}{5}$  and so is  $5\frac{2}{5}$  to X. This means Y loses  $5\frac{2}{5}$ .

Since the above solution occurs at the intersection of  $l_2$  and  $l_3$ , strategy  $X_1$  will not be used by X.

Suppose X uses strategy  $X_2$  for a fraction  $p$  of the time and strategy  $X_3$  for a fraction  $(1 - p)$  of the time. Then using:

$$(\text{gain for X if Y uses } Y_1) = (\text{gain for X if Y uses } Y_2)$$

$$6p + 3(1 - p) = 5p + 7(1 - p)$$

$$3p + 3 = -2p + 7$$

or 
$$5p = 4$$

So 
$$p = \frac{4}{5} \quad \text{and} \quad (1 - p) = \frac{1}{5}$$

X will therefore not use strategy  $X_1$  at all, will use  $X_2$  a fraction  $\frac{4}{5}$  of the time and use  $X_3$  a fraction  $\frac{1}{5}$  of the time.

X's gain will be  $(3p + 3)$  when  $p = \frac{4}{5}$ , that is  $5\frac{2}{5}$ , the value of the game to X.



- (a) Obtain the linear programming problem which when solved will give the optimal strategy for B.
- (b) Solve the linear programming problem to obtain B's optimal strategy and the value of the game.
- (c) Deduce the optimal strategy for A.

## 4.7 Dominance

The principle of **dominance** can be used to reduce the size of games by eliminating strategies that would never be used. A strategy for a player can be eliminated if the player can always do as well or better playing another strategy. In other words, **a strategy can be eliminated if all its game outcomes are the same or worse than the corresponding game outcomes of another strategy.**

### Example 8

Let us look again at the payoff table given in Example 1.

|   |                |                |                |                |
|---|----------------|----------------|----------------|----------------|
|   |                | Y              |                |                |
|   |                | Y <sub>1</sub> | Y <sub>2</sub> | Y <sub>3</sub> |
| X | X <sub>1</sub> | 8              | 12             | 9              |
|   | X <sub>2</sub> | 13             | 9              | 8              |
|   | X <sub>3</sub> | 11             | 14             | 10             |

For player X's strategies we employ **the dominance rule for rows:**

- Every value in the dominating row must be equal to, or *greater than*, the corresponding value in the dominated row.

Applying this rule to the above table we can see that row 3, that is strategy X<sub>3</sub>, dominates row 1, that is strategy X<sub>1</sub>, since 11 > 8, 14 > 12 and 10 > 9.

Row 1 and strategy X<sub>1</sub> can therefore be eliminated. The reduced payoff matrix obtained is:

|   |                |                |                |                |
|---|----------------|----------------|----------------|----------------|
|   |                | Y              |                |                |
|   |                | Y <sub>1</sub> | Y <sub>2</sub> | Y <sub>3</sub> |
| X | X <sub>2</sub> | 13             | 9              | 8              |
|   | X <sub>3</sub> | 11             | 14             | 10             |

For player Y's strategies we employ **the dominance rule for columns:**

- Every value in the dominating column must be equal to, or *less than*, the corresponding value in the dominated column.



**Note:** The rule for rows involves 'greater than' but the rule for columns involves 'less than'.

Applying the column rule to the reduced payoff table above we see that column 3, that is strategy  $Y_3$ , **dominates** column 2, that is strategy  $Y_2$ , since  $8 < 9$  and  $10 < 14$ .

Column 2 can therefore be eliminated. The reduced payoff matrix is then:

|       | $Y_1$ | $Y_3$ |
|-------|-------|-------|
| $X_2$ | 13    | 8     |
| $X_3$ | 11    | 10    |

We have reduced the original  $3 \times 3$  game to a  $2 \times 2$  game. Clearly it is easier to handle the smaller game. This game has a saddle point since:

$$\begin{aligned}\max(\text{row minimum}) &= 10 \\ \min(\text{column maximum}) &= 10\end{aligned}$$

### Example 9

Consider the following payoff table:

|   |     | B  |    |     |
|---|-----|----|----|-----|
|   |     | I  | II | III |
| A | I   | 8  | 6  | 2   |
|   | II  | 11 | 8  | 4   |
|   | III | 4  | 1  | 7   |

- Apply the principle of dominance to this payoff table.
- Determine player A's optimal strategy.
- Determine player B's optimal strategy.
- State the value of the game.

(a) Consider first the rows of the table. Row 2 dominates row 1 since  $11 > 8$ ,  $8 > 6$  and  $4 > 2$ .

So we can eliminate row 1.

The reduced table is then:

|   |     | B  |    |     |
|---|-----|----|----|-----|
|   |     | I  | II | III |
| A | II  | 11 | 8  | 4   |
|   | III | 4  | 1  | 7   |



Consider now the columns.

Column 2 dominates column 1 since  $8 < 11$  and  $1 < 4$ .  
So we can eliminate column 1 and obtain the reduced table:

|   |     | B  |     |
|---|-----|----|-----|
|   |     | II | III |
| A | II  | 8  | 4   |
|   | III | 1  | 7   |

(b) Suppose A uses strategy II for a fraction  $p$  of the time and strategy III for a fraction  $(1 - p)$  of the time. Then we require that:

$$8p + (1 - p) = 4p + 7(1 - p)$$

so

$$7p + 1 = -3p + 7$$

and

$$10p = 6$$

or

$$p = \frac{3}{5} \text{ and } (1 - p) = \frac{2}{5}$$

The optimal strategy for A is then:

never use strategy I

use strategy II for  $\frac{3}{5}$  of the time

use strategy III for  $\frac{2}{5}$  of the time

(c) Suppose B uses strategy II for a fraction  $q$  of the time and strategy III for a fraction  $(1 - q)$  of the time. Then we require that:

$$8q + 4(1 - q) = q + 7(1 - q)$$

so

$$4q + 4 = -6q + 7$$

and

$$10q = 3$$

or

$$q = \frac{3}{10} \text{ and } (1 - q) = \frac{7}{10}$$

The optimal strategy for B is then:

never use strategy I

use strategy II for  $\frac{3}{10}$  of the time

use strategy III for  $\frac{7}{10}$  of the time

(d) The value of the game is given, from (b), by  $(7p + 1)$  when  $p = \frac{3}{5}$ , so  $v = \frac{26}{5} = 5\frac{1}{5}$ .

We could also calculate it from (c) as  $(4q + 4)$  when  $q = \frac{3}{10}$ , which again gives  $v = 5\frac{1}{5}$ .



## 4.8 Summary of how to solve a two person zero-sum game

- To end this chapter we present a summary of the procedure for finding the optimal strategies and the value of a two person zero-sum game.

- Step 1 Check to see if the game has a saddle point. If not go to step 2.
- Step 2 Eliminate as many rows and columns as possible using the principle of dominance.
- Step 3 Solve the resulting game by an appropriate method. If we have a  $2 \times 2$ ,  $2 \times 3$  or  $3 \times 2$  game these may be solved graphically as we have seen. Otherwise solve the game by using linear programming methods.

### Exercise 4E

- 1 Consider the following payoff table:

|   |     | B   |     |     |
|---|-----|-----|-----|-----|
|   |     | I   | II  | III |
| A | I   | 1.8 | 1.1 | 1.3 |
|   | II  | 1.3 | 1.5 | 1.8 |
|   | III | 1.3 | 1.3 | 1.8 |

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- (a) Apply the principle of dominance to this payoff table.
- (b) Determine player A's optimal strategy.
- (c) Determine player B's optimal strategy.
- (d) State the value of the game.

- 2 Consider the following payoff table:

|   |     | Y  |    |     |
|---|-----|----|----|-----|
|   |     | I  | II | III |
| X | I   | -2 | 4  | 5   |
|   | II  | 0  | -3 | 6   |
|   | III | 5  | 1  | -6  |

1 1/2 3  
2 -4 -5  
0 3 -6  
Vikate minon

- (a) Apply the principle of dominance to this payoff table.
- (b) Determine player X's optimal strategy.
- (c) Determine player Y's optimal strategy.
- (d) State the value of the game.



3 Consider the following payoff matrix:

|   |     | Q  |    |
|---|-----|----|----|
|   |     | I  | II |
| P | I   | 2  | 4  |
|   | II  | 1  | 2  |
|   | III | 3  | 0  |
|   | IV  | -1 | 6  |

- Apply the principle of dominance to this payoff table.
- Determine player Q's optimal strategy.
- Hence determine player P's optimal strategy.
- What is the value of the game?

### SUMMARY OF KEY POINTS

- A **two person game** is one in which only two parties can play.
- A **zero-sum game** is one in which the sum of the losses for one player is equal to the sum of the gains for the other player.
- A two person zero-sum game for which  $\max \text{ over all rows (row minimum)} = \min \text{ over all columns (column maximum)}$  is said to have a **saddle point**. The common value of both sides of the above equation is the **value  $v$**  of the game to the row player.
- Dominance rule for rows**  
If every entry in a given row  $R_1$  is **less than or equal to** the corresponding entry in another given row  $R_2$ , then  $R_1$  is said to be **dominated** by  $R_2$  and may therefore be eliminated.
- Dominance rule for columns**  
If every entry in a given column  $C_1$  is **greater than or equal to** the corresponding entry in another given column  $C_2$ , then  $C_1$  is said to be **dominated** by  $C_2$  and may therefore be eliminated.
- To solve a two person zero-sum game.
  - Check to see if the game has a saddle point. If not go to step 2.
  - Eliminate as many rows and columns as possible using the principle of dominance.
  - Solve the resulting game by an appropriate method.