

Revisiting the basics

In this course we are going to talk about different quantum platforms: atoms in tweezers, trapped ions, superconducting qubits and photonics systems

The goal of this first class is to recap the definitions of the qubit, its representation on the Bloch sphere, the concept of measurement, mixed states, the interaction picture and the DiVincenzo's criteria to build a quantum computer.

The qubit

Before jumping into the qubit let's say a few words about the bit! The bit is an abstraction of any two-state classical system. Think of a light switch that can be turned on and off, a coin that we can see heads or tails... in general all sort of systems can be used to encode one bit of information: a variable that can take one out of two possible values, $x \in \{0, 1\}$

Information theory would not be very interesting if it could only speak of known bits, like the switch that we remember having turned on. We want to make predictions about future events, based on our current partial information: for instance, we may toss a coin that we know is biased. To express this notion of uncertainty about a random bit, we represent it as a random variable X . This is just an object that is associated with two things: a set of possible outcomes (0 or 1 in the case of a bit), which we call the alphabet (or range) $\mathcal{X} = \{0, 1\}$ and a probability distribution P_X that gives the probability of each outcome. In the case of a fair coin, or any other uniformly random bit, this would

just be $P_X(0) = P_{\text{ROB}} [X=0] = \frac{1}{2}$ AND $P_X(1) = P_{\text{ROB}} [X=1] = \frac{1}{2}$.

Finally, what operations can we perform on a single bit? When it comes to reversible operations, all we can do is flip the bit. In other words, we can apply a NOT operation (or gate), which maps 0 to 1 and vice-versa.

The qubit is just a generalization of the bit. However, we say that these states correspond to vectors

in \mathbb{C}^2

$$\begin{cases} '0' \rightarrow |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ '1' \rightarrow |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{cases}$$

EQUIVALENT TO A SPIN $\frac{1}{2}$ SYSTEM

$|0\rangle \equiv |\downarrow\rangle$
 $|1\rangle \equiv |\uparrow\rangle$

a qubit can also be in any superposition of these two states,

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \quad \text{with } |\alpha|^2 + |\beta|^2 = 1$$

Because the coefficients are normalized, we can express them in terms of

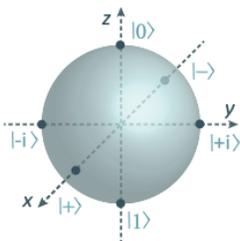
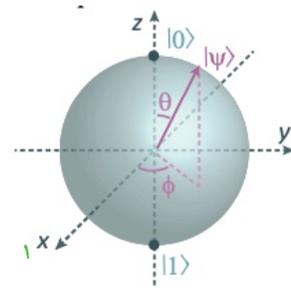
angles

$$|\psi\rangle = e^{i\gamma} \left[\cos\frac{\theta}{2} |0\rangle + e^{i\varphi} \sin\frac{\theta}{2} |1\rangle \right]$$

γ IS IRRELEVANT, $\theta \in [0, \pi]$, $\varphi \in [0, 2\pi]$



IT IS A GLOBAL PHASE \Rightarrow DOES NOT COUNT WHEN WE CALCULATE PROBABILITIES



OTHER IMPORTANT STATES

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

So what can we do with a qubit? A bit could encode a single unit of information (0 or 1), but it sounds like we might encode infinite information on a qubit: we imposed no restrictions on the two angles, which may take any real value within that interval.

Seems a little over-powerful for an object described as the minimal unit of quantum information. As it happens, there is a catch. Although we may prepare any state we want, we can only read one out of

two outcomes.

Measurements

When reading a bit we can obtain either 0 or 1. With qubits, we still only obtain one bit of information, but with a fundamental difference: we can choose the basis of measurement.

The probability of reading out a given state $|\varphi\rangle$ when measuring a qubit in state $|\psi\rangle$ is given by the

overlap between the two states, measured by $\text{Prob}[\varphi]_{\psi} = |\langle\varphi|\psi\rangle|^2$

IN THE COMPUTATIONAL BASIS $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$

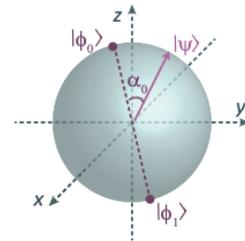
$$\Rightarrow \text{Prob}[0]_{\psi} = |\langle 0|\psi\rangle|^2 = \cos^2\frac{\theta}{2} = |\alpha|^2$$

$$\text{Prob}[1]_{\psi} = |\langle 1|\psi\rangle|^2 = \sin^2\frac{\theta}{2} = |\beta|^2$$

IT WORKS ALSO FOR ANY OTHER ORTHONORMAL BASIS $\{|\varphi_0\rangle, |\varphi_1\rangle\}$

$$\text{Prob}[\varphi_0]_{\psi} = |\langle\varphi_0|\psi\rangle|^2 = \cos^2\frac{\alpha_0}{2}$$

$$\text{Prob}[\varphi_1]_{\psi} = |\langle\varphi_1|\psi\rangle|^2 = \cos^2\left(\frac{\pi}{2} - \frac{\alpha_0}{2}\right)$$



More generally, a measurement is defined by a set of operators $\{M_m\}$ which act on the state space of the system. The probability of a measurement result m is

$$P(m) = \langle\psi|M_m^\dagger M_m|\psi\rangle \quad \text{with} \quad \sum_m M_m^\dagger M_m = \mathbb{1} \quad \text{BECAUSE} \quad \sum P(m) = 1$$

EXAMPLE: COMPUTATIONAL BASIS

$$M_0 = 10 \times 01 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad M_1 = 11 \times 11 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

For $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$$P(0) = \underbrace{\langle\psi|}_{\alpha^*} \underbrace{10 \times 01}_{1} \underbrace{|\psi\rangle}_{\alpha} = \alpha^* \alpha = |\alpha|^2$$

Mixed states

All the discussion we had so far is valid for pure states, however, in real world scenarios due to noise and errors states tend to be a statistical mixture of two or more states. For example, a flawed source of qubits which is supposed to produce state $|\psi\rangle$, may produce state $|\chi\rangle$ with probability p . We can thus write this mixture using the density matrix ρ , which generalize the concept of pure states.

$$\rho = (1-p)|\psi\rangle\langle\psi| + p|\chi\rangle\langle\chi|$$

more generally $\rho = \sum_n P_n |\psi_n\rangle\langle\psi_n|$ NOTE THAT ρ CAN ALSO REFER TO PURE STATES.
 $\rho = |\psi\rangle\langle\psi|$

THE MEASUREMENT THEN TAKES THE FORM OF

$$P(m) = \text{Tr}(M_m \rho)$$

DIFFERENCES BETWEEN MIXED AND SUPERPOSITION STATES

CASE 1: SUPERPOSITION. $|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \Rightarrow \rho_+ = |+\rangle\langle+| = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \frac{\langle 0| + \langle 1|}{\sqrt{2}}$
 $= \frac{|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|}{2}$

IN THE COMPUTATIONAL BASIS: $\rho_+ = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

IN THE $\{|+\rangle, |-\rangle\}$ BASIS $\rho_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

CASE 2: FULLY MIXED STATE

$\rho_n = \frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2}$ SO IN COMPUTATIONAL BASIS $\rho_n = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

IN $\{| \pm \rangle\}$ BASIS. WRITE $|0\rangle = \frac{|+\rangle + |-\rangle}{\sqrt{2}}$, $|1\rangle = \frac{|+\rangle - |-\rangle}{\sqrt{2}}$

$\Rightarrow \rho_n = \frac{|+\rangle\langle+| + |-\rangle\langle-|}{2} \Rightarrow \rho_n = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

IN THE CASE OF PURE STATES THERE IS INTERFERENCE (OFF-DIAGONAL TERMS)!

The trace of a density matrix is 1.

$$\text{Tr}(\rho) = 1 \quad \text{WHY? } \rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \Rightarrow$$

$$\frac{1}{d} \leq \text{Tr}(\rho^2) \leq 1$$

↑ HILBERT SPACE DIMENSION

$$\text{Tr}(\rho) = \sum_i p_i \text{Tr}(|\psi_i\rangle\langle\psi_i|) = \sum_i p_i = 1$$

↑ POLES STAYS

Bloch Sphere representation

In the Bloch representation, pure states lie on the surface of a sphere. Mixed states are inside the

sphere. In fact, if two pure states $|\chi\rangle, |\psi\rangle$ have the Bloch vectors \vec{t}, \vec{p} , respectively, then the Bloch

vector of mixed state $\rho = p|\chi\rangle\langle\chi| + (1-p)|\psi\rangle\langle\psi|$ is

$$\vec{r} = p\vec{t} + (1-p)\vec{p} \quad \leftarrow \text{WEIGHTED AVERAGE OF } \vec{t} \text{ AND } \vec{p}$$

More generally we can write any state as

$$\rho = \frac{1}{2} \left(\mathbb{1} + \vec{r} \cdot \vec{\sigma} \right)$$

WHERE $\vec{r} \cdot \vec{\sigma} = r_x \sigma_x + r_y \sigma_y + r_z \sigma_z$

↑ PAULI MATRICES

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$|\vec{r}| \leq 1$$

DiVincenzo's Criteria

So far we introduced the concept of qubit and how it can be used for encoding information. We also

said that despite we can encode an infinite amount of information, measurements allow us to retrieve

only one bit of information. For completeness we also showed what happens if our system is noisy

and causes unwanted mixtures between different pure states. A major point of this course is to learn

about the many physical systems which have realized basic digital quantum computation/simulation.

To compare the various platforms we are going to use the DiVincenzo's criteria. In 2000, David

DiVincenzo, while working at IBM, published a paper outlining the necessary criteria for a physical

system to successfully implement quantum information processing. The 5 core criteria are

- (1) A **scalable** physical system with well characterized qubit
- (2) The ability to **initialize** the state of the qubits
- (3) **Long relative decoherence times**, much longer than the gate operations time
- (4) A **universal** set of **quantum gates** with high fidelity
- (5) Qubit specific **measurement** capabilities

He also added two supplementary criteria relating to the transmission and movement of information:

- (6) The ability to **convert** from stationary to mobile, "flying" qubits and back.
- (7) The ability to **faithfully transmit** flying qubits from one location to another.

Many of these concept could still be unclear or undefined. For example decoherence has to do with noise (environmental or imperfect control) and, loosely speaking is a measure of how well one can control the qubit. Said in just a few words, noise affects the purity of a qubit inducing mixing.

Throughout the course we will talk about different channels of decoherence that are specific to each quantum platforms.

The second DiVincenzo criteria tells us that we need to be able to **initialize** our qubits. Using our qubit model of the Bloch sphere, this means most simply that we need to be able to put our qubit vector into any position on that sphere (a fiducial starting state that is typically $|\psi\rangle$). It should be quite easy to convince yourself that there are 4 unique rotation operations which allow the vector to transform from any state into any other state in the Bloch sphere. These 4 rotations operations are rotations about each Cartesian axis, and an identity rotation which does not alter the vector. In fact, with a little thought you will see that only two rotation axis are required for universal state creation.

Universal gates

The language of quantum mechanics is Linear Algebra and the dynamics are governed by the time

dependent Schrödinger equation $i\hbar \frac{d}{dt} |\psi\rangle = \hat{H} |\psi\rangle$

the general solution for a time-independent Hamiltonian is $|\psi(t)\rangle = e^{-\frac{i\hat{H}t}{\hbar}} |\psi_0\rangle$ INITIAL STATE

As it turns out, the Pauli matrices when exponentiated are perfect rotation matrices about the

respective axis

$$R_x(\theta) = e^{-i\theta\sigma_x/2} = \begin{pmatrix} \cos\theta/2 & -i\sin\theta/2 \\ -i\sin\theta/2 & \cos\theta/2 \end{pmatrix}$$

$$R_y(\theta) = e^{-i\theta\sigma_y/2} = \begin{pmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{pmatrix}$$

$$R_z(\theta) = e^{-i\theta\sigma_z/2} = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}$$

why? Let's show it quickly for $R_x(\theta)$...

$$e^{-i\theta\sigma_x/2} = 1 - i\frac{\theta\sigma_x}{2} - \frac{1}{2!} \left(\frac{\theta\sigma_x}{2}\right)^2 + \frac{i}{3!} \left(\frac{\theta\sigma_x}{2}\right)^3 + \frac{1}{4!} \left(\frac{\theta\sigma_x}{2}\right)^4 + \dots$$

$$\begin{matrix} \nearrow \\ \sigma_x^2 = \mathbb{1} \end{matrix} \begin{matrix} \cos\theta/2 \mathbb{1} & -i\sin\theta/2 \sigma_x \end{matrix}$$

The take-away message here, is that a Hamiltonian with Pauli operators present indicates a physical

system allowing for complete single qubit initialization and manipulation. We therefore have some

kind of guide for what to look for in a physical system for single qubit manipulation.

Let's now show another important thing that will become relevant later for example when we talk about two qubit gates with Rydberg atoms.

$$R_x(2\pi) = R_y(2\pi) = R_z(2\pi) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \rightarrow \text{ANY STATE GOES BACK ON ITSELF PICKING A PHASE } -1!$$

SPECIAL ROTATIONS:

X : $|0\rangle \rightarrow |1\rangle, |1\rangle \rightarrow |0\rangle$ LIKE THE NOT GATE IN CLASSICAL COMPUTING
 $R_y(\pi)$
T ROTATION ABOUT THE X AXIS

Y : $|0\rangle \rightarrow i|1\rangle, |1\rangle \rightarrow -i|0\rangle$
 $R_x(\pi)$

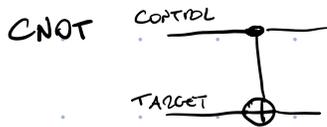
Z : $|0\rangle \rightarrow |0\rangle, |1\rangle \rightarrow -|1\rangle$ PHASE FLIP.
 $R_z(\pi)$

H: HADAMARD GATE $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ π ROTATION ABOUT $\frac{1}{\sqrt{2}} \begin{pmatrix} \vec{x} + \vec{z} \end{pmatrix}$ AXIS.
 $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ VECTOR
 $|0\rangle \rightarrow |+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$
 $|1\rangle \rightarrow |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$

The next thing we need to find are quantum gates which act on two or more qubits. Mathematically, a quantum gate is simply an operator that acts on the system of qubits. For single qubit manipulations these transformations acted on a 2 dimensional space. Our Bloch sphere analysis of a general qubit allowed us to find all the operations we needed. However, extending the Bloch sphere to multiple qubits is impossible. When we start talking about a larger number of qubits, the dimensionality of the system grows. The dimension of the system is related to the total number of states realizable. For 2 qubits there are four possible basis states $\{00,01,10,11\}$.

Also in the case of gates with more than one qubit we require unitarity which implies that operations are reversible. $U^\dagger U = I$

There are a group of quantum gates called control gates. A control gate, such as the controlled NOT gate (CNOT) performs a logic operation if and only if, then. That is a control gate has an input qubit, the control bit, which controls if a particular operation will or will not occur on a target qubit.



$$\begin{aligned} |00\rangle &\rightarrow |00\rangle \\ |01\rangle &\rightarrow |01\rangle \\ |10\rangle &\rightarrow |11\rangle \\ |11\rangle &\rightarrow |10\rangle \end{aligned}$$

$$U_{\text{CNOT}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

4x4 matrix

This can be understood as if and only if the control qubit is 1, then NOT target qubit.

Similarly the controlled phase gate is CZ

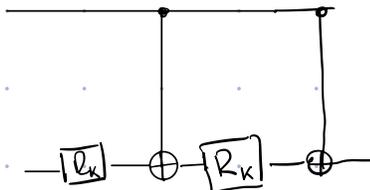


$$\begin{aligned} |00\rangle &\rightarrow |00\rangle \\ |01\rangle &\rightarrow |01\rangle \\ |10\rangle &\rightarrow |10\rangle \\ |11\rangle &\rightarrow -|11\rangle \end{aligned}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

A set of quantum gates is universal if any quantum operation acting on N qubits can be implemented by a circuit using only those gates and ancillas (additional support qubits), for all N.

Theorem: CNOT and universal single qubit gates form a universal set. Notice that this set is not finite (we need rotations for all angles)!



Time-Evolution pictures in QM

To finish off with the basics let's also recall the different time evolution pictures in quantum mechanics. We recalled the Schrödinger equation which is a homogeneous partial differential equation, linear and only to first order in t . So an initial state evolves deterministically for all later times. All observable quantities are then predicted by the scalar product of a bra and a ket, or by matrix elements of operators. These predictions are therefore invariant under unitary transformations of the kets and operators. Sometimes it is a matter of taste, but often of practicality, to transform the state and solve for the dynamics in a different "picture". Below I illustrate this for the Schrödinger, Heisenberg and interaction pictures.

Schrödinger picture

In the Schrödinger picture, operators (observables and others) are generally constant in time while the state vectors evolve. The unitary time evolution operator $U(t)$ relates initial and later states,

$|\psi(t)\rangle = U(t) |\psi(0)\rangle$ It is determined by the Schrödinger equation for the time evolution operator,

$$i\hbar \partial_t U(t) = H(t) U(t) \quad \text{with} \quad U(0) = 1$$

In the case that the Hamiltonian commutes with itself at different times, the solution is given by

$$U(t) = \exp\left(\int_0^t dt' \frac{H(t')}{i\hbar}\right) \quad \text{Otherwise, if the Hamiltonian does not commute with itself}$$

at different times, the differential equation can be formally solved by a Dyson series or perturbatively

by a Magnus expansion

$$U(t) = \exp\left\{ \int_0^t dt_1 \overset{H_1 = \frac{H(t_1)}{i\hbar}}{H_1} + \frac{1}{2} \int_0^t dt_1 \int_0^{t_1} dt_2 [H_1, H_2] + \frac{1}{6} \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 ([H_1, [H_2, H_3]] + [H_3, [H_2, H_1]]) + \dots \right\}$$

Heisenberg picture

The Heisenberg picture is a formulation in which the operators incorporate a dependency on time,

but the state vectors are time-independent. It differs from the Schrödinger picture only by a basis change with respect to the time-evolution operator:

$$|\psi\rangle_{\text{H}} = U^\dagger(t) |\psi(t)\rangle = |\psi(0)\rangle$$

HEISENBERG

$$A^\dagger(t) = U^\dagger(t) A(t) U(t)$$

$$\Rightarrow i\hbar \frac{d}{dt} A^\dagger(t) = [A^\dagger(t), H^\dagger(t)] + i\hbar (\partial_t A^\dagger(t))^\dagger$$

Interaction picture

The interaction picture (also known as the Dirac picture) is an intermediate representation between the Schrödinger and the Heisenberg picture in the sense that here both, state vectors and operators, carry part of the time dependence. It is useful in dealing with changes to the wave functions and observables of a composite system that are only due to the interactions.

The Schrödinger-picture Hamiltonian is split as $H = H_0 + H_1(t)$

↑ SINGLE SYSTEM TERMS
← TIME DEPENDENCE INTERACTION TERMS

TRANSFORMATION W.R.T. $U_0(t) = e^{-iH_0 t/\hbar}$ YIELDS

$$|\psi^\dagger(t)\rangle = U_0^\dagger(t) |\psi(t)\rangle$$

$$A^\dagger(t) = U_0^\dagger(t) A(t) U_0(t)$$

THE STATES EVOLVE ACCORDING TO TRANSFORMED SCHRÖDINGER EQUATION.

$$i\hbar \partial_t |\psi^\dagger(t)\rangle = i\hbar \partial_t [U_0^\dagger(t) |\psi(t)\rangle] = i\hbar \dot{U}_0^\dagger(t) |\psi(t)\rangle + U_0^\dagger(t) i\hbar \partial_t |\psi(t)\rangle$$

$$= i\hbar \dot{U}_0^\dagger(t) |\psi(t)\rangle + U_0^\dagger(t) H |\psi(t)\rangle = [i\hbar \dot{U}_0^\dagger(t) U_0(t) + U_0^\dagger(t) H U_0(t)] |\psi^\dagger(t)\rangle$$

$\dot{U}_0^\dagger(t) U_0(t)$
 $\frac{i}{\hbar} H_0 U_0^\dagger$
 $U_0^\dagger H_0 U_0 + U_0^\dagger H_1 U_0$

$$= [-H_0 + \underbrace{U_0^\dagger(t) H_0 U_0(t)}_{H_0} + \underbrace{U_0^\dagger(t) H_1 U_0(t)}_{H_1^\dagger}] |\psi^\dagger(t)\rangle$$

$$\Rightarrow i\hbar \sigma_t |\psi^I(t)\rangle = H_1^I |\psi^I(t)\rangle$$

with $H_1^I(t) = U_0^\dagger(t) H_1 U_0(t) = e^{iH_0 t/\hbar} H_1 e^{-iH_0 t/\hbar}$

ANALOGOUSLY ONE CAN SHOW THAT TIME INDEPENDENT OPERATORS EVOLVE AS

$$i\hbar \frac{dA^I}{dt} = [A^I(t), H_0]$$

EXAMPLE

$$H = H_0 + H_1(t) \quad H_0 = \hbar \omega_c (a^\dagger a + \frac{1}{2}) + \frac{\hbar \omega_e}{2} \sigma_z \leftarrow \text{Atom in a cavity}$$

$$H_1 = \sum_{n,m} c^n e^{+im} e^{-in}$$

$$H_1^I(t) = e^{iH_0 t/\hbar} H_1 e^{-iH_0 t/\hbar}$$

$$= \sum_{n,m} e^{iH_0 t/\hbar} e^{+im} e^{-in} e^{-iH_0 t/\hbar}$$

$e^{-iH_0 t/\hbar} e^{+im} e^{-in} e^{iH_0 t/\hbar} \leftarrow \text{OK BECAUSE } = \mathbb{1}$ REPEAT FOR ALL TERMS

$$= \sum_{n,m} e^{iH_0 t/\hbar} e^{-iH_0 t/\hbar} e^{+im} e^{-in} e^{iH_0 t/\hbar} e^{-iH_0 t/\hbar} =$$

$- e^I(t)$

$$= \sum_{n,m} (e^I(t))^m (e^{+I})^n$$

HOW TO CALCULATE $e^{+I}(t) = ?$

$$e^{+I}(t) = e^{iH_0 t/\hbar} e^{+} e^{-iH_0 t/\hbar}$$

HADAMARD'S LEMMA AND BAKER-CAMPBELL-HAUSDORFF EXPANSION

$$= e^{+} + \frac{i}{\hbar} [H_0, e^{+}] + \frac{i^2}{2! \hbar^2} [H_0, [H_0, e^{+}]] + \dots$$

$$[H_0, e^{+}] = [\hbar \omega_c (a^\dagger a + \frac{1}{2}) - \hbar \omega_e e^+ e^+ e^-]$$

$$[e^+, e^-] = e^+ e^- - e^- e^+ = -1$$

$$\dot{e}^+ = e^+ - 1$$

$$\Rightarrow [H_0, e^+] = \hbar \omega_c e^+ e^+ - \hbar \omega_c e^+ e^+ + \hbar \omega_c e^+ = +\hbar \omega_c e^+$$

$$\Rightarrow \boxed{e_{\pm}^+(t) = e^{i\omega_c t} e^{\pm}}$$