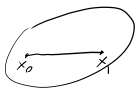
2 october

Def X a vector space on $K = \mathbb{C}, \mathbb{R}$
Then $\Omega \subseteq X$ is a convex subset if
$\forall\ x_0, x_1 \in \Omega$ we have

$$x_t = (1-t)x_0 + t x_1 \in \Omega. \qquad t \in [0,1]$$



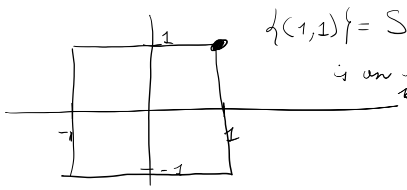
Example If X is normed $\|\cdot\|$
then $D_X(0,1) = \{x \in X : \|x\| < 1\}$
is convex
$\|x_0\| < 1$, $\|x_1\| < 1$ $\qquad 0 \le t \le 0$

$$\|x_t\| = \|(1-t)x_0 + t x_1\| \le (1-t)\overbrace{\|x_0\|}^{<1} + t\overbrace{\|x_1\|}^{<1}$$
$$< (1-t) + t = 1$$

Def X vector space $\Omega \overset{\subseteq X}{\text{convex}}$
$S \subseteq \Omega$ is an extreme subset of Ω if

whenever $x_0, x_1 \in \Omega$ and there is a $t \in (0,1)$ s.t.
$x_t \in S \Rightarrow x_0, x_1 \in S$

Example $\mathbb{R}^2 = \{(x,y) : x, y \in \mathbb{R}\}$
$\|(x,y)\| = \max\{|x|, |y|\}$
$\overline{D_{\mathbb{R}^2}(0,1)}$



$\{(1,1)\} = S$
is an extreme set

In the particular case in which
S is an extreme set of the form $S = \{y\}$
then both S and y are called extreme points

$$\mathrm{Ext}(\Omega) = \{x \in \Omega : x \text{ is an extreme point of } \Omega\}$$

Exercise 4.4 Let $A \subseteq \mathbb{R}^d$ open

$$\overline{D_{L^1(A)}(0,1)} = \{f \in L^1(A) : \|f\|_{L^1(\Omega)} \le 1\}$$

$$\mathrm{Ext}\left(\overline{D_{L^1(A)}(0,1)}\right) = \phi$$

If X is a TVS and $\Omega \subseteq X$ is convex then $\mathring{\Omega}$ and $\overline{\Omega}$ are convex.

Def a) A TVS X is locally convex for any nbhd U of 0 there exists a convex nbhd V of 0 with $V \subseteq U$.

b) When X is metrizable and locally convex then it is called a Fréchet space.

Lemma X TVS locally convex. Then for any nbhd U of 0 $\exists$ a nbhd V of 0 s.t. V is convex, balanced and $V \subseteq U$.

Pf We start with U. We can assume that U is convex. . . .

Def Let X be a vector space

A function $p : X \to [0, +\infty)$ is a seminorm if

a) $p(x+y) \le p(x) + p(y) \quad \forall x, y \in X$

b) $p(\lambda x) = \lambda p(x)$ for all $x \in X$ and $\lambda > 0$

Exercise If $p : X \to \mathbb{R}$ is a seminorm then $C = \{ x \in X : p(x) < 1 \}$ (1)

is a convex subset of X.

Lemma Let X be a TVS and C a convex and open subspace of X with $C \ni 0$. Then $\exists$ a seminorm $p: X \to \mathbb{R}$ s.t. (1) is true.

p is called the Minkowsky functional (or gauge) of C.

Pf Let $x \in X$

$$p(x_0) = \inf\{a > 0 : \tfrac{x_0}{a} \in C\}$$

$$\mathbb{C} \times X \longrightarrow X$$
$$\cup\!\!/$$
$$\mathbb{R} \times X \nearrow$$
$$(0, x_0) \longrightarrow 0 \cdot x_0 = 0$$

[illegible] $\exists\ \delta > 0$ and a nbhd V of x_0 $\forall$ U nbhd of 0 in X

s.t. $\lambda x \in U$ $\forall$ $|\lambda| \le \delta$ and $x \in V$

If I consider $U = C$

I conclude that for all $0 < |\lambda| \le \delta$

$\lambda x_0 \in C$ also for $\lambda \in \mathbb{R}$, $0 < \lambda < \delta$

So for $\frac{1}{a} = \lambda$

$\frac{1}{a} x_0 \in C$ $\quad \forall x_0 \in X$

$$p(x_0) = \inf\{a > 0 : \tfrac{1}{a} x_0 \in C\} \neq \emptyset$$

A well defined function $p: X \longrightarrow \mathbb{R}$ remains defined

$p(0) = 0$

We prove that

$$p(x) < 1 \iff x \in C$$

Let $x \in C$. Then

$$1 \in \{a > 0 : \tfrac{x}{a} \in C\} \Rightarrow p(x) \le 1$$

If $\lambda > 0$ $\quad p(\lambda x) = \lambda p(x)$ $\quad \forall x$

$$p(\lambda x) = \inf\{a > 0 : \tfrac{\lambda x}{a} \in C\} =$$
$$= \inf\{\lambda\alpha > 0 : \tfrac{\cancel{\lambda} x}{\cancel{\lambda}\alpha} \in C\}$$
$a = \lambda\alpha$

$$= \lambda\, \underbrace{\inf\{\alpha > 0 : \tfrac{x}{\alpha} \in C\}}_{p(x)}$$

$p(\lambda x) = p(x)$.

Let $x \in C$. We show $p(x) < 1$

If $1 \cdot x \in C$ by the fact that C we conclude that in fact $\lambda \cdot x \in C$ for $|\lambda - 1|$ sufficiently small.

$\exists\ \varepsilon > 0$ s.t. $|\lambda - 1| \le \varepsilon \Rightarrow \lambda \cdot x \in C$

In particular $(1+\varepsilon) x \in C$

$p((1+\varepsilon)x) \le 1$

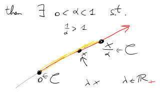
$(1+\varepsilon) p(x) \le 1$

$$p(x) \le \frac{1}{1+\varepsilon} < 1$$

Let x be s.t. $p(x) < 1$. We want to show that $x \in C$

$$p(x) = \inf\{\alpha > 0 : \tfrac{x}{\alpha} \in C\}$$

If $p(x) < 1$ then $\exists\ 0 < \alpha < 1$ s.t. $\frac{x}{\alpha} \in C$.



$0, \frac{x}{\alpha} \in C$

$x = \alpha \frac{x}{\alpha} + (1-\alpha)\, 0$ $\quad \alpha \in (0,1)$

Since C is convex $\Rightarrow x \in C$

We need to prove subadditivity

$$p(x+y) \le p(x) + p(y) \quad \forall x, y \in X$$

x, y $\quad \varepsilon > 0$

$$p\left(\frac{x}{p(x)+\varepsilon}\right) < 1, \quad p\left(\frac{y}{p(y)+\varepsilon}\right) < 1$$

$$\frac{x}{p(x)+\varepsilon}, \quad \frac{y}{p(y)+\varepsilon} \in C$$

$\forall\ t \in [0, 1]$

$$t \frac{x}{p(x)+\varepsilon} + (1-t) \frac{y}{p(y)+\varepsilon} \in C$$

$$t = \frac{p(x)+\varepsilon}{p(x)+p(y)+2\varepsilon} \in [0,1]$$

$$1 - t = \frac{\cancel{p(x)} + p(y) + 2\varepsilon - \cancel{p(x)} - \varepsilon}{p(x)+p(y)+2\varepsilon} = \frac{p(y)+\varepsilon}{p(x)+p(y)+2\varepsilon}$$

$$= \frac{x+y}{p(x)+p(y)+2\varepsilon} \in C$$

$$p\left(\frac{x+y}{p(x)+p(y)+2\varepsilon}\right) < 1$$

$$p(x+y) < p(x) + p(y) + 2\varepsilon \quad \forall \varepsilon > 0$$

$$\Rightarrow p(x+y) \le p(x) + p(y)$$

$\square$

Remark If C is balanced

($x \in C \Rightarrow \lambda x \in C$ $\forall\ |\lambda| \le 1$)

then we have $2'$) $p(\lambda x) = |\lambda| p(x)$ $\quad \forall \lambda \in \mathbb{K}$

And viceversa if ~~It~~ I start with a seminorm $p: X \longrightarrow \mathbb{R}$ such that 2) is replaced by $2'$) then $C = \{x : p(x) < 1\}$ is balanced.

Lemma Let X be a locally convex TVS.
Then $\exists$ a family $\{p_j\}_{j \in J}$ of continuous
$p_j : X \to \mathbb{R}$ which are seminorms
s.t. the sets of the form

$$\{x \in X : \; p_j(x) < \varepsilon\} \qquad j, \varepsilon > 0$$

form a subbasis of nbhds of 0 in X.

That is s.t. for any nbhd U of 0 in X $\exists$ $j_1, \dots, j_n \in J$ and $\varepsilon_1 > 0, \dots, \varepsilon_n > 0$

s.t. $$\bigcap_{l=1}^{n} p_{j_l}^{-1}\left([0, \varepsilon_j)\right) \subset U$$

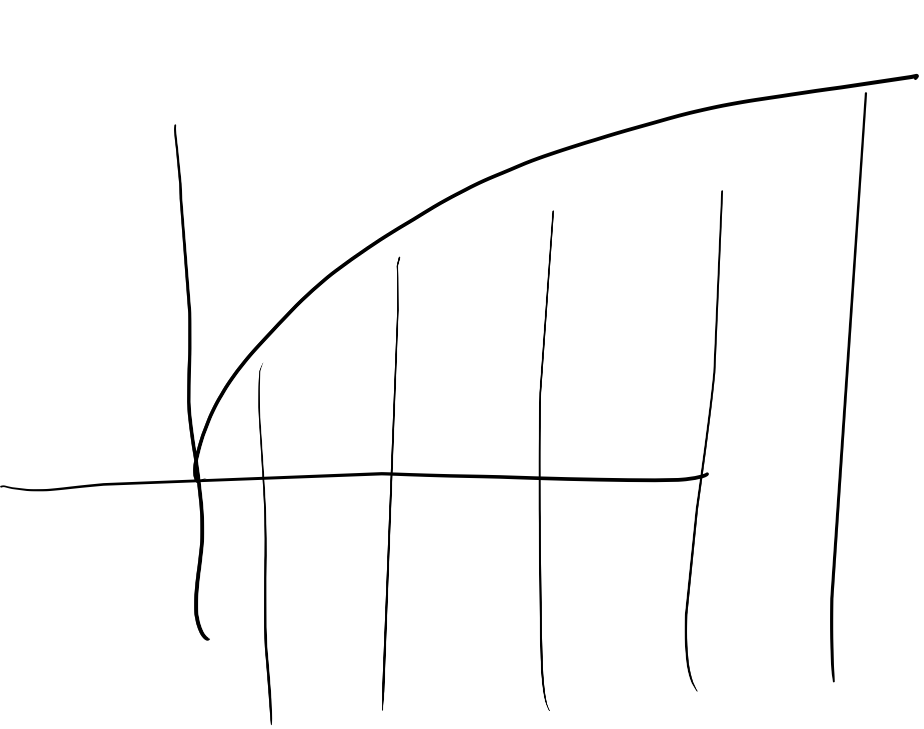
Def $f : X \to K$ is homogeneous of order $\alpha \geq 0$ if $f(\lambda x) = \lambda^\alpha f(x)$ $\forall \lambda \geq 0$ and $x \in X$

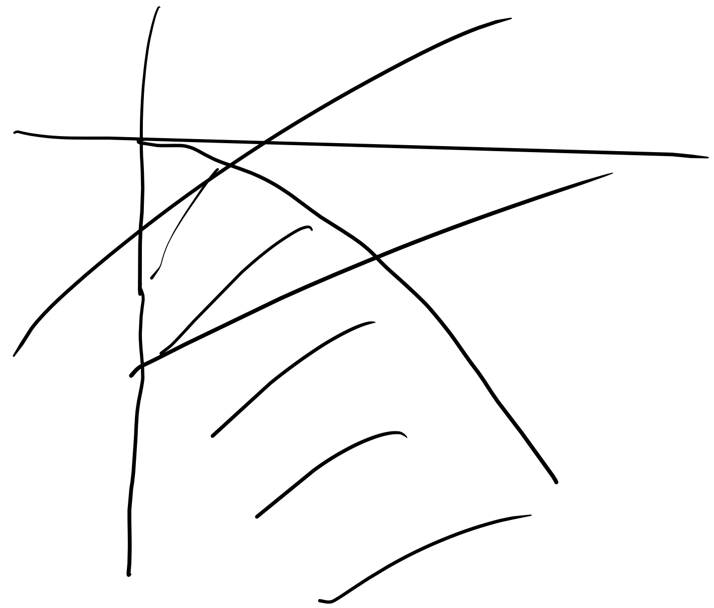
Exercise 4.2.4. Let $f : X \to K$ be homogeneous of order $\alpha = 1$ and X a T.V.S locally convex with $\{p_j\}_{j \in J}$ a subbasis of seminorms. Then f is continuous in 0 iff $\exists$ $j_1, \dots, j_m \in J$ and a constant $C > 0$ s.t.

$$|f(x)| \leq C \left(p_{j_1}(x) + \dots + p_{j_m}(x)\right) \qquad \forall x \in X.$$

Exercise $f : [0,+\infty) \to [0,+\infty)$

$f(0) = 0$ and f concave





Show that $f(x+y) \le f(x) + f(y)$

$\forall\, x, y \ge 0.$

$f(x) = x^p \qquad 0 < p < 1$

$(a+b)^p \le a^p + b^p$

Example $L^p(0,1)$ $0<p<1$

$$d(f,g) = \int_0^1 |f(t)-g(t)|^p\,dt$$ it is metric

$$d(f,g) \le d(f,h) + d(h,g) \quad \forall\ f,g,h \in L^p$$

$$d(f,g) = \int_0^1 \left|\big(f(t)-h(t)\big)+\big(h(t)-g(t)\big)\right|^p dt$$

$$\le \int_0^1 |f(t)-h(t)|^p\,dt + \int_0^1 |h(t)-g(t)|^p\,dt$$

If V is an open convex nbhd of 0

$$V = L^p(0,1)$$

Let V be a convex open nhd of 0

Let $f \in L^p(0,1)$. $\exists\ \varepsilon_0>0$ s.t. $D(0,\varepsilon_0) \subseteq L^p(0,1)$

$$n^{p-1}\int_0^1 |f(t)|^p\,dt \le \varepsilon_0 \qquad n >> 1$$

$[0,1]$

$t_0 < t_1 < \dots < t_n = 1$ s.t.

$$\int_{t_{i-1}}^{t_i} |f(t)|^p\,dt = \frac{1}{n}\int_0^1 |f(t)|^p\,dt$$

$$g_j(t) = n\,\chi_{[t_{j-1},t_j]}(t)\ f$$