October 3

$0<p<1$ $\qquad L^p(0,1) = \{ f : \int_0^1 |f(x)|^p dx < +\infty \}$

$d(f,g) = \int_0^1 |f(x)-g(x)|^p dx$ is a metric

Claim

If $V \ni 0$ is open and convex in $L^p(0,1)$

then $V = L^p(0,1)$.

Pf $D_{L^p(0,1)}(0, \varepsilon_0) \subset V$ $\qquad \varepsilon_0 > 0$

Pick any $f \in L^p(0,1)$. Choose $n \in \mathbb{N}$ s.t.

$$n^{p-1} \int_0^1 |f(x)|^p dx < \varepsilon_0$$

and decompose $[0,1]$

$t_0 = 0 < \dots < t_n = 1$ in n intervals

$[t_0, t_1], \dots,$ so that $\int_{t_{i-1}}^{t_i} |f(t)|^p dt = \frac{1}{n}\int_0^1 |f(t)|^p dt$

$t \longrightarrow \underbrace{\int_0^t |f(t')|^p dt'}_{F(t)}$ is continuous and increasing

$F(t) = \frac{1}{n}\int_0^1 |f(t)|^p dt$ $\qquad t_1$

$F(t) = \frac{2}{n}\int_0^1 |f(t)|^p dt$ $\qquad t_2$

$g_j(t) = n \cdot \chi_{[t_{j-1}, t_j]}(t) \, f(t)$ $\qquad j = 1, \dots, n$

$$\int_0^1 |g_j(t)|^p dt = n^p \int_{t_{j-1}}^{t_j} |f(t)|^p dt =$$

$$= n^{p-1} \int_0^1 |f(t)|^p dt < \varepsilon_0$$

$g_j \in D_{L^p(0,1)}(0, \varepsilon_0) \subseteq V$ for $j = 1, \dots, n$

By the convexity of V we have

$$\frac{g_1 + \dots + g_n}{n} \in V$$

$$= \frac{\chi_{[0,t_1]} + \dots + \chi_{[t_{n-1},1]}}{\cancel{n}} \cancel{n} f(t)$$

$= f$

$V = L^p(0,1)$ $\qquad \mathbb{R}$

$(L^p(0,1))' = \{ T : T$ is continuous linear map $L^p(0,1) \to \mathbb{R} \} = 0$

More generally if X is a locally convex TVS then

$T : L^p(0,1) \longrightarrow X$ linear and continuous $\Rightarrow T \equiv 0$

Consider V an open convex nbhd of $0 \in X$.

Then $T^{-1}V$ is open an convex in $L^p(0,1)$

Then $T^{-1}V = L^p(0,1)$

Then $\underbrace{T\, L^p(0,1)}_{R(T)} \subseteq V$

$R(T) \subseteq V$ $\qquad \forall$ open convex nbhd of $0 \in X$.

$R(T) \subseteq \bigcap_{\substack{\forall \text{ open convex} \\ \text{nbhd of } 0 \in X}} V = \{0\}$

$R(T) = \{0\} \iff T \equiv 0$

Given two normed spaces

$$X \xrightarrow{T} Y \quad \text{linear}$$

Lemma (The following ~~two~~ three ~~th~~ are equivalent

1) T is continuous

2) T is bounded

3) The following quantity is finite.

$$\|T\|_{\mathcal{L}(X,Y)} = \sup_{x \in D_X(0,1)\setminus\{0\}} \frac{\|Tx\|_Y}{\|x\|_X} =$$

$$\| \quad M < +\infty$$

$$= \sup_{x \in X \setminus \{0\}} \frac{\|Tx\|_Y}{\|x\|_X}$$

$$= \sup_{\|x\|_X = 1} \|Tx\|_Y < +\infty$$

What happens is that

$\forall \; x \in D_X(0,1)\setminus\{0\}$

$$\|Tx\|_Y \leq M$$

$$\frac{\|Tx\|_Y}{\|x\|_X} \leq \sup_{x' \in D_X(0,1)\setminus\{0\}} \frac{\|Tx'\|_Y}{\|x'\|_X} =: M < \infty$$

$$\|Tx\|_Y \leq M \overbrace{\|x\|_X}^{<1} \leq M$$

$$T\, D_X(0,1) \subseteq D_Y(0, M)$$

Suppose X is locally convex

This means that there is a sub-basis $\{p_j\}_{j\in J}$

of seminorms $p_j(\lambda x) = |\lambda| p_j(x) \quad \forall \lambda \in \mathbb{K}$.

Recall that X is metrizable iff 0 has a countable sub-basis of nbhds.

It is then possible to conclude that X locally convex is metrizable iff it has a countable sub-basis of seminorms

$(X, \{p_j\}_J)$ with $\operatorname{card} J \leq \operatorname{card} \mathbb{N}$.

$J \hookrightarrow \mathbb{N}$

then the following metric induces on X its topology

$$d(x,y) = \sum_{j\in J} 2^{-j} \frac{p_j(x-y)}{1+p_j(x-y)} \leq 2$$

≤ 1

$3 \implies 1$

$M = \|T\|_{\mathcal{L}(X,Y)} < +\infty$

Need to show T is continuous in 0.
We will show that $\forall \varepsilon > 0 \; \exists \delta > 0$ s.t.

$\|x\|_X < \delta \implies \|Tx\|_Y < \varepsilon$

$+\infty > M = \sup_{x \in X \setminus \{0\}} \frac{\|Tx\|_Y}{\|x\|_X}$. This implies

that $\|Tx\|_Y \leq M \|x\|_X$ $\quad (*)$

$$\frac{\|Tx\|_Y}{\|x\|_X} \leq \sup_{x' \in X \setminus \{0\}} \frac{\|Tx'\|_Y}{\|x'\|_X} = M$$

$(*)$ Is telling us that T is Lipschitz at 0
$\implies$ continuous at 0

ε $\quad \boxed{\delta = \frac{\varepsilon}{M}}$

$\|x\|_X < \delta \implies \|Tx\|_Y \leq M \|x\|_X < M\delta = \varepsilon$

$(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$

T_n $\mathcal{L}(X, Y)$

$\|T\|_{\mathcal{L}(X,Y)}$ is a norm.

When Y is a B-space then also $\mathcal{L}(X, Y)$ is complete.

$X' = \mathcal{L}(X, K)$

$$\|f\|_{X'} = \sup_{\|x\|_X = 1} |f(x)|$$

$$f(x) = \langle f, x\rangle_{X' \times X} = \langle x, f\rangle_{X \times X'}$$

Given a sequence $\{T_n\}$ in $\mathcal{L}(X, Y)$
$\ni T$

there are many ways in which T_n can converge to T.

$$\|T_n - T\|_{\mathcal{L}(X,Y)} \xrightarrow{n \to +\infty} 0$$

this is called "uniform" convergence or convergence in operator norm.

Another way is that

$$\boxed{\lim_{n \to +\infty} T_n x = Tx \quad \forall x \in X}$$

this is called the "strong" convergence

$$\boxed{s\text{-}\lim_{n \to +\infty} T_n = T}$$

Example $\quad 1 \le p < +\infty \quad L^p(\mathbb{R}^d) = \{ f : (\int_{\mathbb{R}^d} |f(x)|^p dx)^{\frac{1}{p}} < +\infty \}$

$$T_n f = \chi_{D_{\mathbb{R}^d}(0,n)} f$$

$$s\text{-}\lim_{n\to+\infty} T_n = 1$$

$$\boxed{\lim_{n\to+\infty} T_n f = f \quad \text{in } L^p(\mathbb{R}^d)} \quad \forall f$$

I have to show $\lim_{n\to+\infty} \| T_n f - f \|_{L^p(\mathbb{R}^d)} = 0 \quad \forall f$

$$f \longrightarrow n f$$

$$\| T_n f - f \|_{L^p}^p = \| \chi_{D_{\mathbb{R}^d}(0,n)} f - f \|_{L^p(\mathbb{R}^d)}^p =$$

$$= \| \chi_{\mathbb{R}^d \setminus D_{\mathbb{R}^d}(0,n)} f \|_{L^p(\mathbb{R}^d)}^p$$

$$= \int_{\mathbb{R}^d} \underbrace{\chi_{\mathbb{R}^d \setminus D_{\mathbb{R}^d}(0,n)}(x) \, |f(x)|^p}_{\le |f(x)|^p \in L^1(\mathbb{R}^d)} dx$$

$$\forall x \quad \lim_{n\to+\infty} \chi_{\mathbb{R}^d \setminus D_{\mathbb{R}^d}(0,n)}(x) \, |f(x)|^p = 0$$

$$\|T_n f - f\|_{L^p}^p$$

$$\boxed{\lim_{n\to+\infty} \int_{\mathbb{R}^d} \chi_{\mathbb{R}^d \setminus D_{\mathbb{R}^d}(0,n)}(x) \, |f(x)|^p dx} = 0$$

$$= \int_{\mathbb{R}^d} \underbrace{\lim_{n\to+\infty} \chi_{\mathbb{R}^d \setminus D_{\mathbb{R}^d}(0,n)}(x) \, |f(x)|^p}_{0} dx = 0$$

Claim $\| T_n - 1 \|_{\mathcal{L}(L^p(\mathbb{R}^d))} = 1 \qquad \forall n$

$(T_n - 1) f = \chi_{\mathbb{R}^d \setminus D_{\mathbb{R}^d}(0,n)} f = f$

If $\cancel{f} \in C^0_c(\mathbb{R}^d, \mathbb{R})$ iff supp f is inside

$\mathbb{R}^d \setminus D_{\mathbb{R}^d}(0,n)$

$(\|Tx\| \leq \|T\| \, \|x\|)$

$\| (T_n - 1) f \|_{L^p(\mathbb{R}^d)} = \| f \|_{L^p(\mathbb{R}^d)}$

$\leq \| T_n - 1 \|_{\mathcal{L}(L^p)} \| f \|_{L^p(\mathbb{R}^d)}$

$\| T_n - 1 \|_{\mathcal{L}(L^p)} \geq 1$

In general for any multiplier I have $Tf = mf \quad \forall f$

$\| T f \|_{L^p(\mathbb{R}^d)} \leq \| m \|_{L^\infty(\mathbb{R}^d)} \| f \|_{L^p(\mathbb{R}^d)}$

$\sup_{f \in L^p(\mathbb{R}^d) \setminus \{0\}} \dfrac{\| T f \|_{L^p(\mathbb{R}^d)}}{\| f \|_{L^p(\mathbb{R}^d)}} \leq \| m \|_{L^\infty(\mathbb{R})}$

In fact it is an equality.

$\| (T_n - 1) f \|_{L^p(\mathbb{R}^d)} = \| \chi_{\mathbb{R}^d \setminus D_{\mathbb{R}^d}(0,n)} f \|_{L^p(\mathbb{R}^d)} \leq \| f \|$

$\| T_n - 1 \|_{\mathcal{L}(L^p(\mathbb{R}^d))} \leq 1$

$\| T_n - 1 \|_{\mathcal{L}(L^p(\mathbb{R}^d))}$

Convolution. Let $\rho \in L^1(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} \rho(x)\,dx = 1$

and let $\rho_n(x) = n^d \rho(nx)$

Then for any $1 \le p < +\infty$

$$\rho_n * f(x) = \int_{\mathbb{R}^d} \rho_n(x-y) f(y)\,dy$$

$$L^p(\mathbb{R}^d) \ni f \longrightarrow \rho_n * f \in L^p(\mathbb{R}^d)$$

$$\|\rho_n * f\|_{L^p(\mathbb{R}^d)} \le \|\rho_n\|_{L^1(\mathbb{R}^d)} \|f\|_{L^p(\mathbb{R}^d)}$$

It can be shown $s-\lim_{n\to\infty} \rho_n * \sqcup = 1$

But again it is not true that $\rho_n * \sqcup \longrightarrow 1$ in operator norm. $p < +\infty$

Example (Schwartz)

$$S(\mathbb{R}^d,\mathbb{C}) = \left\{ \phi \in C^\infty(\mathbb{R}^d,\mathbb{C}) \;:\; P_{\alpha\beta}(\phi) = \sup_{x\in\mathbb{R}^d} |x^\beta \partial_x^\alpha \phi(x)| < +\infty \right.$$

for all multi-indexes α and β $\}$

$$S(\mathbb{R}^d,\mathbb{C}) \supset C_c^\infty(\mathbb{R}^d,\mathbb{C})$$

It is easy to see that the $P_{\alpha\beta}$ are seminorms

$$P_{\alpha\beta}(f+g) \le P_{\alpha\beta}(f) + P_{\alpha\beta}(g)$$

$$P_{\alpha\beta}(\lambda f) = |\lambda| \; P_{\alpha\beta}(f) \qquad \mathbb{N}_0 = \mathbb{N}\cup\{0\}$$

$$\left\{ \cancel{S}(\mathbb{R}^d,\mathbb{C}),\; \{P_{\alpha\beta}\}_{(\alpha,\beta)\in\mathbb{N}_0^d\times\mathbb{N}_0^d} \right\}$$

A structure of Fréchet space remain defined. It is not a normed space

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$d(x,y)$

$\|x\| = d(x,0)$

$\|x+y\| \le \|x\| + \|y\|$

$d(x+y,0) \le d(x,0) + d(y,0)$

$d(x+y,0) = d(x,-y) \le d(x,0) + d(0,-y)$

$\|\lambda x\| = |\lambda| \; \|x\| \xrightarrow{\lambda\to+\infty} +\infty$ if $x \ne 0$

$d(x,y) \qquad \dfrac{d(x,y)}{1+d(x,y)}$