$S(\mathbb{R}^d, \mathbb{C})$

$\{ S(\mathbb{R}, \mathbb{C}), P_{\alpha\beta} \}$

$P_{\alpha\beta}(f) = | x^\alpha f^{(\beta)}(x) |_{L^\infty(\mathbb{R})}$

$P_{\alpha\beta}(\lambda f) = |\lambda| P_{\alpha\beta}(f) \quad \forall \lambda \in \mathbb{C}$

show that there is no norm $\|\cdot\|$ in $S(\mathbb{R}^d, \mathbb{C})$

Suppose there is a norm, I t

$\|\lambda x\| = |\lambda| \, \|x\|$ so $\|\cdot\|$ is hom of degree 1 and is a continuous function

Then $\exists$ $P_{\alpha_1\beta_1}, \dots, P_{\alpha_m\beta_m}$ and a $C>0$

s.t. $\forall x$

$\|f\| \le C \left(P_{\alpha_1\beta_1}(f) + \dots + P_{\alpha_m\beta_m}(f) \right)$

$\forall f \in S(\mathbb{R}, \mathbb{C})$.

On the other hand, any $P_{\alpha\beta}$ is continuous as well. in $(S(\mathbb{R}, \mathbb{C}), \|\cdot\|)$

$\forall P_{\alpha\beta}$ $\exists$ $C_{\alpha\beta} > 0$ s.t.

$P_{\alpha\beta}(f) \le C_{\alpha\beta} \|f\| \le C C_{\alpha\beta} \left(P_{\alpha_1\beta_1}(f) + \dots + P_{\alpha_m\beta_m}(f) \right)$

If this happens $P_{0\beta}$ f

$$| f^{(\beta)}(0) | \le | f^{(\beta)} |_{L^\infty(\mathbb{R})} \le C C_{0\beta} \left(| x^{\alpha_1} f^{(\beta_1)} |_{L^\infty(\mathbb{R})} + \dots + | x^{\alpha_m} f^{(\beta_m)} |_{L^\infty(\mathbb{R})} \right)$$

$\forall f$

$\chi \in C_c^\infty(\mathbb{R}^d)$

$\chi_\lambda(x) = \chi\left(\frac{x}{\lambda}\right) \quad \forall \lambda > 0$

$\chi_\lambda^{(\beta)}(x) = \frac{1}{\lambda^\beta} \chi^{(\beta)}\left(\frac{x}{\lambda}\right)$

$$\frac{1}{\lambda^\beta} | \chi^{(\beta)}(0) | \le C C_{0\beta} \sum_{j=1}^m \lambda^{-\beta_j} | x^{\alpha_j} \chi^{(\beta_j)}(\tfrac{\cdot}{\lambda}) |_{L^\infty(\mathbb{R})}$$

$$= C C_{0\beta} \sum_{j=1}^m \lambda^{-\beta_j + \alpha_j} \left| \left(\frac{x}{\lambda}\right)^{\alpha_j} \chi^{(\beta_j)}\left(\frac{\cdot}{\lambda}\right) \right|_{L^\infty(\mathbb{R})}$$

$$\frac{1}{\lambda^\beta} | \chi^{(\beta)}(0) | \le C C_{0\beta} \sum_{j=1}^m \lambda^{-\beta_j + \alpha_j} | x^{\alpha_j} \chi^{(\beta_j)} |_{L^\infty(\mathbb{R})}$$

$\beta > -\beta_j + \alpha_j \quad \forall j$

$\lambda \to 0^+$ A contradiction.

$\{X, \{p_n\}_{n\in\mathbb{N}}\}$ $\qquad p_n(\lambda x) = |\lambda|\, p_n(x) \quad \forall \lambda \in K$

$$d(x,y) = \sum_{n=1}^{\infty} 2^{-n} \frac{p_n(x-y)}{1+p_n(x-y)}$$

$d(x,y) \le d(x,z) + d(z,y) \quad \forall x,y,z.$

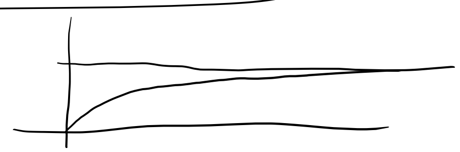
$f(p_n(x-y)) \le f\big(p_n(x-z) + p_n(z-y)\big) \le f(p_n(x-z)) + f(p_n(z-y))$

$p_n(x-y) \le p_n(x-z) + p_n(z-y)$ $\qquad \forall$

$\Downarrow$?

$$\frac{p_n(x-y)}{1+p_n(x-y)} \le \frac{p_n(x-z)}{1+p_n(x-z)} + \frac{p_n(z-y)}{1+p_n(z-y)}$$

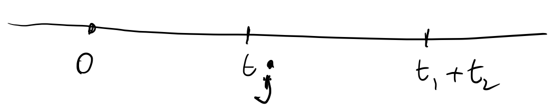
$$f(t) = \frac{t}{1+t}$$



For $t \geq 0$ $\quad f$ is concave $\quad f(0)=0$

$$f(p_n(x-y)) \le f(p_n(x-z)) + f(p_n(z-y))$$

$$f(t_1+t_2) \le f(t_1) + f(t_2) \quad \forall\, t_1, t_2 \geq 0.$$



$$t_j = \frac{t_j}{t_1+t_2}(t_1+t_2) + \left(1 - \frac{t_j}{t_1+t}\right) 0$$

$$\boxed{f(t_j) \geq \frac{t_j}{t_1+t_2} f(t_1+t_2)} + \left(1 - \frac{t_j}{t_1+t_2}\right) \overbrace{f(0)}^{0}$$

$$f(t_1) + f(t_2) \geq f(t_1+t_2) \overbrace{\left(\frac{t_1}{t_1+t_2} + \frac{t_2}{t_1+t_2}\right)}^{1}$$

$$d(x,0) = \sum_{n=0}^{\infty} 2^{-n} \frac{P_n(x)}{1+P_n(x)}$$

$\forall r>0$

$D_r = \{ x \in X : d(x,0) < r \}$ $\{D_r\}_{r>0}$

Is a basis of nbhds of 0 (X, d)

take subbasis $\mathcal{U}$ for $\{X, (P_n)_{n\in\mathbb{N}}\}$

$\mathcal{U}$ is formed by sets of form

U_ε'' $\{ x : P_{n_1}(x) < \varepsilon, \dots, P_{n_k}(x) < \varepsilon \}$ $\forall \varepsilon > 0$

$\forall U \in \mathcal{U}$ $\exists D_r$ s.t. $U \supset D_r$

and viceversa $\forall D_r$ $\exists U \in \mathcal{U}$

s.t. $D_r \supset U$.

Let us pick D_r. We need a $U \in \mathcal{U}$ with

$U \subset D_r$

$$x \in D_r \iff r > d(x,0) = \sum_{n=1}^{\infty} 2^{-n} \frac{P_n(x)}{1+P_n(x)}$$

Let N be such that $\frac{1}{2^N} < \frac{r}{2}$

and let $U = \{ x : P_1(x) < \frac{r}{2}, \dots, P_N(x) < \frac{r}{2} \}$

$\in \mathcal{U}$

Let $x \in U \overset{?}{\Rightarrow} x \in D_r$

$$d(x,0) = \sum_{n=1}^{N} 2^{-n} \frac{P_n(x)}{1+P_n(x)} + \sum_{n=N+1}^{+\infty} 2^{-n} \frac{P_n(x)}{1+P_n(x)}$$

$$\leq \sum_{n=1}^{N} 2^{-n} P_n(x) + \sum_{n=N+1}^{+\infty} 2^{-n}$$

$$< \frac{r}{2} \underbrace{\sum_{n=1}^{\infty} 2^{-n}}_{1} + 2^{-N} < \frac{r}{2} + \frac{r}{2} = r$$

Let $U \in \mathcal{U}$

$\varepsilon > 0$ $\quad U = \{ x : \; p_1(x) < \varepsilon, \ldots, p_N(x) < \varepsilon \}$

Want a $D_r \subset U$.

$x \in D_r$

$$r > \sum_{j=1}^{\infty} 2^{-j} \frac{p_j(x)}{1 + p_j(x)}$$

$$2^{-j} \frac{p_j(x)}{1+p_j(x)} < r \qquad 2^{-j} p_j(x) < r + r\, p_j(x)$$

$$(2^{-j} - r)\, p_j(x) < r$$

$$p_j(x) < \frac{r}{2^{-j} - r}$$

If $\quad 2^{-j} > r$

Want $r \leq 2^{-j} \quad \forall \; j = 1, \ldots, N$

$r < 2^{-N}$ s.t. $\quad \dfrac{r}{2^{-j} - r} < \varepsilon \;\; \forall$

$j = 1, \ldots N$

$$\frac{r}{2^{-N} - r} < \varepsilon \qquad r < 2^{-N}$$

$$r < 2^{-N}\varepsilon - \varepsilon r$$

$$(1+\varepsilon)\, r < 2^{-N}\varepsilon$$

$$\boxed{r < \frac{2^{-N}\varepsilon}{1+\varepsilon}} \implies d(x,0) < r$$

$\implies p_j(x) < \varepsilon$

$\forall \; j = 1, \ldots, N.$

Def X B.-space $T \in \mathcal{L}(X)$. $K = \mathbb{C}$

The resolvent set of T

$$\rho(T) = \{ z \in \mathbb{C} : (T - z) \text{ is invertible and } (T-z)^{-1} \in \mathcal{L}(X) \}$$

$(T-z)^{-1} =: R_T(z)$ ($R_T(z)$ is the resolvent computed in z)

$\sigma(T) = \mathbb{C} \setminus \rho(T)$ is called the spectrum of T.

Exercise $\lambda \in \mathbb{C}$ is an eigenvalue of T if $\ker(T - \lambda) \neq 0$.

Then if λ is an eigenvalue we have $\lambda \in \sigma(T)$

$\rho(T)$ is open in $\mathbb{C}$

$\sigma(T)$ closed $\mathbb{C}$ $\qquad A : X \to X$

<u>Lemma</u> Let $A \in \mathcal{L}(X)$ with $\|A\| < 1$

Then $(1+A)^{-1} = \sum_{n=0}^{\infty} (-1)^n A^n$

<u>Pf</u> $\lim_{N \to +\infty} \sum_{n=0}^{N} (-1)^n A^n$ exists if $\|A\| < 1$

$N < M$ $\qquad \|A B\| \leq \|A\| \, \|B\|$

$$\left\| \sum_{n=N}^{M} (-1)^n A^n \right\| \leq \sum_{n=N}^{M} \|A^n\| \leq$$

$$\leq \sum_{n=N}^{M} \|A\|^n$$

From $\|A\| < 1$ we conclude that $\left\{ \sum_{n=0}^{N} (-1)^n A^n \right\}_{N \in \mathbb{N}}$

is a Cauchy sequence and so converges

$$B = \sum_{n=0}^{\infty} (-1)^n A^n$$

Need now $\quad B(1+A) = (1+A)B = 1$

$$B(1+A) = \lim_{N \to +\infty} \sum_{n=0}^{N} (-1)^n A^n \, (1+A)$$

$$= \lim_{N \to +\infty} \left(1 + (-1)^N A^{N+1}\right) = 1 + \underbrace{\lim_{N \to +\infty} (-1)^N A^{N+1}}_{=0}$$

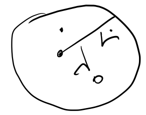
$$\|\cancel{(-1)^N} A^{N+1}\| \leq \|A\|^{N+1} \xrightarrow{N \to +\infty} 0$$

because $\|A\| < 1$

$$1 = B(1+A) = (1+A) B$$

Suppose $\lambda_0 \in \rho(T)$

$|\lambda - \lambda_0| < r$



$$T - \lambda = T - \lambda_0 + (\lambda - \lambda_0) =$$

$$= (T - \lambda_0)\left(1 + (T-\lambda_0)^{-1}(\lambda - \lambda_0)\right)$$

$$(T-\lambda)^{-1} = \left(1 + (T-\lambda_0)^{-1}(\lambda-\lambda_0)\right)^{-1}(T-\lambda_0)^{-1}$$

$$\|(\lambda-\lambda_0)(T-\lambda_0)^{-1}\| < 1$$

$$|\lambda - \lambda_0| \, \|R_T(\lambda_0)\| < 1$$

If $\boxed{|\lambda - \lambda_0| < \frac{1}{\|R_T(\lambda_0)\|}}$

$$\Longrightarrow \lambda \in \rho(T)$$

$\rho(T)$ is open $\Longleftrightarrow$ $\sigma(T)$ is closed.

If $|\lambda| > \|T\| \Rightarrow \lambda \in \rho(T)$

$$T - \lambda = \lambda\left(\frac{T}{\lambda} - 1\right) = -\lambda\left(1 - \frac{T}{\lambda}\right)$$

$$(T-\lambda)^{-1} = -\lambda^{-1}\left(1 - \frac{T}{\lambda}\right)^{-1}$$

$$\left\|\frac{T}{\lambda}\right\| = \frac{1}{|\lambda|}\|T\| < 1$$

$$\sigma(T) \neq \emptyset.$$