Solution problem set 5

1. Find the general and the definite solution of the following differential equations. Check the validity of your answers.
   a. $\frac{dy}{dt} + 4y = 12; \quad y(0) = 2$
   b. $\frac{dy}{dt} - 2y = 0; \quad y(0) = 9$
   c. $\frac{dy}{dt} + 10y = 15; \quad y(0) = 0$
   d. $2\frac{dy}{dt} + 4y = 6; \quad y(0) = 1.5$

Solution

a. Particular integral: $\frac{dy}{dt} = 0 \rightarrow y_p = 3$

Complementary function: $\frac{dy}{dt} + 4y = 0 \rightarrow y_c = Ae^{-4t}$

General solution: $y = 3 + Ae^{-4t}$

Using initial condition $y(0) = 2$: $y(0) = 3 + Ae^{-4\cdot 0} = 2 \rightarrow A = -1$

Definite solution: $y = 3 - e^{-4\cdot t}$

b. This differential equation is homogeneous

General solution: $y = Ae^{2t}$

Using initial condition $y(0) = 9$: $y(0) = Ae^{2\cdot 0} = 9 \rightarrow A = 9$

Definite solution: $y = 9e^{2\cdot t}$

c. Particular integral: $\frac{dy}{dt} = 0 \rightarrow y_p = 1.5$

Complementary function: $\frac{dy}{dt} + 10y = 0 \rightarrow y_c = Ae^{-10t}$

General solution: $y = 1.5 + Ae^{-10t}$

Using initial condition $y(0) = 0$: $y(0) = 1.5 + Ae^{-10\cdot 0} = 0 \rightarrow A = -1.5$

Definite solution: $y = 1.5 - 1.5\, e^{-10\cdot t}$

d. Particular integral: $\frac{dy}{dt} = 0 \rightarrow y_p = 1.5$

Complementary function: $2\frac{dy}{dt} + 4y = 0 \rightarrow y_c = Ae^{-2t}$

General solution: $y = 1.5 + Ae^{-2t}$

Using initial condition $y(0) = 1.5$: $y(0) = 1.5 + Ae^{-2\cdot 0} = 1.5 \rightarrow A = 0.5$

Definite solution: $y = 1.5$

2. Solve the following first order linear differential equations

   a. $\frac{dy}{dt} + 2ty = t; y(0) = 1.5$

   b. $\frac{dy}{dt} + t^2 y = 5t^2;\ \ y(0) = 6$

   c. $2\frac{dy}{dt} + 12y + 2e^t = 0; y(0) = \frac{6}{7}$

   d. $\frac{dy}{dt} + y = t$; y(0)=1

Solution

To find the general solution, in all four differential equations, we apply th3e formula:

$$y = e^{-\int u(t)dt}(A + \int w(t)e^{\int u(t)dt}dt)$$

(the differential equation is $\frac{dy}{dt} + u(t)y = w(t)$)

a. $u(t) = 2t \quad w(t) = t$
   $\int u(t)dt = \int 2t\, dt = t^2 + k_1$ where k_1 is an arbitrary constant.
   Replacing in the formula we get:
   $y = e^{-(t^2+k_1)}(A + \int t\, e^{(t^2+k_1)}dt) = e^{-(t^2+k_1)}\left(A + \frac{1}{2}e^{(t^2+k_1)} + k_2\right) = Be^{-t^2} + \frac{1}{2}$ where k_2 and B are arbitrary constants.
   Using initial condition $y(0) = 1.5$: $\ y(0) = Be^0 + \frac{1}{2} = 1.5 \rightarrow B = 17$
   Definite solution: $y = e^{-t^2} + \frac{1}{2}$

b. $u(t) = t^2 \quad w(t) = 5t^2$
   $\int u(t)dt = \int t^2\, dt = \frac{1}{3}t^3 + k_1$ where k_1 is an arbitrary constant.
   Replacing in the formula we get:
   $y = e^{-\left(\frac{1}{3}t^3+k_1\right)}\left(A + \int 5t^2\, e^{\left(\frac{1}{3}t^3+k_1\right)}dt\right) = e^{-\left(\frac{1}{3}t^3+k_1\right)}\left(A + 5e^{\left(\frac{1}{3}t^3+k_1\right)} + k_2\right) = Be^{-\frac{1}{3}t^3} + 5$
   where k_2 and B are arbitrary constants.
   Using initial condition $y(0) = 6$: $\quad y(0) = Be^{-\frac{1}{3}0^3} + 5 = 6 \rightarrow B = 1$
   Definite solution: $y = e^{-\frac{1}{3}t^2} + 5$

c. We rewrite the equation as: $\frac{dy}{dt} + 6y = -e^t$.
   $u(t) = 6 \quad w(t) = -e^t$
   $\int u(t)dt = \int 6\, dt = 6t + k_1$ where k_1 is an arbitrary constant.
   Replacing in the formula we get:
   $y = e^{-(6t+k_1)}(A - \int e^t\, e^{(6t+k_1)}dt) = e^{-(6t+k_1)}t\left(A - \frac{1}{7}e^{(7t+k_1)} + k_2\right) = Be^{-6t} - \frac{1}{7}e^t$
   where k_2 and B are arbitrary constants.
   Using initial condition $y(0) = \frac{6}{7}$: $\quad y(0) = Be^{-6\cdot 0} - \frac{1}{7}e^0 = \frac{6}{7} \rightarrow B = 1$
   Definite solution: $y = e^{-6t} - \frac{1}{7}e^t$

d. $u(t) = 1 \quad w(t) = t$
   $\int u(t)dt = \int 1\, dt = t + k_1$ where k_1 is an arbitrary constant.
   Replacing in the formula we get:

$y = e^{-(t+k_1)}\left(A + \int t\, e^{(t+k_1)} dt\right) = e^{-(t+k_1)}\left(A + (t-1)e^{(t+k_1)} + k_2\right) = Be^{-t} + (t-1)$
where k_2 and B are arbitrary constants (hint: the first integral is solved using integration by parts)
Using initial condition $y(0) = 1$: $y(0) = Be^{-0} + (0-1) = 1 \rightarrow B = 2$
Definite solution: $y = 2e^{-t} + (t-1)$

3. Verify that each of the following differential equation is exact, then solve it.

a. $3y^2t\,dy + (y^3 + 2t)\,dt = 0$

b. $t(1+2y)dy + y(1+y)\,dt = 0$

c. $\frac{dy}{dt} + \frac{2y^4t+3t^2}{4y^3t^2} = 0$

Solution

a. $M = 3y^2t \quad N = y^3 + 2t$

$\frac{dM}{dt} = 3y^2 \ \frac{dN}{dy} = 3y^2$ → then it is an exact differential equation

$$F(y,t) = \int 3y^2t\,dy = y^3t + g(t)$$

$$N = \frac{d\,F(y,t)}{dt} = y^3 + g'(t)$$

$$y^3 + g'(t) = y^3 + 2t$$

$$g'(t) = 2t$$

$$g(t) = \int g'(t)\,dt = \int 2t\,dt = t^2 + k$$

$$F(y,t) = y^3t + t^2 + k = c$$

$$y = \sqrt{\frac{K - t^2}{t}}$$

where k, c, K are arbitrary constants.

b. $= t(1+2y) \quad N = y(1+y)$

$\frac{dM}{dt} = (1+2y) \ \frac{dN}{dy} = (1+2y)$ → then it is an exact differential equation

$$F(y,t) = \int t(1+2y)\,dy = ty + ty^2 + g(t)$$

$$N = \frac{d\,F(y,t)}{dt} = y + y^2 + g'(t)$$

$$y + y^2 + g'(t) = y(1+y)$$

$$g'(t) = 0$$

$$g(t) = \int g'(t)\,dt = \int 0\,dt = k$$

$$F(y,t) = ty + ty^2 + k = c$$

where k is an arbitrary constant.

Solving by y we get the general solution.

c. The equation is rewritten as:

$$dy + \frac{2y^4t + 3t^2}{4y^3t^2}dt = 0$$

Then $M = 1$ and $N = \frac{2y^4t+3t^2}{4y^3t^2}$

It is straightforward that $\frac{dM}{dt} = 0$ and $\frac{dN}{dy} \neq 0$ then the equation is not exact.

4. Are the following differential equations exact? If not try y, t, y^2 as possible integrating factors

a. $2(t^3+1)\,dy + 3yt^2\,dt = 0$
b. $4y^3t\,dy + (2y^4+3t)dt = 0$

Solution

a. $M = 2(t^3+1) \quad N = 3yt^2$
$\frac{dM}{dt} = 6t^2 \quad \frac{dN}{dy} = 3t^2$ then, it is not exact

Integrating factor y

$M = 2(t^3+1)y \quad N = 3y^2t^2$
$\frac{dM}{dt} = 6yt^2 \quad \frac{dN}{dy} = 6yt^2$ then, it is exact

Integrating factor t

$M = 2(t^4+t) \quad N = 3yt^3$
$\frac{dM}{dt} = 8t^3+2 \quad \frac{dN}{dy} = 3t^3$ then, it is not exact

Integrating factor y^2

$M = 2y^2(t^3+1) \quad N = 3y^3t^2$
$\frac{dM}{dt} = 6y^2t^2 \quad \frac{dN}{dy} = 9y^2t^2$ then, it is not exact

b. $M = 4y^3t \quad N = 2y^4+3t$
$\frac{dM}{dt} = 4y^3 \quad \frac{dN}{dy} = 8y^3$ then, it is not exact

Integrating factor y

$M = 4y^4t \quad N = 2y^5+3ty$

$\frac{dM}{dt} = 4y^4 \quad \frac{dN}{dy} = 10y^4+3t$ then, it is not exact

Integrating factor t

$M = 4y^3t^2 \quad N = 2y^4t+3t^2$
$\frac{dM}{dt} = 8y^3t \quad \frac{dN}{dy} = 8y^3\mathrm{t}$ then, it is exact

Integrating factor y^2

$M = 4y^5t \quad N = 2y^6+3ty^2$

$\frac{dM}{dt} = 4y^5 \quad \frac{dN}{dy} = 12y^5+6ty$ then, it is not exact

5. Applying the method to solve an exact differential equation to the general exact differential equation $M\ dy + N\ dt = 0$ derive the following formula for the general solution of an exact differential equation $\int M\ dy + \int N\ dt - \int \frac{d \int M dy}{dt}\ dt = c$. Verify that this formula is the solution of exact differential equation $M\ dy + N\ dt = 0$

- $F(y,t) = \int M\ dy + g(t)$
- $\frac{dF}{dt} = \frac{d \int M\ dy}{dt} + g'(t) = N$
- $g'(t) = N - \frac{d \int M\ dy}{dt}$
- $g(t) = \int g'(t)dt = \int Ndt - \int \frac{d \int M\ dy}{dt} dt$

Replacing in the first line

- $F(y,t) = \int M\ dy + \int Ndt - \int \frac{d \int M\ dy}{dt} dt$

Then the solution is given by:

$$\int M\ dy + \int Ndt - \int \frac{d \int M\ dy}{dt} dt = c$$