Solution Problem set 2

Integration

1. Find the indefinite integral of:
   a. $x+1$
   b. $-2x^2$
   c. xe^x

a) $\int x+1 \, dx = \frac{1}{2}x^2 + x + c$

b) $\int -2x^2 \, dx = -2\int x^2 dx = -2\,\frac{1}{3}x^3 = -\frac{2}{3}x^3$

c) $\int x e^x dx$ we solve it by parts.

$u(x) = x \quad \frac{dv(x)}{dx} = e^x \rightarrow \frac{du(x)}{dx} = 1 \quad v(x) = e^x$

$\int x e^x dx = x \cdot e^x - \int 1 \cdot e^x dx = x e^x - e^x + c$

$= e^x(x-1) + c$

check:

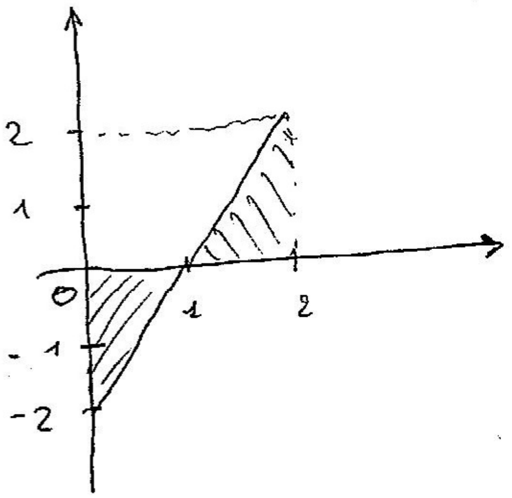
$$\frac{d\, e^x(x-1)+c}{dx} = e^x(x-1) + e^x = x e^x$$

2. Find the definite integral between 0 and 2 of:

   a. $2x - 2$

   b. $e^{0.5x}$

a) $$\int_0^2 2x-2 \, dx = \left| x^2 - 2x \right|_0^2 = 4-4-0+0 = 0$$



it is equal to 0 because is the sum of a "negative" and a "positive" area of equal absolute value

b) $$\int_0^2 e^{0.5x} \, dx = 2e^{0.5x} \Big|_0^2 = 2e - 2 = 2(e-1)$$

3. Which of these improper integrals exists?

a. $\int_0^2 x^2\,dx$

b. $\int_0^\infty e^{-2x}\,dx$

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a) it is not an improper integral
Upper and lower limits are finite and the integrand is well defined over The interval $[0, 2]$

$$\int_0^2 x^2\,dx = \frac{1}{3}x^3\Big|_0^2 = \frac{1}{3}2^3 - \frac{1}{3}0^3 = \frac{8}{3}$$

b) it is improper because The uper limit is ∞
we have to compute the limit

$$\lim_{b\to\infty} \int_0^b e^{-2x}\,dx$$

To start solve the integral $\int_0^b e^{-2x}\,dx$.

it is $-\frac{1}{2}e^{-2x}\Big|_0^b = -\frac{1}{2}e^{-2b} + \frac{1}{2}e^{-20} = \frac{1}{2} - \frac{1}{2}e^{-2b}$

then the limit is:

$$\lim_{b\to\infty} \frac{1}{2} - \frac{1}{2}e^{-2b} = \frac{1}{2}$$

Then the integral converges

4. Use the substitution method to integrate the following

$$\int_0^2 \frac{3x^2}{(x^3+1)^2}dx$$

Let be $U(x) = x^3 + 1$

$$\frac{dU}{dx} = 3x^2 \rightarrow dx = \frac{dU}{3x^2}$$

the upper limit will be $2^3 + 1 = 9$

the lower limit will be $0^3 + 1 = 1$

Replacing

$$\int_1^9 \frac{3x^2}{U^2} \frac{dU}{3x^2} = \int_1^9 \frac{1}{U^2} dU$$

$$= -\frac{1}{U}\Big|_1^9 = -\frac{1}{9} + 1 = \frac{8}{9}$$

Check

a) $-\frac{1}{U} = -\frac{1}{x^3+1}$ taking its derivative wrt x we get the integrand

b) $-\frac{1}{x^3+1}\Big|_0^2 = -\frac{1}{9} + 1 = \frac{8}{9}$ same result.

5. Integrate by parts

$$\int_2^5 \frac{3x}{(x+1)^2}\,dx$$

Let be

$$\frac{dv(x)}{dx} = \frac{1}{(x+1)^2} \qquad \text{and} \qquad u(x) = 3x$$

then

$$v(x) = -\frac{1}{x+1} \qquad \text{and} \qquad \frac{du(x)}{dx} = 3$$

The integral is re-written as

$$-\frac{3x}{x+1}\Big|_2^5 - \int_2^5 -\frac{3}{x+1}\,dx =$$

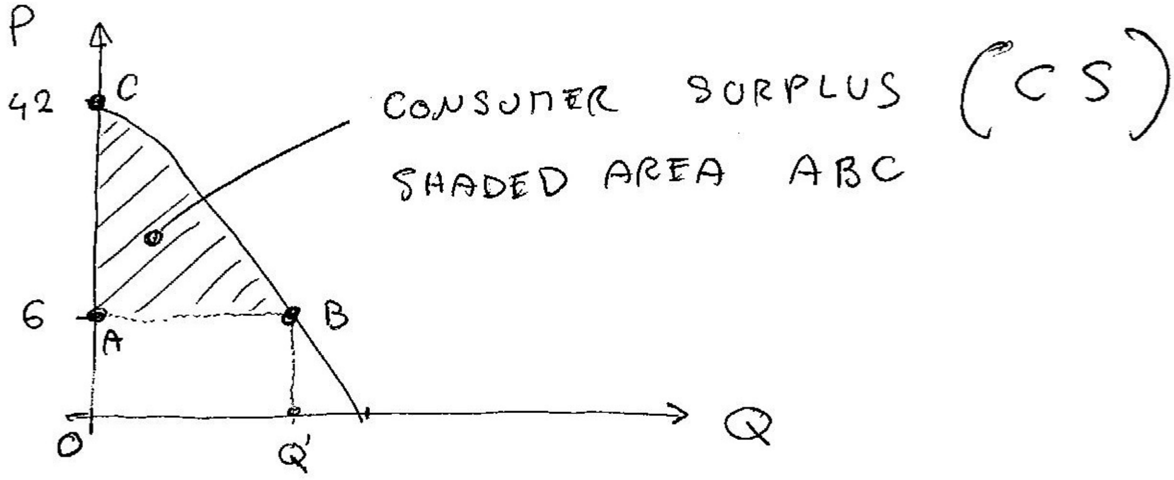
$$= -\frac{3x}{x+1}\Big|_2^5 + 3\int_2^5 \frac{1}{x+1}\,dx =$$

$$= -\frac{3x}{x+1}\Big|_2^5 + 3\ln(x+1)\Big|_2^5 =$$

$$= \frac{-15}{6} + \frac{6}{3} + 3\ln(6) - 3\ln(3) =$$

$$= -\frac{1}{2} + 3\ln(2) = -\frac{1}{2} + \ln(8)$$

6. Given the demand function $P = 42 - 5Q - Q^2$. Find the consumer surplus at the equilibrium price of $p = 6$.



Find Q' (the quantity demande at price $p = 6$)

You have to solve $6 = 42 - 5Q - Q^2$.

Solutions are 4 and −9

The good solution is the first one

Then $Q' = 4$.

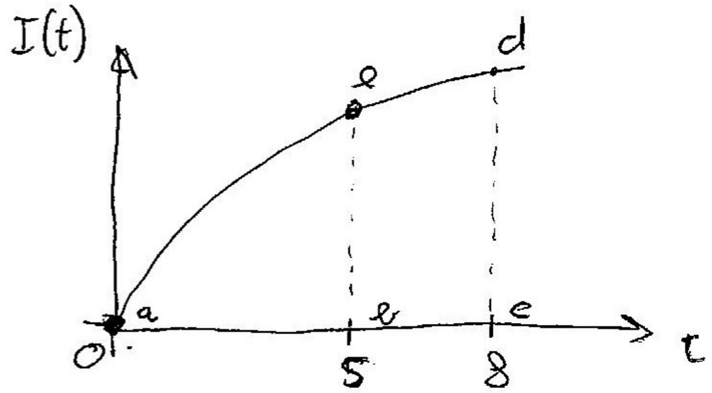
$$CS = \underbrace{\int_0^4 42 - 5Q - Q^2}_{\text{area } OQ'BC} - \underbrace{6 \cdot 4}_{\text{area } OQ'BA}$$

$$= 42Q - 2{,}5Q^2 - \frac{1}{3}Q^3 \Big|_0^4 - 24 =$$

$$= 168 - 40 - \frac{64}{3} - 24 = \frac{248}{3}$$

7. Given Investment $I(t) = 9t^{0.5}$ find the level of capital stock in a) after 8 years b) between years 5 and 8.

accumulated



a) it is given by the area acda

$$\int_0^8 9t^{0.5}\,dt = 6t^{\frac{3}{2}}\Big|_0^8 = 6 \cdot 8^{\frac{3}{2}} = 13.58$$

b) it is given by the area bcde

$$\int_5^8 9t^{0.5}\,dt = 6t^{\frac{3}{2}}\Big|_5^8 = 6 \cdot 8^{\frac{3}{2}} - 6 \cdot 5^{\frac{3}{2}} = 6.87$$