Esempio $|x| : \mathbb{R} \to \mathbb{R}$

$|x| \in C^0(\mathbb{R})$

Si tratta di dimostrare che $\forall x_0 \in \mathbb{R}$

$\lim_{x \to x_0} |x| = |x_0|$ cioè

$$\boxed{\forall \varepsilon > 0 \ \exists \delta_\varepsilon > 0 \ \text{t.c.} \ 0 < |x - x_0| < \delta_\varepsilon \Rightarrow \big| |x| - |x_0| \big| < \varepsilon} \quad (1)$$

Preliminarmente osserviamo che vale

$$\boxed{\big| |x| - |y| \big| \le |x - y| \qquad \forall x, y \in \mathbb{R}.} \quad (2)$$

La (2) si può scrivere nella forma seguente

$$-|x-y| \boxed{\le} |x| - |y| \boxed{\le} |x - y| \quad (3)$$

$\forall x, y.$

Dimostriamo (3), cominciamo col dimostrare

$|x| - |y| \le |x - y| \iff$ ~~$|y| \le |y - x| + |x|$~~ $|x| \le |x - y| + |y|$ $*$ $\forall x, y$

e l'ultima segue dalla disug. triangolare perché

$$\boxed{|x| = |(x-y) + y| \le |x - y| + |y|}$$

Dimostriamo infine

$-|x - y| \le |x| - |y| \iff |y| \le |x - y| + |x| =$

$\iff |y| \le |y - x| + |x|$ quest'ultima si ottiene da

da $*$ scambiando di ruolo x e y.

Conclusione: si ha $\big| |x| - |y| \big| \le |x - y| \quad \forall x, y \in \mathbb{R}$

Vogliamo dimostrare la (1)

$$\boxed{\forall \varepsilon > 0 \ \exists \delta_\varepsilon > 0 \ \text{t.c.} \ |x - x_0| < \delta_\varepsilon \Rightarrow \big| |x| - |x_0| \big| < \varepsilon} \quad (1)$$

$\delta_\varepsilon = \varepsilon$

δ_ε $\qquad \big| |x| - |x_0| \big| < \varepsilon$

Posso dire che $\big| |x| - |x_0| \big| \le |x - x_0| < \delta_\varepsilon$

Se scelgo $\delta_\varepsilon = \varepsilon$ la (1) è vera

Teoremi. Siano $f, g : X \to \mathbb{R}$

$x_0 \in X'$ e $\lim\limits_{x \to x_0} f(x) = a$, $\lim\limits_{x \to x_0} g(x) = b$

1) $\lim\limits_{x \to x_0} \left(f(x) + g(x)\right) = a + b$ solvo coi indeterminati

2) $\lim\limits_{x \to x_0} f(x)\, g(x) = a\, b$ //

3) $\lim\limits_{x \to x_0} \dfrac{f(x)}{g(x)} = \dfrac{a}{b}$ solvo $b = 0$ e $(a, b) = (\pm\infty, \pm\infty)$

4) Se $f(x) \leq g(x)$ $\forall x \in X$
allora $a \leq b$

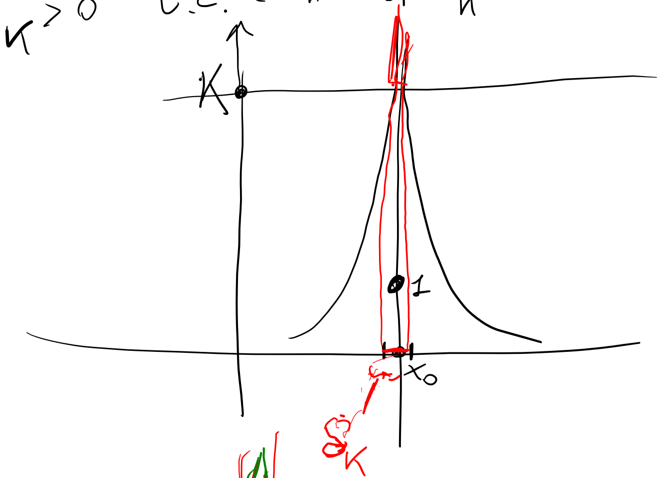
5) Se $f(x) \leq h(x) \leq g(x)$ $\forall x \in X$ e $a = b$

$\Rightarrow$ $\lim\limits_{x \to x_0} h(x) = a = b$.

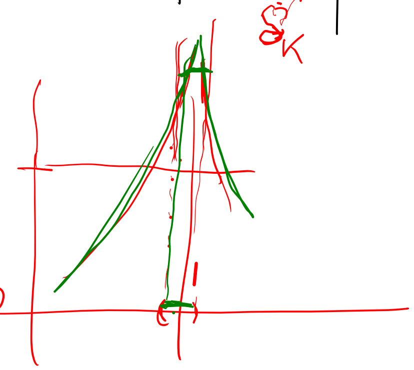
Def $f : X \to \mathbb{R}$, $x_0 \in X'$

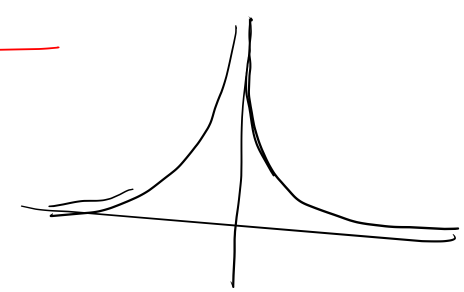
Diciamo che $\lim_{x \to x_0} f(x) = +\infty$ se

$\forall\, K > 0 \ \exists\, \delta_K > 0$ t.c. $0 < |x - x_0| < \delta_K$ e $x \in X \Rightarrow f(x) > K$



$$f(x) = \begin{cases} \frac{1}{x^2} & \text{per } x \neq 0 \\ 0 & \text{per } x = 0 \end{cases}$$





$$\lim_{x \to 0} \frac{1}{x^2} = +\infty$$

Voglio

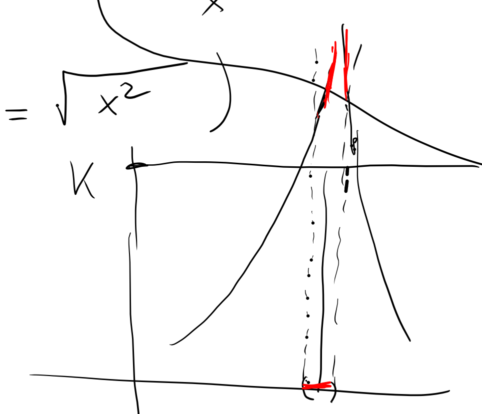
$\forall\, K > 0 \ \exists\, \delta_K > 0$ t.c. $0 < |x| < \delta_K \Rightarrow \frac{1}{x^2} > K$

$\frac{1}{x^2} > K \iff \frac{1}{K} > x^2 = |x|^2$

$\frac{1}{K} > |x|^2 \iff \frac{1}{\sqrt{K}} > |x|$

$\delta_K = \frac{1}{\sqrt{K}}$

$\left(|x| = \sqrt{x^2} \right)$



Teorema $\sin x$ e $\cos x$ $\in C^0(\mathbb{R})$.

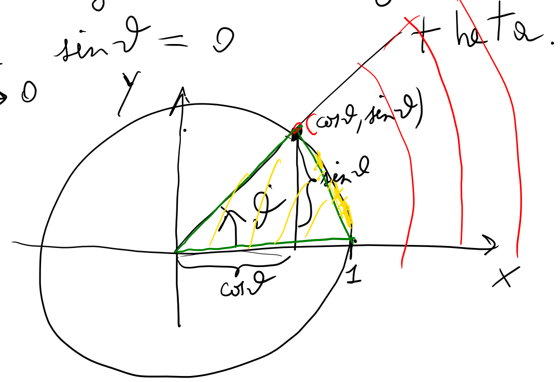
Dim

1) Cominciamo col dimostrare che $\sin x$ e' continua nello 0

cioe' $\lim_{x \to 0} \sin x = \sin 0 = 0$. Applicheremo i carabinieri

ma prima guardiamo al significato geometrico del seno.

$\lim_{\vartheta \to 0} \sin \vartheta = 0$



Area (triangolo verde) $= \frac{\sin\vartheta}{2}$

Area (fetta torta) $= \frac{\vartheta}{2}$

Area (triangolo) < area (fetta torta)

$\frac{\sin\vartheta}{\cancel{2}} < \frac{\vartheta}{\cancel{2}}$ $\qquad 0 < \vartheta < \frac{\pi}{2}$

$|\sin\vartheta| < |\vartheta|$ $\qquad 0 \le |\vartheta| < \frac{\pi}{2}$

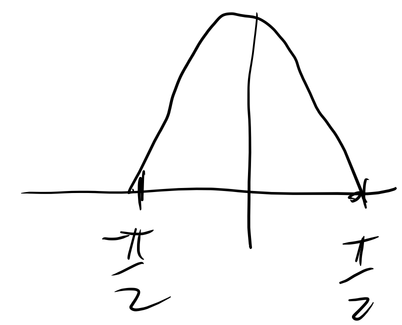
$|\sin(-\vartheta)| = |-\sin\vartheta| = |\sin\vartheta|$

$-|\vartheta| < \sin\vartheta < |\vartheta|$ $\qquad \vartheta \to 0$

$-|\vartheta| \xrightarrow{\vartheta \to 0} 0$, $\quad \sin\vartheta \xrightarrow{\text{per i carab.}} 0$, $\quad |\vartheta| \xrightarrow{\vartheta \to 0} |0| = 0$

Conclusione $\lim_{\vartheta \to 0} \sin\vartheta = \lim_{x \to 0} \sin x = 0 = \sin(0)$

2) $\lim_{x \to 0} \cos x = \cos 0 = 1$. Per $|x| < \frac{\pi}{2}$



$$\cos x = \sqrt{1 - \sin^2 x}$$

$$\cos^2 x = 1 - \sin^2 x$$

$$\lim_{x \to 0} \cos^2 x = \lim_{x \to 0} \left(1 - \sin^2 x\right) =$$

$$= \lim_{x \to 0} 1 - \lim_{x \to 0} \sin^2 x$$

$$= 1 - 0 = 1$$

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\cos 2x = 2\cos^2 x - 1$$

$$\lim_{x \to 0} \cos(2x) = \lim_{x \to 0} \left(2 \underset{\downarrow}{\underset{1}{\cos^2 x}} - 1\right) = 1$$

$y = 2x$ $\left(= \lim_{y \to 0} \cos(y) = 1 \right.$

3) $\lim_{x \to x_0} \sin x = \sin x_0 \quad \forall x_0 \in \mathbb{R}$

$$x = h + x_0$$

$$\lim_{x \to x_0} \operatorname{sen} x = \lim_{h \to 0} \operatorname{sen}(h + x_0) =$$

$$= \lim_{h \to 0} \Big(\underbrace{\sin(h)\cos(x_0)}_{\to 0} + \underbrace{\cos(h)}_{\to 1} \sin(x_0) \Big) = \sin x_0$$

4) $\lim_{x \to x_0} \cos x = \cos x_0 \quad \forall x_0 \in \mathbb{R}$

$$\Big(\lim_{h \to 0} \cos(\cancel{h} + x_0) = \lim_{h \to 0} \Big(\underbrace{\cos(h)\cos(x_0)}_{\to \cos(x_0)} - \overbrace{\operatorname{sen}(h)\sin(x_0)}^{\to 0} \Big) = \cos x_0$$

$$\lim_{x\to+\infty}\left(\sqrt[n]{x^2+1}-\sqrt[n]{x^2+\cancel{x}+1}\right)=? \qquad \forall\ n\geq 2$$

$n=2$

$$\lim_{x\to+\infty}\left(\sqrt{x^2+1}-\sqrt{x^2+\cancel{x}+\cancel{1}}\right)=?$$

$$a^2-b^2=(a-b)(a+b)$$

$$\left(\overbrace{\sqrt{x^2+1}}^{(a}-\overbrace{\sqrt{x^2+x+1}}^{b)}\right)\frac{\overbrace{\sqrt{x^2+1}}^{a}+\overbrace{\sqrt{x^2+x+1}}^{b}}{\sqrt{x^2+1}+\sqrt{x^2+x+1}}$$

$$=\frac{\cancel{x^2}+\cancel{1}-(\cancel{x^2}+x+\cancel{1})}{\sqrt{x^2+1}+\sqrt{x^2+x+1}}$$

$$=\frac{-x}{\sqrt{x^2+1}+\sqrt{x^2+x+1}}$$

$$=\frac{-x}{\sqrt{x^2\left(1+\frac{1}{x^2}\right)}+\sqrt{x^2\left(1+\frac{1}{x}+\frac{1}{x^2}\right)}}=$$

$$=\frac{-\cancel{x}}{\cancel{x}\sqrt{1+\frac{1}{x^2}}+\cancel{x}\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}}$$

$$=\frac{-1}{\underbrace{\sqrt{1+\frac{1}{x^2}}}_{\downarrow\ 1}+\underbrace{\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}}_{\downarrow\ 1}}\xrightarrow{x\to+\infty}\frac{-1}{2}$$

$$\lim_{x\to+\infty}\left(\sqrt[n]{x^2+1}-\sqrt[n]{x^2+\cancel{x}+1}\right)=? \qquad \forall\ n\geq 2$$

$$a^n-b^n=(a-b)\left(a^{n-1}+a^{n-2}b+\cdots+ab^{n-2}+b^{n-1}\right)$$

$$=(a-b)\sum_{k=1}^{n}a^{n-k}b^{k-1}$$

$$\left(\sqrt[n]{x^2+1}-\sqrt[n]{x^2+x+1}\right)\frac{\sum_{k=1}^{n}(x^2+1)^{\frac{n-k}{n}}(x^2+\cancel{x}+1)^{\frac{k-1}{n}}}{\sum_{k=1}^{n}(x^2+1)^{\frac{n-k}{n}}(x^2+x+1)^{\frac{k-1}{n}}}$$

$$=\frac{\cancel{x^2}+\cancel{1}-(\cancel{x^2}+x+\cancel{1})}{\sum_{k=1}^{n}(x^2+1)^{\frac{n-k}{n}}(x^2+x+1)^{\frac{k-1}{n}}} \qquad n>2$$

$$=\frac{-x}{\sum_{k=1}^{n}\left(x^2\left(1+\frac{1}{x^2}\right)\right)^{\frac{n-k}{n}}\left(x^2\left(1+\frac{1}{x}+\frac{1}{x^2}\right)\right)^{\frac{k-1}{n}}}$$

$$=\frac{-x}{\sum_{k=1}^{n}x^{2\left(\frac{n-\cancel{k}}{n}+\frac{\cancel{k}-1}{n}\right)}\left(1+\frac{1}{x^2}\right)^{\frac{n-k}{n}}\left(1+\frac{1}{x}+\frac{1}{x^2}\right)^{\frac{k-1}{n}}}$$

$$x^{2\,\frac{n-1}{n}}$$

$$=\frac{-x}{x^{2\frac{n-1}{n}}}\cdot\frac{1}{\sum_{k=1}^{n}\underbrace{\left(1+\frac{1}{x^2}\right)^{\frac{n-k}{n}}}_{\to 1}\underbrace{\left(1+\frac{1}{x}+\frac{1}{x^2}\right)^{\frac{k-1}{n}}}_{\to 1}}\ \to n$$

$$=-x^{\cancel{1}-2\left(\frac{n-1}{n}\right)}$$

$$=-x^{1-2\left(1-\frac{1}{n}\right)}=-x^{-1+\frac{2}{n}}\xrightarrow{x\to+\infty}-\frac{1}{(+\infty)^{1-\frac{2}{n}}}=0$$