$X \quad \mathbb{C}$

$T \in \mathcal{L}(X)$

$$\mathbb{C} \neq \rho(T) \supseteq \mathbb{C} \setminus \overline{D_{\mathbb{C}}(0, \|T\|)}$$

$$\underset{z}{\ni} \longrightarrow R_T(z) = (T - z)^{-1}$$

This is an analytic function $\in H(\rho(T), \mathcal{L}(X))$

If $z_0 \in \rho(T)$ $\quad D_{\mathbb{C}}(z_0, r)$ $\quad r > 0$ small

$$R_T(z) = (T - z)^{-1} = (T - z_0 + z_0 - z)^{-1} =$$

$$= (1 - (z - z_0) R_T(z_0))^{-1} R_T(z_0)$$

$$= \sum_{n=0}^{\infty} R_T^n(z_0) (z - z_0)^n \quad R_T(z_0)$$

$$= \boxed{\sum_{n=0}^{\infty} R_T^{n+1}(z_0) (z - z_0)^n}$$

$$|z - z_0| < \frac{1}{\| R_T(z_0) \|}$$

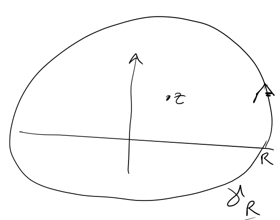
The theory for $H(\rho(T), \mathcal{L}(X))$ is the same as the theory for $H(\rho(T), \mathbb{C})$

For example we have Liouville theorem:

If $f \in H(\mathbb{C}, \mathcal{L}(X))$ is bounded, that is, $\exists\, C > 0$ s.t.

$$\|f(z)\| \leq C \quad \forall z \in \mathbb{C}$$

then $f(z) \equiv A \in \mathcal{L}(X)$ for A a fixed operator



$$f(z) = \frac{1}{2\pi i} \int_{\gamma_R} \frac{f(w)}{w - z}\, dw$$

$$f'(z) = \frac{1}{2\pi i} \int_{\gamma_R} \frac{f(w)}{(w - z)^2}\, dw \qquad \gamma_R(t) = R e^{it} \quad t \in [0, 2\pi]$$

$$|f'(z)| \leq \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{f(w)\, R i e^{it}}{(R e^{it} - z)^2} \right| dt$$

$$\leq \frac{1}{2\pi} R \int_0^{2\pi} \frac{C}{(R - |z|)^2}\, dt$$

$$|f'(z)| \leq C \frac{R}{(R - |z|)^2} \xrightarrow{R \to \infty} 0 \qquad |z| < R$$

$\sigma(T) \neq \emptyset$

Because if $\exists\ T \in \mathcal{L}(X)$ with $\sigma(T) = \emptyset$

then $R_T \in H(\mathbb{C}, \mathcal{L}(X))$

It is easy to see it is bounded

$$z \to \|(T-z)^{-1}\| \in C^0(\mathbb{C}, \mathbb{R})$$

$$\lim_{z\to\infty} \|(T-z)^{-1}\| = 0$$

$$(T-z)^{-1} = z^{-1}\Big(\frac{T}{z} - 1\Big)^{-1}$$

$z \to \infty$, $\frac{T}{z} \to 0$

$$\Big\|\frac{T}{z}\Big\| = \frac{1}{|z|}\|T\| \xrightarrow[z\to\infty]{0} 0$$

$\therefore \exists\ C > 0$ s.t.

$$\|R_T(z)\| \leq C \quad \forall z \in \mathbb{C}$$

Then $R_T(z) = A \in \mathcal{L}(X)$

$$\|A\| = \lim_{z\to\infty} \|R_T(z)\| = 0 \Rightarrow A \equiv 0.$$

Absurd $\quad (T-z)R_T(z) = 1$

$$(T-z)\,0 = 0$$

$\Omega \subseteq \mathbb{R}^d$ open

$L^p(\Omega) \circlearrowleft$ given a function $m(x)$ in Ω

$f \longrightarrow m f = T_m f$

for this to be bounded we need $\|m\|_{L^\infty(\Omega)} < +\infty$

and then $\|T_m\|_{\mathcal{L}(L^p(\Omega))} = \|m\|_{L^\infty(\Omega)}$

These are the analogues of the diagonal ~~operators~~ matrices in finite dimension.

$\mathbb{R}^2 \longrightarrow \mathbb{R}^2 \qquad A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$

$A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a x \\ b y \end{pmatrix}$

$\mathbb{R}^2 =$ space of functions $\{1,2\} \to \mathbb{R}$ $\quad f(1) = x,\ f(2) = y$

$m : \{1,2\} \to \mathbb{R} \qquad m(1) = a$

$m(2) = b$

$f \longrightarrow m f.$

$(mf)(1) = m(1) f(1) = a x$

$(mf)(2) = m(2) f(2) = b y$

$L^p(\Omega) \subset \mathbb{R}^\Omega$

$T : H \circlearrowleft \qquad \begin{array}{c} x \\ \downarrow \\ Tx \end{array} \qquad H \xrightarrow{U} L^2(\Omega, d\mu) \qquad \begin{array}{c} Ux \\ \downarrow \\ mUx \end{array}$

$T \circlearrowleft \qquad \circlearrowleft \ m \in L^\infty(\Omega)$

$m \in C^0([0,1])$

$T_m f = m f$ in $L^p([0,1])$

$1 \le p < \infty$

$\rightarrow \boxed{\sigma(T_m) = m([0,1])}$

$L^p(\mathbb{R})$ $\quad m \in BC^0(\mathbb{R})$

$\sigma(T_m) = \overline{m(\mathbb{R})} \subseteq \mathbb{C}$

I will use: $\|T_{m_1}\|_{\mathcal{L}(L^p([0,1]))}$ is bounded iff $m_1 \in L^\infty([0,1])$ and $\|T_{m_1}\|_{\mathcal{L}(L^p([0,1]))} = \|m_1\|_{L^\infty([0,1])}$

If $z \notin m([0,1])$

$(T_m - z) f = (m(x) - z) f(x)$

$m([0,1])$ is a compact subspace of $\mathbb{C}$.

Then $\underbrace{\operatorname{dist}(z, m([0,1]))}_{d_z} = \inf_{x \in [0,1]} |z - m(x)| > 0$

$\Rightarrow \left| \frac{1}{m(x) - z} \right| \le \frac{1}{d_z} < \infty$

so $\frac{1}{m(\cdot) - z}$ is bounded multiplier in $L^p([0,1])$

$\frac{1}{m(x)-z} (m(x) - z) f = f \quad \forall f$

$R_{T_m}(z)$ $\quad T_m - z$

$T_{m_1} T_{m_2} = T_{m_1 m_2} = T_{m_2} T_{m_1}$

$\sigma(T_m) \subseteq m([0,1])$, $\supseteq$

$z_0 \in m([0,1])$ We need to show is not in the resolvent set of T_m. $\frac{1}{m(x) - z_0}$ will be unbounded

x in $[0,1]$

$f \xrightarrow{T} x f \qquad \sigma(T) = [0,1]$

There are no eigenvalues: No $f \neq 0$ s.t.

$\boxed{x f(x) = \lambda f(x)}$ for some fixed $\lambda \in [0,1]$

$(x - \lambda) f(x) = 0 \quad$ a.e. $x \in [0,1]$

$\Rightarrow f(x) = 0$ a.e. $\Rightarrow f = 0$

$K \subset\subset \mathbb{C}$

Find a B space X and a $T \in \mathcal{L}(X)$ s.t.

$\sigma(T) = K$. card $K = \infty$

$X = \ell^p(\mathbb{N}, \mathbb{C})$ $1 \le p < +\infty$

$= \left\{ f : \mathbb{N} \to \mathbb{C} : \left(\sum_{n=1}^{+\infty} |f(n)|^p \right)^{\frac{1}{p}} < +\infty \right\}$ $\mathbb{N} = \{1, 2, \dots\}$ $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$

$m \in \ell^\infty(\mathbb{N}, \mathbb{C})$

$T_m f(n) = m(n) f(n) \quad \forall n \in \mathbb{N}.$

$\sigma(T) = K$

$K = \{z_0\}$

$f \to z_0 f$

I consider a countable dense subset of K

It is the image of my $n \longrightarrow m(n)$ T_m

Each $m(n)$ is an eigenvalue $\Rightarrow m(n) \in \sigma(T_m)$

e_n $\ell^p(\mathbb{N}, \mathbb{C})$ $e_n(k) = \begin{cases} 0 & k \neq n \\ 1 & k = n \end{cases}$

$e_n = (0, \dots, 0, \underset{n}{1}, 0, \dots\dots)$

$e_1 = (1, 0, 0 \dots \dots 0)$

$T_m f(k) = m(k) f(k) \quad \forall k \in \mathbb{N}$

$T_m e_n(k) = m(k) e_n(k) = m(n) e_n(k) \quad \forall k$

$T_m e_n = m(n) e_n$ $e_n \in \ker(T_m - m(n)) \neq \{0\}$

So $\overline{\{m(n)\}_{n \in \mathbb{N}}} \subseteq \sigma(T_m)$

$K \subseteq \sigma(T_m)$

Now one needs to prove that

$\mathbb{C} \setminus K \subseteq \rho(T_m)$ $\Rightarrow K = \sigma(T_m)$

If $z \notin K$ $(T_m - z) f(k) = (m(k) - z) f(k)$

$0 < \operatorname{dist}(z, K) \le |m(n) - z| \quad \forall n \in \mathbb{N}$

$\left\| \frac{1}{m(k) - z} \right\|_{\ell^\infty(\mathbb{N})} \le \frac{1}{\operatorname{dist}(z, K)} < +\infty$

Thur. 23 class is cancelled

Stages in Austria

Nov 6. Thursday at 4.15 in Aula Seminari

stage at BCAM (Bilbao)

presentation by Prof. F. Fanelli.

CERN

$T \in \mathcal{L}(X)$

$$e^T = \sum_{n=0}^{\infty} \frac{T^n}{n!}$$

$\|T^n\| \leq \|T\|^n$

$\|AB\| \leq \|A\| \, \|B\|$

$$S_N = \sum_{n=0}^{N} \frac{T^n}{n!}$$

$M > N$

$$\|S_M - S_N\| = \left\| \sum_{n=N+1}^{M} \frac{T^n}{n!} \right\| \leq \sum_{n=N+1}^{M} \frac{\|T^n\|}{n!}$$

$$\leq \sum_{n=N+1}^{+\infty} \frac{\|T\|^n}{n!} \xrightarrow{N \to +\infty} 0$$

$$\frac{d}{dt} e^{tT} = T e^{tT} = e^{tT} T$$

$$\begin{cases} \dot{x} = Tx + f(t) \\ x(0) = x_0 \in X \end{cases}$$

$T \in \mathcal{L}(X)$

$f : C^0(\mathbb{R}, X)$

$$x(t) = e^{tT} x_0 + \int_0^t e^{(t-s)T} f(s)\, ds$$

Duhamel formula

$\dot{x} - Tx = f$

$T = 0$

$\dot{x} = f$

$x(t) - x(0) = \int_0^t f(t)\, dt$

$x(t) = x_0 + \int_0^t f(t)\, dt$

$$e^{-tT}(\dot{x} - Tx) = e^{-tT} f(t)$$

$$\underbrace{e^{-tT}\dot{x}(t) - e^{-tT} T x(t)} = e^{-tT} f(t)$$

$$\frac{d}{dt} e^{-tT} x(t) = e^{-tT} f(t)$$

$$e^{-tT} x(t) - x_0 = \int_0^t e^{-sT} f(s)\, ds$$

e^{tT}

$$x(t) = e^{tT} x_0 + e^{tT} \int_0^t e^{-sT} f(s)\, ds$$

$$= e^{tT} x_0 + \int_0^t e^{tT} e^{-sT} f(s)\, ds$$

$$= e^{tT} x_0 + \int_0^t e^{(t-s)T} f(s)\, ds$$

In general $e^A e^B \neq e^{A+B}$