Lezione 10 - 14/10/2025

Numeri complessi

$\mathbb{R}^2 = \{(a,b) : a,b \in \mathbb{R}\}$

su $\mathbb{R}^2$ mettiamo una operazione $\#$ "somma"

$$(a,b) \# (a',b') = (a+a', b+b')$$

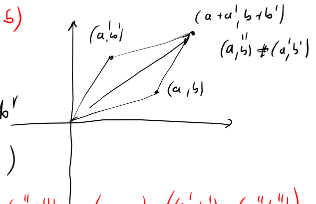
la somma di numeri reali

è associativa

$\exists$ neutro : è $(0,0)$

$\exists$ opposto : l'opposto di (a,b) è $(-a,-b)$

è commutativa



su $\mathbb{R}^2$ mettiamo una operazione $\circ$ "prodotto"

$$(a,b)\circ(a',b') = (aa'-bb', ab'+ba')$$

è associativa $\quad \big((a,b)\circ(a',b')\big)\circ(a'',b'') = (a,b)\circ\big((a',b')\circ(a'',b'')\big)$

esiste neutro $\quad (1,0)$

$$(a,b)\circ(1,0) = (a\cdot 1 - b\cdot 0,\ a\cdot 0 + b\cdot 1) = (a,b)$$

esiste il reciproco

infatti prendo $(a,b) \neq (0,0)$

e vedo se c'è un (x,y) t.c.

$$(a,b)\circ(x,y) = (1,0)$$

$$(ax-by,\ ay+bx) = (1,0)$$

$$\begin{cases} ax - by = 1 \\ ay + bx = 0 \end{cases}$$

c'è una sola soluzione $\quad x = \dfrac{a}{a^2+b^2}, \quad y = \dfrac{-b}{a^2+b^2}$

cioè $\quad (a,b)\circ\left(\dfrac{a}{a^2+b^2}, \dfrac{-b}{a^2+b^2}\right) = (1,0)$

$$\left(\dfrac{a}{a^2+b^2}, \dfrac{-b}{a^2+b^2}\right) = (a,b)^{-1}$$

- è commutativa
- vale la proprietà distributiva

$$(a,b)\circ\big((a',b')\#(a'',b'')\big) = \big((a,b)\circ(a',b')\big)\#\big((a,b)\circ(a'',b'')\big)$$

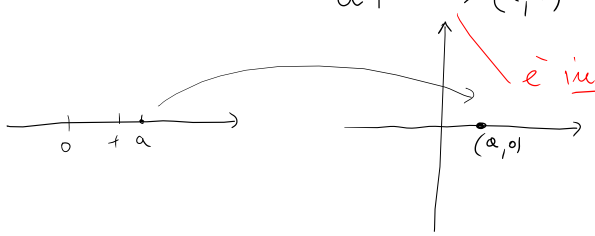
l'insieme $(\mathbb{R}^2, \#, \circ)$ (l'insieme con le 2 operazioni)

lo chiamo <u>spazio dei numeri complessi</u>

<u>oss</u>. considero

$$\mathbb{R} \longrightarrow \mathbb{C}$$
$$a \longmapsto (a,0)$$

è iniettiva



$$b \longmapsto (b,0)$$

$$a+b \longmapsto (a+b,0) = (a,0)\#(b,0)$$
$$(a\cdot b) \longmapsto (a b,0) = (a,0)\circ(b,0)$$

ora identifico i numeri reali con i numeri complessi che stanno sull'asse x

quindi $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \underbrace{\mathbb{R} \subseteq \mathbb{C}}$

i reali stanno nei complessi

$$\begin{array}{ccc} \mathbb{R} & & \mathbb{C} \\ \Downarrow & & \Downarrow \end{array}$$

identifico a con $(a,0)$

e osservo $(a,0)\circ(x,y) = (ax - 0\cdot y,\ ay + 0\cdot x)$

$$\overset{!}{=} (ax,\ ay)$$

$$\overset{!}{=} a(x,y)$$

identifico (a,b) con $a(1,0) + b(0,1)$

moltiplicazione per uno scalare

infine $(1,0) = \mathbb{1}$ unità reale

$(0,1) = i$ unità immaginaria

in questo modo $(a,b) = a\cdot\mathbb{1} + b\cdot i = \underline{\underline{a+ib}}$

notazione

il numero complesso (a,b) lo rappresento con $a+ib$

b è la parte immaginaria

a è la "parte reale"

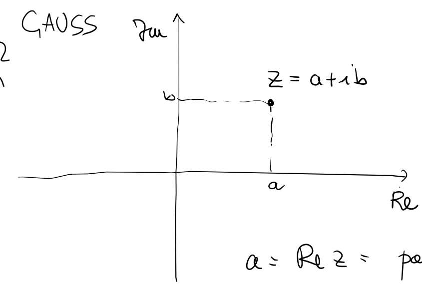
unità immaginaria

$i^2 = i\circ i = (0,1)\circ(0,1) = (0-1,\ 0+0) = (-1,0) = -1(1,0) = -1$

$$\boxed{i^2 = -1}$$

piano di GAUSS

è il piano $\mathbb{R}^2$ dove i punti sono i numeri complessi



$a = \mathrm{Re}\, z$ = parte reale del numero z

$b = \mathrm{Im}\, z$ = parte immaginaria di z.

osservazione

$(2+3i)\cdot(3+4i) = -6+17i$

dovrei fare

$(2,3)\circ(3,4) =$

$(2\cdot 3 - 3\cdot 4,\ 2\cdot 4 + 3\cdot 3)$

$(6-12,\ 8+9)$

$(-6,\ 17)$

posso fare così

penso a i come a una indeterminata

$2\cdot 3 + 2\cdot 4i + 3i\cdot 3 + 12i^2$

$6 + 17i + 12i^2$ $\quad i^2 = -1$

$6 + 17i - 12 = -6 + 17i$

$i^2 = -1$

$i^3 = -i$

$i^4 = 1$

$\vdots$

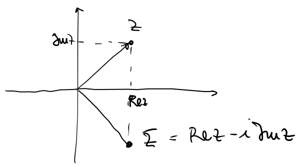
ES. calcolare $\dfrac{1}{3+2i}$ ← è un complesso, voglio scriverlo nella forma $x+iy$

$(3+2i)^{-1}$

$$(3+2i)^{-1} = \frac{3}{9+4} - i\,\frac{2}{9+4} = \underline{\frac{3}{13} - i\,\frac{2}{13}}$$

$$(a+ib)^{-1} = \frac{a}{a^2+b^2} - i\,\frac{b}{a^2+b^2}$$

def. Sia $z = a+ib$ un numero complesso
chiamo $\bar{z}$ = coniugato di z = $a - ib$



$\operatorname{Re} \bar{z} = \operatorname{Re} z$
$\operatorname{Im} \bar{z} = -\operatorname{Im} z$

$\overline{2+3i} = 2 - 3i$

$\bar{2} = 2$

$\bar{i} = -i$

proprietà

- $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

$z_1 = x_1 + iy_1 \qquad \bar{z}_1 = x_1 - iy_1 = x_1 + i(-y_1)$
$z_2 = x_2 + iy_2 \qquad \bar{z}_2 = x_2 - iy_2 = x_2 + i(-y_2)$
$z_1 + z_2 = (x_1+x_2) + i(y_1+y_2)$
$\overline{z_1+z_2} = (x_1+x_2) - i(y_1+y_2) = (x_1 - iy_1) + (x_2 - iy_2) = \bar{z}_1 + \bar{z}_2$

- $\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$

$z_1 \cdot z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$
$\overline{z_1 \cdot z_2} = (x_1x_2 - y_1y_2) - i(x_1y_2 + x_2y_1)$

$\bar{z}_1 \cdot \bar{z}_2 = (x_1x_2 - y_1y_2) + i(-x_1y_2 - x_2y_1) = \underbrace{(x_1x_2 - y_1y_2) - i(x_1y_2 + x_2y_1)}_{= \overline{z_1 z_2}}$

$z = a + ib$
$\bar{z} = a - ib$
$z - \bar{z} = 2ib = i(2b)$

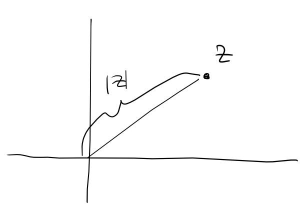
- $z + \bar{z} = 2 \operatorname{Re} z \qquad \operatorname{Re} z = \dfrac{z+\bar{z}}{2}$
$z - \bar{z} = i(2 \operatorname{Im} z) \qquad \operatorname{Im} z = -\dfrac{z-\bar{z}}{2}\, i$

$\boxed{\dfrac{1}{i} = -i}$

- $\overline{(\bar{z})} = z$

def. sia $z \in \mathbb{C}$ con $z = a+ib$
chiamo modulo di z il valore $\sqrt{a^2+b^2}$

$$|z| = \sqrt{a^2+b^2} = \sqrt{(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2}$$



modulo di z = distanza del punto (a,b) dall'origine

oss. sia $a \in \mathbb{R}$
a ha un valore assoluto, $|a|$
a ha un modulo, se penso ad a come numero complesso $|a|$

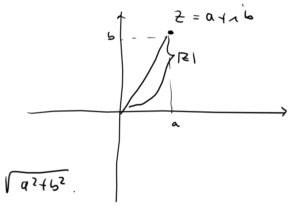
ho $\underbrace{|a|}_{\text{valore assoluto}} = \underbrace{|a|}_{\text{modulo}}$

$\sqrt{a^2+b^2}$ se $b = 0$

$\sqrt{a^2} = |a|$

$z = a + ib$

$|z| = \sqrt{a^2+b^2}$



$|\cdot| : \mathbb{C} \to [0, +\infty[$

$z = a+ib \longmapsto |z| = \sqrt{a^2+b^2}$

- $|\bar{z}| = |z|$

- $z \cdot \bar{z} = |z|^2$

$z \cdot \dfrac{\bar{z}}{|z|^2} = 1 \quad (\text{se } z \neq 0)$

quindi $\quad z^{-1} = \dfrac{1}{z} = \dfrac{\bar{z}}{|z|^2}$

$z = a+ib$

$\bar{z} = a-ib$

$z \cdot \bar{z} = (a+ib)(a-ib) = a^2 + \cancel{iba} - \cancel{iba} - i^2b^2 = a^2+b^2$

$(a,b)^{-1} = \left(\dfrac{a}{a^2+b^2}, \dfrac{-b}{a^2+b^2}\right)$

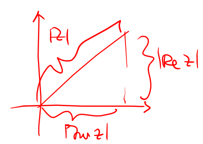
$(a,b) = a+ib$

$\left(\dfrac{a}{a^2+b^2}, \dfrac{-b}{a^2+b^2}\right) = \dfrac{a}{a^2+b^2} - \dfrac{ib}{a^2+b^2} = \dfrac{1}{a^2+b^2}(a-ib)$

- $z = \mathrm{Re}\, z + i\, \mathrm{Im}\, z$

ho $\quad |\mathrm{Re}\, z| \leq |z|, \quad |\mathrm{Im}\, z| \leq |z|$

$|z| = \sqrt{(\mathrm{Re}\, z)^2 + (\mathrm{Im}\, z)^2}$



- $|z_1 z_2| = |z_1|\,|z_2|$

$z_1 = x_1 + iy_1$

$z_2 = x_2 + iy_2$

$z_1 z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$

$|z_1 z_2| = \sqrt{(x_1x_2-y_1y_2)^2 + (x_1y_2+x_2y_1)^2} = \sqrt{x_1^2x_2^2 + y_1^2y_2^2 - \cancel{2x_1x_2y_1y_2} + x_1^2y_2^2 + x_2^2y_1^2 + \cancel{2x_1y_2x_2y_1}}$

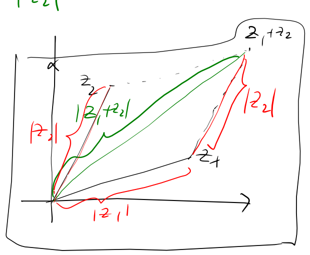
$= \sqrt{x_1^2x_2^2 + y_1^2y_2^2 + x_1^2y_2^2 + x_2^2y_1^2}$

$= \sqrt{x_1^2(x_2^2+y_2^2) + y_1^2(x_2^2+y_2^2)} = \sqrt{(x_1^2+y_1^2)(x_2^2+y_2^2)}$

$= \sqrt{x_1^2+y_1^2} \cdot \sqrt{x_2^2+y_2^2} = |z_1| \cdot |z_2|$

- <u>disuguaglianza triangolare</u>

$$\boxed{|z_1+z_2| \leq |z_1| + |z_2|}$$



$|z_1+z_2|^2 = (z_1+z_2)\overline{(z_1+z_2)}$

$= (z_1+z_2)(\bar{z}_1+\bar{z}_2)$

$= \underbrace{z_1\bar{z}_1}_{|z_1|^2} + z_2\bar{z}_1 + z_1\bar{z}_2 + \underbrace{z_2\bar{z}_2}_{|z_2|^2}$

$= |z_1|^2 + z_2\bar{z}_1 + z_1\bar{z}_2 + |z_2|^2$

$= |z_1|^2 + z_2\bar{z}_1 + \overline{z_2\bar{z}_1} + |z_2|^2$

$= |z_1|^2 + 2\mathrm{Re}(z_2\bar{z}_1) + |z_2|^2$

$\leq |z_1|^2 + 2|z_2\bar{z}_1| + |z_2|^2$

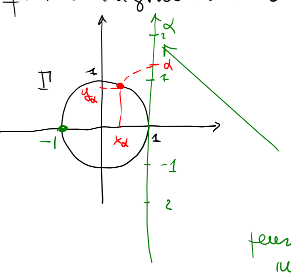
$\leq |z_1|^2 + 2|z_1||z_2| + |z_2|^2 = (|z_1|+|z_2|)^2$

$\overline{z_1\bar{z}_2} = \bar{z}_1\bar{\bar{z}}_2 = \bar{z}_1 z_2$

$\mathrm{Re}\, a \leq |\mathrm{Re}\, a| \leq |a|$

conclusione $\quad |z_1+z_2|^2 \leq (|z_1|+|z_2|)^2 \quad$ <u>da cui la tesi</u>

funzioni trigonometriche



Γ = circonferenza unitaria

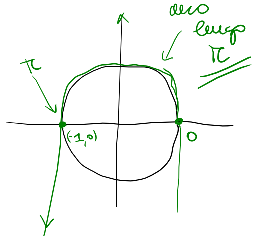
$= \{(x,y) : x^2+y^2=1\}$

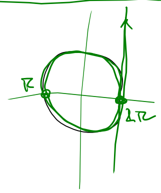
asse reale in cui ci sono tutti i numeri di $\mathbb{R}$, parallelo asse y stesso orientamento

fermo all'asse come a un filo e lo avvolgo su Γ

arco lungo π

si conservano le lunghezze





costruisco una funz.

$$\mathbb{R} \longrightarrow \Gamma$$

$$\alpha \longmapsto (x_\alpha, y_\alpha)$$

$x_\alpha = \cos\alpha$, $y_\alpha = \operatorname{sen}\alpha$

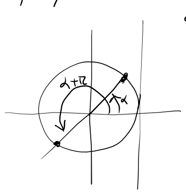
$\cos : \mathbb{R} \longrightarrow [-1,1]$

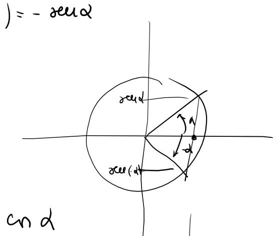
$\operatorname{sen} : \mathbb{R} \longrightarrow [-1,1]$

1) $\forall \alpha \in \mathbb{R}$, $(\cos\alpha)^2 + (\operatorname{sen}\alpha)^2 = 1$

2) $\forall \alpha \in \mathbb{R}, \forall k \in \mathbb{Z}$, $\cos(\alpha + 2k\pi) = \cos\alpha$
$\operatorname{sen}(\alpha+2k\pi) = \operatorname{sen}\alpha$

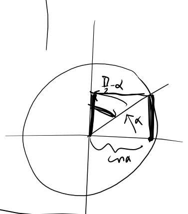
3) $\forall \alpha \in \mathbb{R}$, $\cos(\alpha+\pi) = -\cos\alpha$
$\operatorname{sen}(\alpha+\pi) = -\operatorname{sen}\alpha$





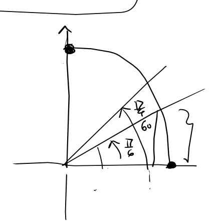
4) $\forall \alpha$, $\cos(-\alpha) = \cos\alpha$
$\operatorname{sen}(-\alpha) = -\operatorname{sen}\alpha$

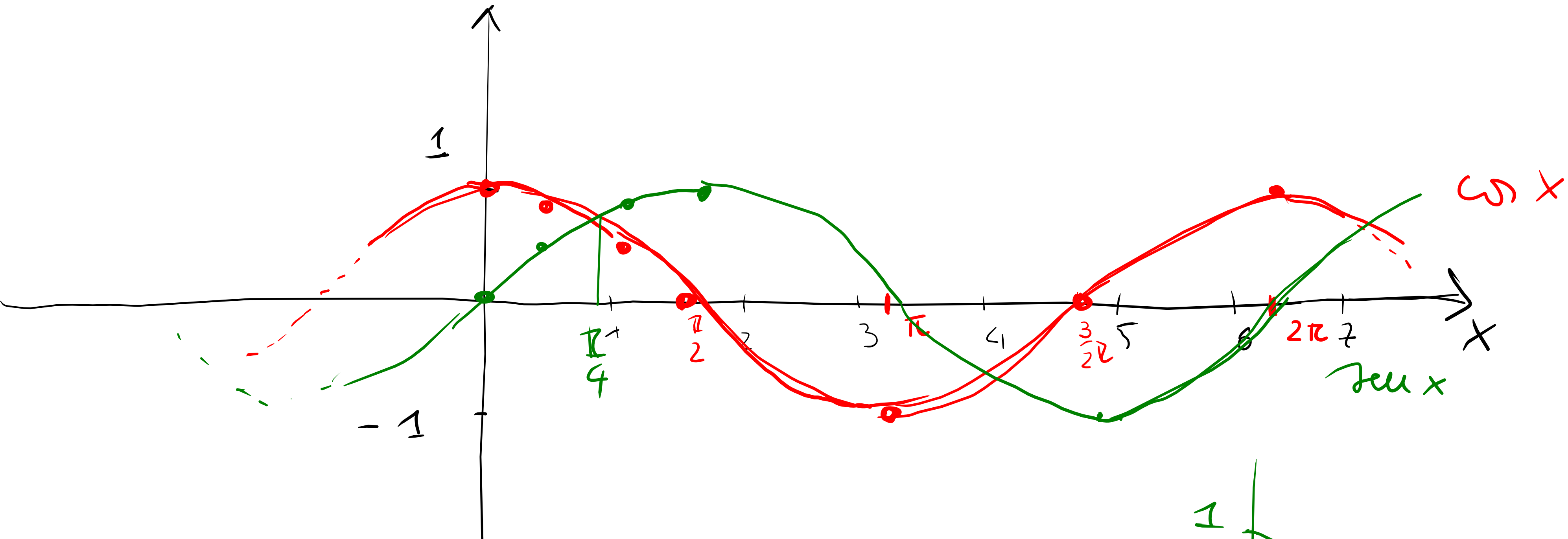
5) $\forall \alpha$ $\cos\left(\frac{\pi}{2}-\alpha\right) = \operatorname{sen}\alpha$
$\operatorname{sen}\left(\frac{\pi}{2}-\alpha\right) = \cos\alpha$



$$\boxed{\begin{array}{ll} 6)\ \forall \alpha, \beta & \cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \operatorname{sen}\alpha\operatorname{sen}\beta \\ & \operatorname{sen}(\alpha\pm\beta) = \operatorname{sen}\alpha\cos\beta \pm \operatorname{sen}\beta\cos\alpha \end{array}}$$

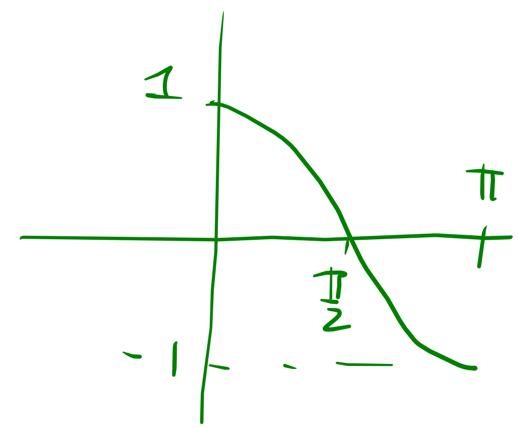
α	$\cos\alpha$	$\operatorname{sen}\alpha$
0	1	0
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{2}$	0	1





$$\cos\big|_{[0,\pi]} : [0,\pi] \longrightarrow [-1,1]$$

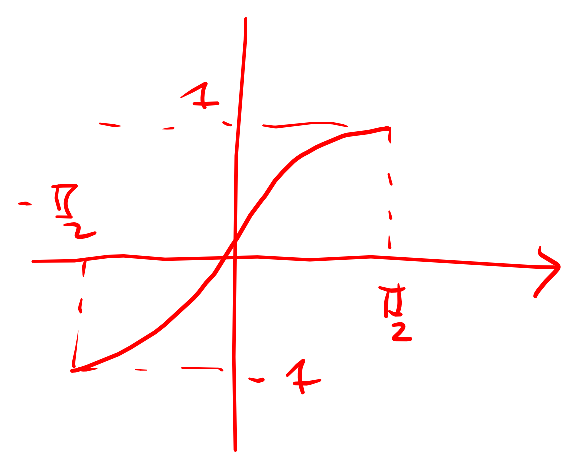
biiettiva



$$\left(\cos\big|_{[0,\pi]}\right)^{-1} = \arccos$$

$$\operatorname{sen}\big|_{[-\frac{\pi}{2},\frac{\pi}{2}]} : \left[-\frac{\pi}{2},\frac{\pi}{2}\right] \longrightarrow [-1,1]$$

è biiettiva



$$\left(\operatorname{sen}\big|_{[-\frac{\pi}{2},\frac{\pi}{2}]}\right)^{-1} = \operatorname{arcsen}$$