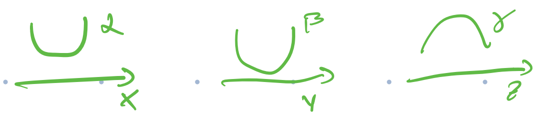
Trapping ions in a Paul trap

In an ion trap charged particles can be confined. At the center of an ideal Paul trap, the net force on an ion vanishes and the electric potential, consisting of a static and an oscillating part at frequency Ω_{RF} is quadratic in the Cartesian coordinates.

An electric potential $\phi(x,y,z)$ fulfills Laplace's equation at every instant in time

$$\nabla^2 \phi(x,y,z) = 0$$

So it is not possible to create confining potentials along all three spacial dimensions with only static electric fields $\phi = \alpha x^2 + \beta y^2 + \gamma z^2 \Rightarrow \nabla^2 \phi = 0 \rightarrow 2\alpha + 2\beta + 2\gamma = 0$



AT LEAST ONE DIRECTION SHOULD BE ANTI CONFINED.

which excludes the generation of a momentary local minimum. Thus the potential can only trap in a dynamical way.

Given these constraints different electrode geometries can realize a trap. In a linear Paul trap, confinement along one direction (also called the axial direction, here defined as the z axis) is purely static $\Rightarrow$ CURVATURE OF ϕ ALONG $\hat{z}$ IS POSITIVE

Constraining the potential to only quadratic terms the overall potential close to the center may be written as

$$\phi(x,y,z) = \underbrace{\frac{1}{2} U_{DC}\left(-\alpha_{DC} x^2 - \beta_{DC} y^2 + \gamma_{DC} z^2\right)}_{\text{STATIC}} + \underbrace{\frac{1}{2} U_{RF} \cos(\Omega_{RF} t)\left(\alpha_{RF} x^2 - \beta_{RF} y^2\right)}_{\text{RADIO FREQUENCY}}$$

$\alpha, \beta, \gamma > 0$

Here the coordinate system is already chosen such that x and y align with the principal axes of the pseudopotential (see later). As all three axes of the potential are independent, the equations of motion may be derived separately for each axis. In one dimension for a single ion the equation of mo tion are

$$\ddot{u} = \frac{F(t)}{m} = -\frac{e}{m}\frac{d\phi}{du}$$

CHARGE

MASS

IN HERE WE ARE ASSUMING AN ION OF CHARGE $+1e$

AXIAL DIRECTION $\longrightarrow$ SIMPLE HARMONIC OSCILLATOR WITH FREQUENCY

$$\omega_z = \sqrt{\frac{e\,\gamma_{DC}\,U_{DC}}{m}}$$

RADIAL DIRECTION $\quad \ddot{u}_\pm = \pm\frac{e}{m}\left[\delta_{DC}\,U_{DC} + \delta_{RF}\,U_{RF}\cos(\Omega_{RF}t)\right]u_\pm$

$u_+ = x,\ u_- = y$

α_{DC} FOR u_+

β_{DC} FOR u_-

α_{RF} OR β_{RD}

This last differential equation is of the form of Mathieu's differential equation

$$\frac{d^2x}{d\tau} = \left[e_x - q_x\cos(2\tau)\right]x$$

WITH $\quad e_u = -\frac{4e\,\delta_{DC}\,U_{DC}}{m\,\Omega_{RF}^2} \qquad$ AND $\quad q_u = -\frac{2e\,\delta_{RF}\,U_{RF}}{m\,\Omega_{RF}^2}$

PURELY STATIC

DEPENDS ON RF PART

Linear Paul traps are commonly operated in a regime where $\quad |e_u|,\ q_u^2 \ll 1$

The solution of the Mathieu's equations is an exponential series where we only take the first order

$$x(\tau) = B\,e^{i\beta t}\sum_n C_{2n}\,e^{i2n\tau} + C_{-2n}\,e^{-i2n\tau}$$

NOT THE β WE HAD BEFORE

$$u_\pm \simeq A\cos(\omega_\pm t)\left[1 - \frac{q_u}{2}\cos(\Omega_{RF}t)\right]$$

$2BC_0$

$$\omega_\pm = \sqrt{e_u + \frac{q_u^2}{2}}\;\frac{\Omega_{RF}}{2}$$

SECULAR MOTIONAL FREQUENCY

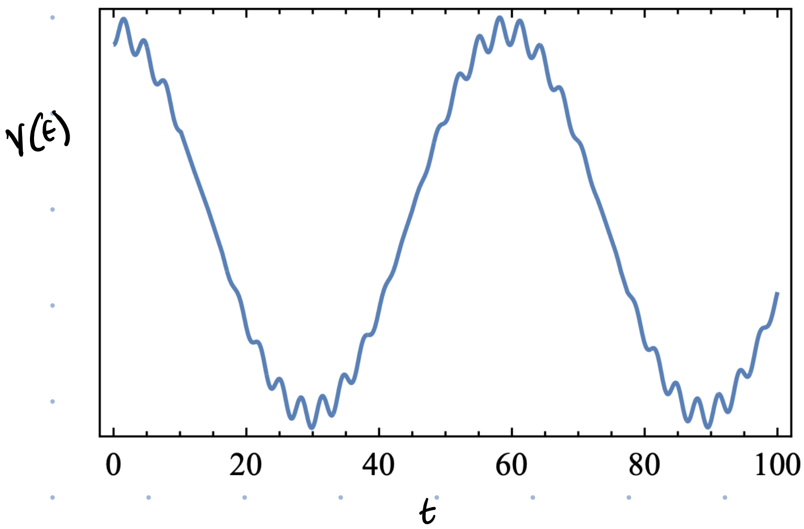
$\omega_\pm \ll \Omega_{RF}$ LOW FREQUENCY MOTION

CALLED $\boxed{\text{SECULAR MOTION}}$

For $a_U \to 0$ (neglect the DC contribution)

$$\boxed{\omega_U = \frac{\sqrt{2}\, e\, S_{RF}\, U_{RF}}{m\, \Omega_{RF}}}$$

$S_{RF} = \begin{cases} \alpha_{RF} & \text{for } \hat{x} \\ \beta_{RF} & \text{for } \hat{y} \end{cases}$ depends on electrode structure

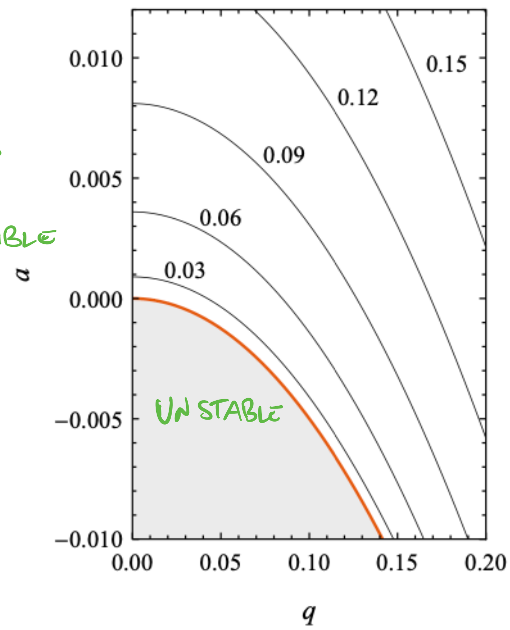


Slow oscillation → secular motion

\+ fast oscillation → micromotion

It is not always possible to find a stable solution to the Mathieu equations. The stability of the solution depends on the parameters a, q

When making an ion trap we always need to check if it will give stable confinement



The pseudopotential approximation for rf ion traps is useful for building an intuition for the rf confinement and as a tool in the simulation of ion traps.

The equation of motion of a particle with charge e and mass m in an oscillating electric field is

$$m\ddot{x} = eE_0 \cos \Omega_{RF} t = F(t)$$

SOLUTION : $x(t) = \bar{x} + \varepsilon(t)$ ← MICROMOTION

↑ TIME AVERAGED REST POSITION

$$\varepsilon(t) = -\varepsilon_0 \cos(\Omega_{RF} t) \quad \text{WITH} \quad \varepsilon_0 = \frac{e E_0(\bar{x})}{m \Omega_{RF}^2}$$

AMPLITUDES OF THE OSCILLATING FIELD AT REST POSITION

TIME AVERAGE OF THE FORCE IN A PERIOD $1/\Omega_{RF}$

$$\langle F(t) \rangle = \langle e E_0 \cos(\Omega_{RF} t) \rangle = 0$$

THE PARTICLE OSCILLATES AROUND A FIXED AVERAGE POSITION $\bar{x}$

ASSUMING A NON VANISHING GRADIENT $\frac{\partial E_0}{\partial x} \neq 0$

$$E_0(x) \simeq E_0(\bar{x}) + \frac{\partial E_0(\bar{x})}{\partial \bar{x}} \varepsilon(t)$$

$$\Rightarrow \langle F(t) \rangle = e \langle \underbrace{E_0(\bar{x}) \cos \Omega_{RF} t}_{\text{AVERAGES TO } 0} + \frac{\partial E_0(\bar{x})}{\partial \bar{x}} \varepsilon(t) \cos(\Omega_{RF} t) \rangle$$

$$= e \frac{\partial E_0(\bar{x})}{\partial \bar{x}} \langle \varepsilon(t) \cos(\Omega_{RF} t) \rangle = e \frac{\partial E_0(\bar{x})}{\partial \bar{x}} \left(-\frac{\varepsilon_0}{2}\right)$$

$$\Rightarrow \langle F(t) \rangle = \bar{F}(\bar{x}) = -e \frac{\partial E_0(\bar{x})}{\partial \bar{x}} \frac{e E_0(\bar{x})}{m \Omega_{RF}^2}$$

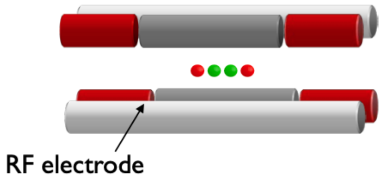
DEFINING $\varphi(\bar{x}) = \frac{e E_0^2(\bar{x})}{4 m \Omega_{RF}^2} \rightarrow \bar{F}(\bar{x}) = -e \frac{\partial \varphi(\bar{x})}{\partial \bar{x}}$

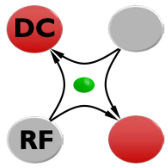
PSEUDO POTENTIAL

$\Rightarrow$ THE PSEUDO POTENTIAL PROVIDES A NET FORCE THAT HOLDS THE ION IN PLACE

$$V_{TOTAL} = U_{STATIC} + \varphi$$

TYPICAL ION TRAP





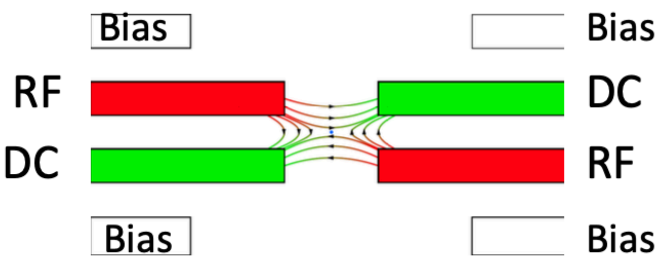
$$V_{RF} = \frac{\alpha V_0}{2R^2}(x^2-y^2)\cos\Omega_{RF} t \longrightarrow \varphi = \left(\frac{\alpha V_0^2}{R^2}\right)^2 \frac{1}{4m\Omega_{RF}^2}(x^2+y^2)$$

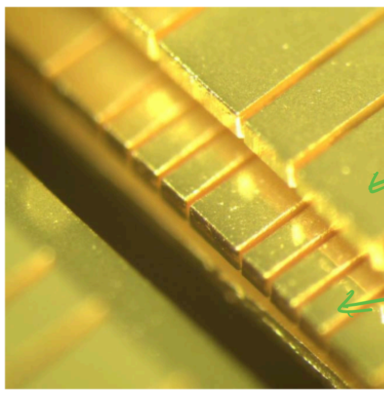
ION-ELECTRODE DISTANCE

$$U_{STATIC} = \frac{\beta U_0}{2Z^2}\left(z^2-(x^2+y^2)/2\right)$$

DISTANCE ION ELECTRODS

Ideally we do not want to have micro motion. To avoid it, the ion need to be in a position where the amplitude of the RF electric field is null (see equation of the psuedopotential)



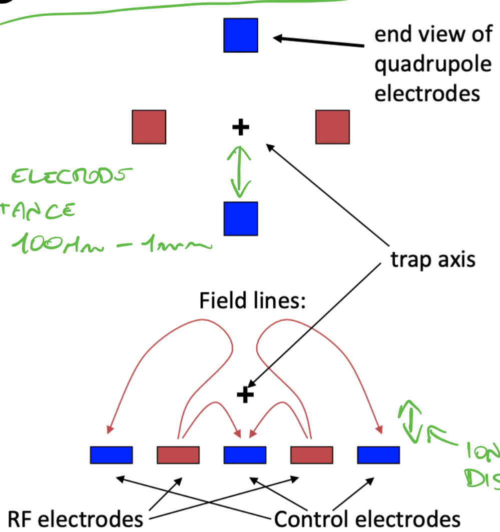


BIAS

DC ELECTRODES

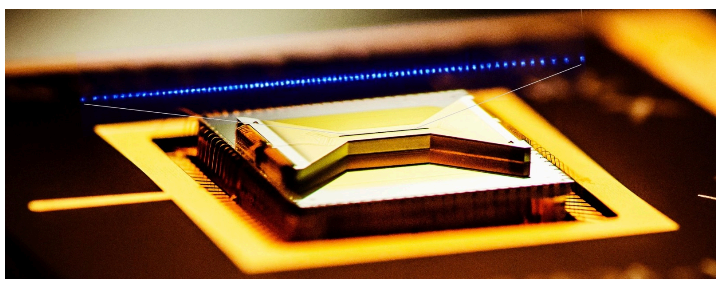
BIAS ELECTRODES ARE USED TO PUSH THE ION IN THE RF NULL → COMPENSATE MICROMOTION

TRAPPING IONS ON A CHIP



ION ELECTRODS DISTANCE 100 µm – 1 mm

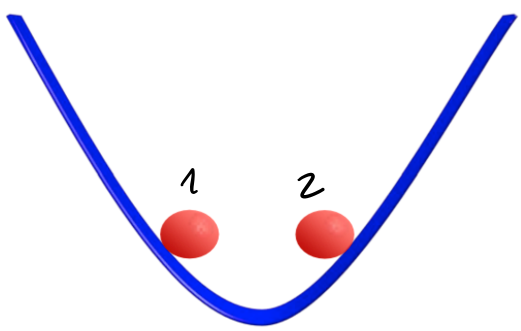
ION-ELECTRODS DISTANCE 10–100 µm



CHALLENGE: SHALLOW TRAP DEPTH, CHARGING OF ELECTRODES (DUE TO LASER TOUCHING THE SURFACES)

Multiple ions in a potential well

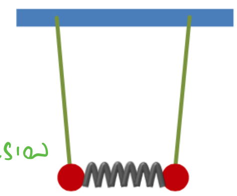
Gate operations between two ions, but also simulations with long ion strings, require two or more ions trapped in the same potential well. Ions will repel each other due to the Coulomb force



COULOMB REPULSION

$$V = \frac{K}{2} z_1^2 + \frac{K}{2} z_2^2 + \frac{\alpha}{|z_1 - z_2|}$$

THIS IS THE SAME SITUATION OF TWO PENDULUMS COUPLED BY A SPRING ← COULOMB REPULSION



⇒ THE MOTIONAL MODES ARE DESCRIBED BY THE NORMAL MODES OF MOTION

↳ 2 INDEPENDENT OSCILLATORS

IF N IONS → N INDEPENDENT MODES

$\hbar\omega_c$  COMMON MODE OSCILLATING

$\hbar\omega_s$  OUT-OF-PHASE MODE OR STRETCH MODE

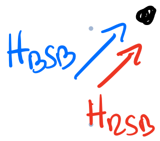
$\omega_c < \omega_s$

This was just for the axial modes. There will also be 2 other modes for each of the radial modes. So in general, for N ions in the same potential well there will be a total of 3N motional modes (N modes for each axis). Each with its own frequency

Because the motion is shared, then we can think of using it as a quantum bus to transfer information from one ion to the other

State dependent forces

One of the most commonly used entangling gates in trapped ions is the Molmer-Sorensen gate (MS gate). To fully understand it we first need to understand the concept of state-dependent force. So let's consider only one ion and apply simultaneously the red and the blue sideband



$$H^{\pm}_{TOT} = H^{\pm}_{RSB} + H^{\pm}_{BSB}$$

$\frac{\hbar\Omega_0}{2}(i\eta)\, e\, \sigma_+ + h.c$

$\frac{\hbar\Omega_0}{2}(i\eta)\, e^{+} \sigma_+ + h.c$

N.B. ASSUMES LD APPROXIMATION

$$\Rightarrow H^{\pm}_{TOT} = \frac{\hbar\Omega_0}{2}\, i\eta\, (e^{+}+e)\,\sigma_+ + \frac{\hbar\Omega_0}{2}(-i\eta)(e^{+}+e)\,\sigma_-$$

$$= \frac{\hbar\Omega_0}{2}\,\eta\, \underbrace{(e^{+}+e)}_{\hat{X}\ \text{POSITION}}\ \underbrace{i(\sigma_+ - \sigma_-)}_{\sigma_Y}$$

$$\Rightarrow H_{TOT} = \frac{\hbar\Omega_0}{2}\,\eta\,\sigma_Y\,\hat{X} = \underset{\text{FORCE}}{\hat{F}}\,\hat{X}$$

THIS FORCE IS STATE DEPENDENT BECAUSE OF $\sigma_Y = |+Y\rangle\langle +Y| - |-Y\rangle\langle -Y|$

$F|+Y\rangle = \text{const}|Y\rangle$

$F|-Y\rangle = -\text{const}|Y\rangle$

THE FORCE PUSHES IN TWO OPPOSITE DIRECTIONS!

Changing the relative phase between the red and blue sideband I can change σ_Y to any combination of σ_X AND σ_Y

Imagine to be in $|+Y\rangle \otimes |n=0\rangle$ $\Rightarrow$ THE TIME EVOLUTION IS

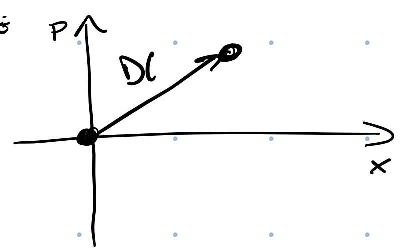
$$U = e^{-\frac{i}{\hbar}Ht} = e^{i(\alpha e^{+} + \alpha^{*} e)}$$

BASICALLY THE EXPONENTIATION OF $\hat{X} = (e + e^{+})$

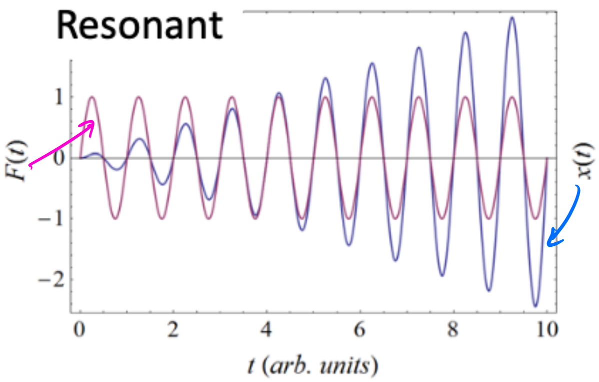
THIS IS THE DISPLACEMENT OPERATOR!!

$$|+Y,0\rangle \longrightarrow |+Y,\alpha\rangle$$

PHASE SPACE

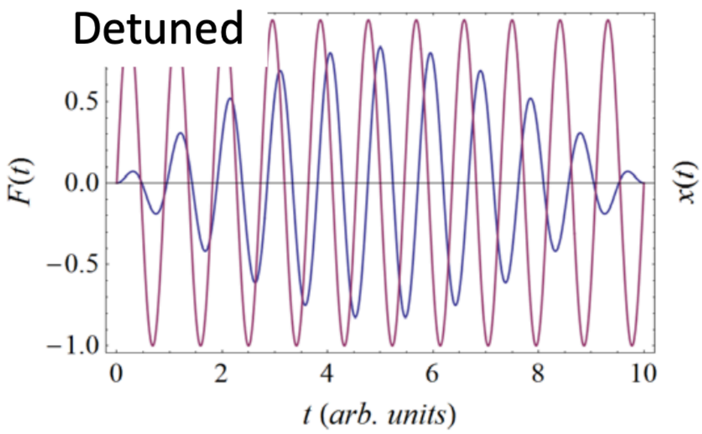


If you think about it it makes sense. What we have done so far is equivalent of a forced harmonic oscillator that is driven on resonance with the oscillator frequency.



⇒ LARGER AND LARGER COHERENT STATE!

What happens if we have a detuned drive? Let's look at the classical case:



IF DETUNING δ, $x(t)=0$

AFTER $t = \dfrac{2\pi}{\delta}$

FOR THE QUANTUM HARMONIC OSCILLATOR:

$\omega_{L_1} = \omega_A + \omega_z + \delta$ ← DETUNED BSB

$\omega_{L_2} = \omega_A - \omega_z - \delta$ ← DETUNED RSB

IN THIS CASE $\delta > 0$. BUT IT IS NOT MANDATORY

$$\Rightarrow \quad H^{I}_{BSB} = \frac{\hbar \Omega_0}{2} (i\eta)\, a^{\dagger} \sigma_{+} e^{-i\delta t} + h.c.$$

$$H^{I}_{RSB} = \frac{\hbar \Omega_0}{2} (i\eta)\, a\, \sigma_{+} e^{i\delta t} + h.c.$$

$$\Rightarrow \quad H^{I}_{TOT}(t) = \frac{\hbar \Omega_0}{2} \eta \left(a^{\dagger} e^{-i\delta t} + a\, e^{i\delta t} \right) \sigma_y$$

PROBLEM: $[H(t), H(t')] \neq 0$ ⇒ TIME EVOLUTION WITH MAGNUS EXPANSION (SEE CLASS 1)

$$[H(t), H(t')] = \left(\frac{\hbar \eta \Omega_0}{2}\right)^2 \left[(a^+ e^{-i\delta t} + a e^{i\delta t}), (a^+ e^{-i\delta t'} + a e^{i\delta t'}) \right]$$

$$= \left(\frac{\hbar \eta \Omega_0}{2}\right)^2 \left(\underbrace{[a^+, a]}_{=1} e^{-i\delta t} e^{i\delta t'} + \underbrace{[a, a^+]}_{=-1} e^{+i\delta t} e^{-i\delta t'} \right)$$

In the Magnus expansion we can only focus on the first term because $[H(t_1), \underbrace{[H(t_2), H(t_3)]}_{\text{CONSTANT}}] = 0$

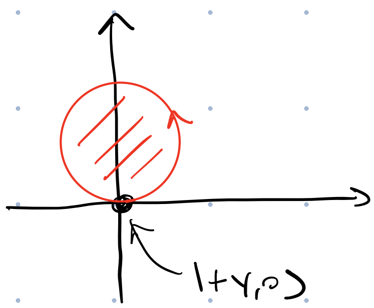
$$\Rightarrow U = \exp\left(\frac{i}{\hbar} \int^t H \, dt - \frac{1}{2\hbar^2} \int^t \int^{t_1} [H(t_1), H(t_2)] \, dt_1 dt_2 \right)$$

FIRST TERM $\quad U = \exp\left(-\int \eta \frac{\Omega_0}{2} (a^+ e^{-i\delta t} + a e^{i\delta t}) \sigma_Y \right)$

THIS IS STILL A DISPLACEMENT OPERATOR BUT IT ROTATES AT δ

SECOND TERM : $\quad \int\int [\ldots] \propto \sigma_Y^2 \Rightarrow \mathbb{1}$ NO STATE DEPENDENCE

$\rightarrow$ JUST A PHASE ϕ



ONE COULD ALSO SHOW THAT THE PHASE ACQUIRED BY THE SECOND TERM IS PROPORTIONAL TO THE AREA SPANNED BY THE LOOP IN PHASE SPACE

$$A \propto \phi$$

FOR THIS REASON ϕ IS CALLED GEOMETRIC PHASE

$$\Rightarrow |\gamma, 0\rangle \xrightarrow[\text{AFTER ONE LOOP}]{} e^{i\phi} |\gamma, 0\rangle$$

This was if we started in an eigenstate of the σ_Y operator. What happens if you start in $|g\rangle$?

Imagine, for simplicity that $H^I_{TOT} \propto \sigma_X$

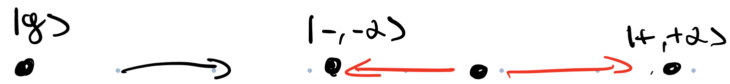
$$|+\rangle = \frac{|g\rangle + |e\rangle}{\sqrt{2}}, \quad |-\rangle = \frac{|g\rangle - |e\rangle}{\sqrt{2}}$$

$$\Rightarrow |g\rangle = \frac{|+\rangle + |-\rangle}{\sqrt{2}}$$

IN THE RESONANT CASE $\delta = 0$ $\Rightarrow$ $|+,0\rangle \longrightarrow |+,\alpha\rangle$

$|-,0\rangle \longrightarrow |-,-\alpha\rangle$

$\Rightarrow$

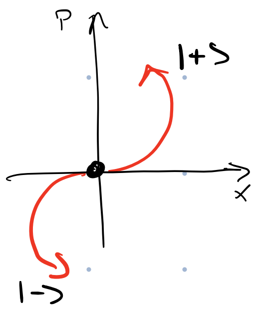


SPLIT THE WAVE PACKET IN TWO

$$|g,0\rangle \longrightarrow \frac{1}{\sqrt{2}}\left(|+,\alpha\rangle + |-,-\alpha\rangle\right) = \left(|g\rangle + |e\rangle\right)|\alpha\rangle + \left(|g\rangle - |e\rangle\right)|-\alpha\rangle$$

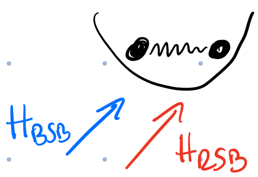
$$= \underbrace{|g\rangle\left(|\alpha\rangle + |-\alpha\rangle\right)}_{\text{MOTIONAL SCHRÖDINGER CAT STATE}} + |e\rangle\left(|\alpha\rangle - |-\alpha\rangle\right)$$

DETUNED CASE



Mølmer–Sørensen gate

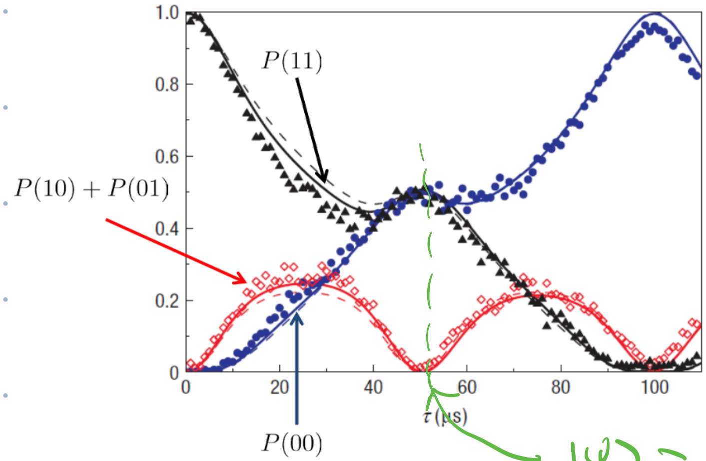
The idea of the Mølmer–Sørensen gate is to use the exactly what we just saw but using two ions. In this case the Hamiltonian becomes



BOTH IONS ARE ADDRESSED BY BOTH HAMILTONIAN

$$\Rightarrow H^{I}_{TOT} = \frac{\hbar \eta \Omega_0}{2} \left(a^{+} e^{-i\delta t} + a e^{i\delta t}\right)\left(\sigma_{x_1} + \sigma_{x_2}\right)$$

$$\Rightarrow \left[H(t), H(t_1)\right] \propto \left(\sigma_{x_1} + \sigma_{x_2}\right)^2 = \underbrace{\sigma_{x_1}^2}_{\mathbb{1}} + \underbrace{\sigma_{x_2}^2}_{\mathbb{1}} + \underbrace{\sigma_{x_1}\sigma_{x_2} + \sigma_{x_2}\sigma_{x_1}}_{\text{COLLECTIVE DRIVING OF THE QUBITS}}$$



IF ONE DOES THE MAGNUS EXPANSION GETS

$$U_{MS}(t) = \hat{D}\left(\alpha(t)\,(\sigma_{x_1}+\sigma_{x_2})\right) e^{-i\phi(t)\,(\sigma_{x_1}+\sigma_{x_2})^2}$$

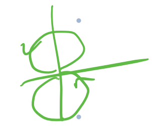
$$\alpha(t) = \frac{\Omega_0}{2\delta}\left(e^{-i\delta t}-1\right)$$

$$\phi(t) = \frac{\Omega_0^2}{4\delta}\left(t - \frac{\sin \delta t}{\delta}\right)$$

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + i|11\rangle\right)$$

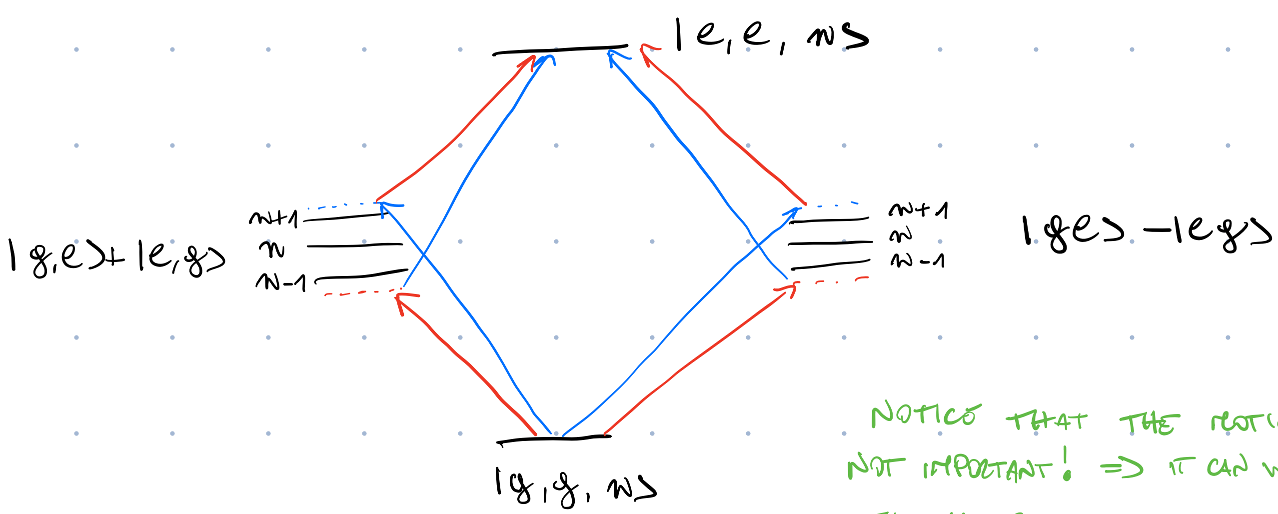
FOR $t_G = \frac{2\pi}{\delta}$ $\quad U(t_G) = \exp\left[-i\,\frac{\Omega_0^2\pi}{2\delta}\,(\sigma_{x_1}+\sigma_{x_2})^2\right]$ $\leftarrow \hat{D}=0$

GO BACK TO THE ORIGIN



FOR $\Omega = \delta/2 \;\longrightarrow\; \boxed{U_{MS} = \exp\left[-i\frac{\pi}{8}(\sigma_{x_1}+\sigma_{x_2})^2\right]}$

An alternative way of seeing the action of an MS gate for ion starting in $|g,g,n\rangle$ is to look at it the energy scale of the the coupled ion systems. Having the red and blue sidebands together creates an interference effects that drive the system collectively from $|g,g,n\rangle$ to $|e e n\rangle$



NOTICE THAT THE MOTIONAL STATE IS NOT IMPORTANT! ⇒ IT CAN WORK ALSO FOR THERMAL STATES.

As an exercise you can try to show that the MS gate unitary operation U_{MS} can be mapped to the CNOT with extra single qubit operations acting on the two ions independently.

This means that the MS gate and single-qubit rotations create the universal set of operations to realize any unitary operation! With this we have now fulfilled the last DiVincenzo's criteria