14 ottobre

Stiamo dimostrando che per $b>1$ la funzione $x \to b^x$ è continua in $\mathbb{R}$.

Nella dimostrazione diamo per scontati i seguenti 4 fatti

1) b^x è crescente: $x_1 < x_2 \Rightarrow b^{x_1} < b^{x_2}$

2) $(b^{x_1})^{x_2} = b^{x_1 x_2}$

3) $b^{x_1+x_2} = b^{x_1} \, b^{x_2}$

4) $b^0 = 1$

Dim della continuità.

Passo I) $\lim_{x\to 0^+} b^x = 1$. L'abbiamo dimostrato osservando che $\lim_{x\to 0^+} b^x = \inf\{b^x : x>0\}$

$\left(\lim_{x\to 0^-} b^x = \sup\{b^x : x<0\} \right)$

Siccome $\forall\, x>0$ si ha $b^0 = 1 < b^x$

seguiva $1 \le \inf\{b^x : x>0\} \le \inf\{b^{\frac{1}{n}} : n\in\mathbb{N}\}$
$= \lim_{n\to+\infty} b^{\frac{1}{n}} = 1$

$n \to b^{\frac{1}{n}}$ è decrescente

$\Rightarrow \lim_{n\to+\infty} b^{\frac{1}{n}} = \inf\{b^{\frac{1}{n}} : n\in\mathbb{N}\}$

$\lim_{x\to 0^+} b^x = \inf\{b^x : x>0\} = 1$

Passo II) $\lim_{x\to 0^-} b^x =$ $\quad y=-x \quad x=-y$

$= \lim_{y\to 0^+} b^{-y} = \lim_{y\to 0^+} \frac{1}{b^y} = \frac{1}{1} = 1$

<u>Osservazione</u> $\lim_{x\to 0^+} b^x = 1$ e $\lim_{x\to 0^-} b^x = 1$

$\Rightarrow \lim_{x\to 0} b^x = 1$, cioè b^x è continua in 0.

Punto III) $\lim_{x \to x_0} b^x = b^{x_0} \quad \forall x_0 \in \mathbb{R}$

$$\lim_{x \to x_0} b^x = \lim_{h \to 0} b^{h+x_0} = \lim_{h \to 0} b^h \, b^{x_0} =$$

$$x = h + x_0 \qquad = b^{x_0}.$$

Teor $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$

Dimostreremo che $\boxed{\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x = e}$ e

$\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x = e$

Lemma $\lim_{x\to+\infty}\left(1+\frac{1}{[x]}\right)^{[x]} = e$

Dim Noi abbiamo già visto il limite

$\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n = e$.

$$\lim_{x\to+\infty}\left(1+\frac{1}{[x]}\right)^{[x]} = \lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n = e$$

$n=[x]$ $\qquad \lim_{x\to+\infty}[x] = +\infty$

$\lim_{x\to+\infty}[x] = +\infty$ vale perché $[x] \le x < [x]+1$

$x-1 < [x]$

$x-1 \xrightarrow{x\to+\infty} +\infty$, $\quad [x] \xrightarrow{x\to+\infty} +\infty$

Dimostriamo ora

$$\lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^x = e$$

$$[x] \leq x < [x] + 1$$

$$\Updownarrow$$

$$\frac{1}{[x]} \geq \frac{1}{x} > \frac{1}{[x]+1}$$

$$1 + \frac{1}{[x]} \geq 1 + \frac{1}{x} > 1 + \frac{1}{[x]+1}$$

$$1 + \frac{1}{[x]+1} < 1 + \frac{1}{x} \leq 1 + \frac{1}{[x]}$$

$$\left(1 + \frac{1}{[x]+1}\right)^{[x]} < \left(1 + \frac{1}{x}\right)^x \leq \left(1 + \frac{1}{[x]}\right)^{[x]+1}$$

$$\frac{\left(1 + \frac{1}{[x]+1}\right)^{[x]+1} \to e}{\underbrace{1 + \frac{1}{[x]+1}}_{\downarrow \atop 1}} < \underbrace{\left(1 + \frac{1}{x}\right)^x}_{\downarrow\, x \to +\infty \atop e} \leq \underbrace{\left(1 + \frac{1}{[x]}\right)^{[x]}}_{\downarrow\, x \to +\infty \atop e} \underbrace{\left(1 + \overset{\to 0}{\frac{1}{[x]}}\right)}_{\downarrow\, x \to +\infty \atop 1}$$

$$\lim_{x \to +\infty} \left(1 + \frac{1}{[x]+1}\right)^{[x]+1} \underset{n = [x]}{=} \lim_{n \to +\infty} \left(1 + \frac{1}{n+1}\right)^{n+1}$$

$$= \lim_{n \to +\infty} \left(1 + \frac{1}{n}\right)^n = e$$

Vogliamo dimostrare $\lim_{x \to -\infty} \left(1+\frac{1}{x}\right)^x = e$

$y = -x$

$$\left(1+\frac{1}{x}\right)^x = \left(1-\frac{1}{y}\right)^{-y} = \left(\frac{y-1}{y}\right)^{-y} = \left(\frac{y}{y-1}\right)^y =$$

$$= \left(\frac{\cancel{y-1}+1}{y-1}\right)^y = \left(1+\frac{1}{y-1}\right)^y =$$

$$= \left(1+\frac{1}{y-1}\right)^{y-1}\left(1+\frac{1}{y-1}\right)$$

$$\lim_{x \to -\infty}\left(1+\frac{1}{x}\right)^x = \lim_{y \to +\infty}\left(1+\frac{1}{y-1}\right)^{y-1}\left(1+\frac{1}{y-1}\right)$$

$$= \lim_{y \to +\infty}\left(1+\frac{1}{y-1}\right)^{y-1} \qquad z = y-1$$

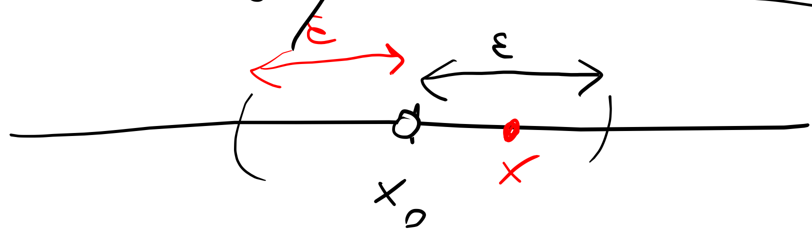
$$= \lim_{z \to +\infty}\left(1+\frac{1}{z}\right)^z = e$$

<u>Esercizio</u> Se $X \subseteq Y \subseteq \mathbb{R}$ allora $X' \subseteq Y'$.

$x_0 \in X'$ significa che

$$\boxed{\forall \varepsilon > 0 \quad \exists\, x \in X \ \text{ t.c. } \ 0 < |x - x_0| < \varepsilon}$$

$$\Downarrow$$
$$x \in Y$$



Punctured disk

siccome $X \subseteq Y$ segue che ogni $x \in X$ è anche un punto di Y

$x \in X \Rightarrow x \in Y$.

Abbiamo appena dimostrato che $x_0 \in X' \Rightarrow x_0 \in Y'$

$\Rightarrow X' \subseteq Y'$.

Funzioni iperboliche

$$\sinh(x) = sh(x) = \frac{e^{x}-e^{-x}}{2}$$

$$\cosh(x) = ch(x) = \frac{e^{x}+e^{-x}}{2}$$

$$\tanh(x) = th(x) = \frac{sh(x)}{ch(x)} = \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$$

$$sh(-x) = -sh(x)$$

$$sh(-x) = \frac{e^{-x}-e^{-(-x)}}{2} = \frac{e^{-x}-e^{x}}{2} = -\frac{e^{x}-e^{-x}}{2}$$
$$= -sh(x)$$

Analogamente
$$ch(-x) = ch(x)$$

$$sh(0) = \frac{e^{0}-e^{-0}}{2} = \frac{1-1}{2} = 0$$

$$ch(0) = \frac{e^{0}+e^{-0}}{2} = \frac{1+1}{2} = 1$$

$$ch(x) > sh(x)$$
$$\Updownarrow$$
$$\frac{\cancel{e^{x}}+e^{-x}}{2} > \frac{\cancel{e^{x}}-e^{-x}}{2} \iff \frac{e^{-x}}{\cancel{2}} > -\frac{e^{-x}}{\cancel{2}} \quad \forall x$$

quest'ultima segue dal fatto che

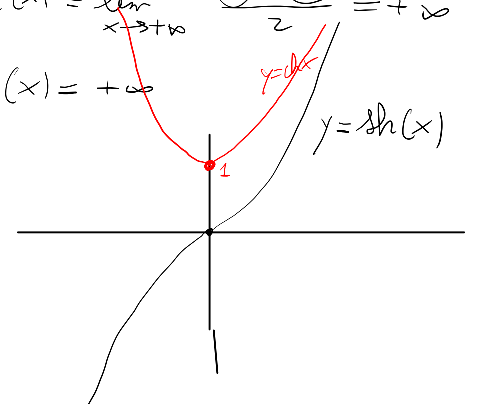
$$e^{-x} > 0 > -e^{-x} \quad \forall x \in \mathbb{R}.$$

$$\lim_{x\to+\infty} e^{x} = +\infty$$

$$\lim_{x\to+\infty} e^{x} = \sup\{e^{x} : x\in\mathbb{R}\} \quad \cancel{+\infty}$$
$$\geq \underbrace{\sup\{e^{n} : n\in\mathbb{N}\}}_{=+\infty} = \lim_{n\to+\infty} e^{n} = +\infty$$

$$\lim_{x\to+\infty} sh(x) = \lim_{x\to+\infty} \frac{\overset{\to+\infty}{e^{x}} - \overset{\to 0}{e^{-x}}}{2} = +\infty$$

$$\lim_{x\to+\infty} ch(x) = +\infty$$



$$\operatorname{ch}^2(x) - \operatorname{sh}^2(x) = 1 \qquad \forall\, x \in \mathbb{R}$$

$$\left(\frac{e^{x}+e^{-x}}{2}\right)^2 - \left(\frac{e^{x}-e^{-x}}{2}\right)^2 = 1$$

$$\frac{(e^{x}+e^{-x})^2}{4} - \frac{(e^{x}-e^{-x})^2}{4} = 1$$

$$\frac{\cancel{e^{2x}} + \cancel{e^{-2x}} + 2}{4} - \frac{\cancel{e^{2x}} + \cancel{e^{-2x}} - 2}{4} = \frac{2}{4} - \frac{-2}{4} =$$

$$= \frac{2}{4} + \frac{2}{4} = 1$$

$$\sin(2x) = 2 \sin x \cos x$$

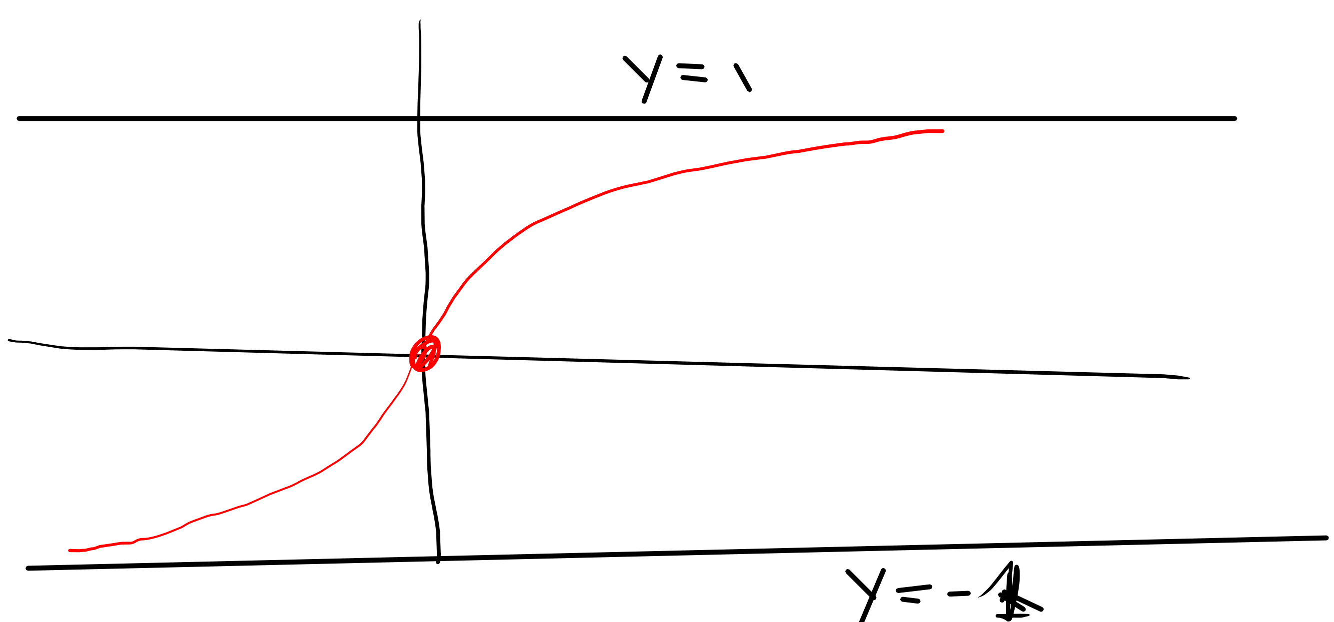
$$\operatorname{sh}(2x) = 2 \operatorname{ch}(x) \operatorname{sh}(x)$$

$$th(x) = \frac{sh(x)}{ch(x)}$$

è dispari

$$\lim_{x \to +\infty} th(x) = \lim_{x \to +\infty} \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}} =$$

$$= \lim_{x \to +\infty} \frac{\cancel{e^{x}}}{\cancel{e^{x}}} = \lim_{x \to +\infty} 1 = 1$$



$sh: \mathbb{R} \to \mathbb{R}$ è biettiva

$x \leftarrow y$

$y = sh\, x$

$$y = \frac{e^{x} - e^{-x}}{2}$$

$$2y = e^{x} - e^{-x}$$

$$e^{x} - e^{-x} - 2y = 0 \qquad e^{x}$$

$$e^{2x} - 2y e^{x} - 1 = 0$$

$$(e^{x})^2 - 2y e^{x} - 1 = 0 \qquad \checkmark$$

$$\boxed{(e^{x})_{\pm} = y \pm \sqrt{y^2 + 1}}$$

$$e^{x} = (e^{x})_{+} = y + \sqrt{y^2 + 1}$$

$$e^{x} = y + \sqrt{y^2 + 1} \qquad lg$$

$$x = lg\left(y + \sqrt{y^2 + 1}\right)$$