

Statistics of the currents in system coupled to two heat baths

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1 Qubit in contact with two heat baths

We consider a system with two energy levels (0 and 1) with energy gap ϵ , in contact with two heat baths at temperature T_L and T_R . Each bath induces transitions between the two states with rates $\omega_{L,R}^\uparrow$ and $\omega_{L,R}^\downarrow$ that satisfy the detailed balance with respect to their own temperature.

The master equation reads

$$d_t \mathbf{p}(t) = \mathbf{K} \cdot \mathbf{p}(t), \quad \mathbf{K} \begin{pmatrix} -\Omega^\uparrow & \Omega^\downarrow \\ \Omega^\uparrow & -\Omega^\downarrow \end{pmatrix}, \quad (1)$$

with $\Omega^\uparrow = \omega_R^\uparrow + \omega_L^\uparrow$, $\Omega^\downarrow = \omega_R^\downarrow + \omega_L^\downarrow$.

We now introduce the joint probability distribution $\phi(x, \Delta S_L^B, \Delta S_R^B, t)$ with $x = 0, 1$ and where ΔS_L^B and ΔS_R^B are the total entropy change in the two baths up to the time t (integrated entropy currents). The generating function for the two integrated quantities read

$$\Psi(x, \lambda_L, \lambda_R, t) = \int d\Delta S_L^B d\Delta S_R^B e^{\lambda_L \Delta S_L^B + \lambda_R \Delta S_R^B} \phi(x, \Delta S_L^B, \Delta S_R^B, t). \quad (2)$$

For the single transition we notice that the elementary entropy variations read

$$\Delta s_{L,R}^B|_{10} = \log \frac{\omega_{L,R}^\uparrow}{\omega_{L,R}^\downarrow} = -\beta_{L,R} \epsilon = -\Delta s_{L,R}^B|_{01}, \quad (3)$$

where we have set $k_B = 1$.

The time evolution of $\Psi(x, \lambda_L, \lambda_R, t)$ is governed by the “tilted” master equation

$$d_t \Psi(t) = \mathbf{L}(\lambda_L, \lambda_R) \cdot \Psi(t), \quad \mathbf{L}(\lambda_L, \lambda_R) = \begin{pmatrix} -\Omega^\uparrow & \Omega_1^\downarrow(\lambda_L, \lambda_R) \\ \Omega_1^\uparrow(\lambda_L, \lambda_R) & -\Omega^\downarrow \end{pmatrix}, \quad (4)$$

with

$$\Omega_1^\uparrow = \omega_L^\uparrow e^{-\lambda_L \beta_L \epsilon} + \omega_R^\uparrow e^{-\lambda_R \beta_R \epsilon}, \quad \Omega_1^\downarrow = \omega_L^\downarrow e^{\lambda_L \beta_L \epsilon} + \omega_R^\downarrow e^{\lambda_R \beta_R \epsilon} \quad (5)$$

The eigenvalues read

$$\mu_\pm(\lambda_L, \lambda_R) = \frac{1}{2} \left(-(\Omega^\uparrow + \Omega^\downarrow) \pm \sqrt{((\Omega^\uparrow - \Omega^\downarrow)^2 + 4\Omega_1^\uparrow(\lambda_L, \lambda_R)\Omega_1^\downarrow(\lambda_L, \lambda_R))} \right). \quad (6)$$

The larger eigenvalue $\mu_+(\lambda_L, \lambda_R)$ is thus our cumulant generating function for the currents

$$\lim_{t \rightarrow \infty} \frac{1}{t} \left\langle e^{\lambda_L \Delta S_L^B + \lambda_R \Delta S_R^B} \right\rangle = \mu_+(\lambda_L, \lambda_R). \quad (7)$$

Notice that while in the following we study the statistic of the currents at long time, solving eq. (4), i.e. finding the eigenvalues and the eigenvectors of the tilted stochastic matrix $\mathbf{L}(\lambda_L, \lambda_R)$, provides the statistic of the currents for any $t \geq 0$.

We now focus on the statistics of only one of the two currents, and set $\lambda_L = 0$. We also change $\lambda_R \rightarrow \lambda_R/\beta_R$, i.e. in the following we will deal with the integrated heat current Q_R^B rather than with the entropy.

Choosing $\omega_{L,R}^\uparrow = \omega_0$ and $\omega_{L,R}^\downarrow = \omega_0 e^{\beta_{L,R} \epsilon}$ we have

$$\begin{aligned} \mu_+(\lambda_R) &= \mu_+(\lambda_L = 0, \lambda_R) = \\ &= \frac{\omega_0}{2} \left[- (2 + e^{\beta_L \epsilon} + e^{\beta_R \epsilon}) + \sqrt{(e^{\beta_L \epsilon} + e^{\beta_R \epsilon})^2 + 4(1 + e^{(\beta_L - \lambda_R)\epsilon} + e^{(\beta_R + \lambda_R)\epsilon})} \right]. \end{aligned} \quad (8)$$

Notice the Gallavotti–Cohen symmetry $\mu_+(\lambda_R) = \mu_+(-\lambda_R - \beta_R + \beta_L)$.

The average current $\langle q_R^B \rangle = \langle Q_R^B/t \rangle$ is given by

$$\langle q_R^B \rangle = \partial_{\lambda_R} \mu_+(\lambda_R)|_{\lambda_R=0} = \epsilon \omega_0 \frac{e^{\beta_R \epsilon} - e^{\beta_L \epsilon}}{2 + e^{\beta_R \epsilon} + e^{\beta_L \epsilon}} \quad (9)$$

Obtaining the explicit expression of rate function $I(q_R^B)$ through Legendre transform is beyond reach. One can however study the statistics of the tails. In the limit $\lambda_R \rightarrow -\infty$, the equation

$$q_R^B = \partial_{\lambda_R} \mu_+(\lambda_R), \quad (10)$$

has solution

$$\lambda_R^* = \beta_L - \frac{2}{\epsilon} \ln \left(\frac{-2q_R^B}{\omega_0 \epsilon} \right), \quad (11)$$

corresponding to large and negative values of q_R^B . Thus the rate function becomes

$$q_R^B \lambda_R^* - \mu_+(\lambda_R^*) \underset{q_R^B \rightarrow -\infty}{\approx} \frac{q_R^B}{\epsilon} \left[2 + \beta_L \epsilon - \ln \left(\frac{-2q_R^B}{\omega_0 \epsilon} \right) \right] \approx -\frac{q_R^B}{\epsilon} \ln \left(\frac{-2q_R^B}{\omega_0 \epsilon} \right) \quad (12)$$

In the limit $\lambda_R \rightarrow +\infty$, the equation

$$q_R^B = \partial_{\lambda_R} \mu_+(\lambda_R), \quad (13)$$

has solution

$$\lambda_R^* = -\beta_R + \frac{2}{\epsilon} \ln \left(\frac{2q_R^B}{\omega_0 \epsilon} \right), \quad (14)$$

corresponding to large and positive values of q_R^B . The rate function becomes

$$q_R^B \lambda_R^* - \mu_+(\lambda_R^*) \underset{q_R^B \rightarrow +\infty}{\approx} -\frac{q_R^B}{\epsilon} \left[2 + \beta_R \epsilon - \ln \left(\frac{2q_R^B}{\omega_0 \epsilon} \right) \right] \approx \frac{q_R^B}{\epsilon} \ln \left(\frac{2q_R^B}{\omega_0 \epsilon} \right) \quad (15)$$

2 Brownian particle coupled to two baths

We consider a 1D Brownian particle in a potential $U(x)$. The particle is, moreover, in thermal contact with two distinct heat reservoirs at temperatures T_1 and T_2 . The heat transferred in time t from the two heat reservoirs is denoted Q_1 and Q_2 ,

Denoting the position of the particle by x and the momentum by p the Langevin equation of motion reads

$$\frac{dx}{dt} = \frac{p}{m}, \quad (16)$$

$$\frac{dp}{dt} = -U'(x) - (\gamma_1 + \gamma_2) \frac{p}{m} + \eta_1 + \eta_2, \quad (17)$$

where the Gaussian white noises η_1 and η_2 are correlated according to

$$\langle \eta_1(t)\eta_1(t') \rangle = 2\gamma_1 k_B T_1 \delta(t-t'), \quad (18)$$

$$\langle \eta_2(t)\eta_2(t') \rangle = 2\gamma_2 k_B T_2 \delta(t-t'), \quad (19)$$

$$\langle \eta_1(t)\eta_2(t') \rangle = 0. \quad (20)$$

The heat flux from the reservoir at temperature T_i , i.e., the rate of work done by the stochastic force $-\gamma_i p/m + \eta_i$ on the particle, is given by

$$\frac{dQ_i}{dt} = -\gamma_i p^2/m^2 + p/m \eta_i. \quad (21)$$

Thus the noise terms in eqs.(16), (17), (21) can be written as

$$\mathbf{B} \cdot \boldsymbol{\xi} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \sqrt{\gamma_1 k_B T} & \sqrt{\gamma_2 k_B T} & 0 \\ 0 & \frac{p}{m} \sqrt{\gamma_1 k_B T} & 0 & 0 \\ 0 & 0 & \frac{p}{m} \sqrt{\gamma_2 k_B T} & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \xi_1 \\ \xi_2 \\ 0 \end{pmatrix}, \quad (22)$$

with $\langle \xi_i(t)\xi_j(t') \rangle = 2\delta_{ij}\delta(t-t')$.

Defining the vectors

$$\mathbf{z} = \begin{pmatrix} x \\ p \\ Q_1 \\ Q_2 \end{pmatrix}, \quad \mathbf{F}(\mathbf{z}) = \begin{pmatrix} p/m \\ -(\gamma_1 + \gamma_2) \frac{p}{m} - U'(x) \\ -\gamma_1 \frac{p^2}{m^2} \\ -\gamma_2 \frac{p^2}{m^2} \end{pmatrix}, \quad (23)$$

the Langevin equation for \mathbf{z} can be written in a compact form as

$$\frac{d\mathbf{z}}{dt} = \mathbf{F}(\mathbf{z}) + \mathbf{B} \cdot \boldsymbol{\xi}. \quad (24)$$

Given a Langevin equation of this type, the general form of the Fokker-Planck equation describing the evolution of the joint probability distribution $P(\mathbf{z}, t)$ reads

$$\partial_t P(\mathbf{z}, t) = - \sum_i \partial_{z_i} [F_i(\mathbf{z})P(\mathbf{z}, t)] + \sum_{ijk} \partial_{z_i} [B_{ik}(\mathbf{z}) (\partial_{z_j} B_{jk}(\mathbf{z})P(\mathbf{z}, t))], \quad (25)$$

see, e.g, section 4.3.6 in [1].

3 Tilted Fokker-Planck operator

We will focus on $Q = Q_1$ in the following, without loss of generality. We also set $k_B = 1$. The Fokker-Planck equation for the joint distribution $P(q, p, Q, t)$, $Q = Q_1$ reads

$$\begin{aligned} \frac{\partial P}{\partial t} = & \{P, H\} + (\gamma_1 T_1 + \gamma_1 T_2) \frac{\partial^2 P}{\partial p^2} + (\gamma_1 + \gamma_2) \frac{\partial(pP)}{\partial p} \\ & + \gamma_1 \frac{\partial}{\partial Q} \left[(p^2 + T_1)P + T_1 p^2 \frac{\partial P}{\partial Q} + 2T_1 p \frac{\partial P}{\partial p} \right], \end{aligned} \quad (26)$$

where $\{P, H\}$ denotes the Poisson bracket

$$\{P, H\} = \frac{\partial P}{\partial p} \frac{\partial H}{\partial x} - \frac{\partial P}{\partial x} \frac{\partial H}{\partial p} = U'(x) \frac{\partial P}{\partial p} - \frac{p}{m} \frac{\partial P}{\partial x}. \quad (27)$$

The heat distribution after having analyzed the Fokker-Planck equation is then given by

$$P(Q, t) = \int dx dp P(x, p, Q, t). \quad (28)$$

Defining the characteristic function with respect to the heat by

$$\Psi(x, p, s, t) = \int dQ P(x, p, Q, t) e^{sQ}, \quad (29)$$

and noticing that in eq. (26) $\partial/\partial Q \rightarrow -s$ and $\partial^2/\partial Q^2 \rightarrow s^2$ we obtain for Ψ

$$\frac{\partial \Psi(s)}{\partial t} = L(s) \Psi(s), \quad (30)$$

where the tilted FP-operator $L(s)$ reads

$$\begin{aligned} L(s) \Psi(s) = & \{\Psi(s), H\} + (\gamma_1 T_1 + \gamma_1 T_2) \frac{\partial^2 \Psi(s)}{\partial p^2} + (\gamma_1 + \gamma_2) \frac{\partial(p\Psi(s))}{\partial p} \\ & - \gamma_1 s \left[(p^2 + T_1) \Psi(s) - s T_1 p^2 \Psi(s) + 2T_1 p \frac{d\Psi(s)}{dp} \right]. \end{aligned} \quad (31)$$

We now consider the case of an unbound particle Brownian particle with $U(x) = 0$ and hence $\{\Psi(x, p, s, t), H\} = -pd\Psi(x, p, s, t)/dx$ and integrating

over the position x which is decoupled from the momentum p we obtain a second order differential equation for Ψ of the Hermite type. By means of the transformation

$$\Psi(p, s, t) = e^{-A(s)p^2/2m} \tilde{\Psi}(p, s, t), \quad A(s) = \frac{\gamma_1 + \gamma_2 - 2s\gamma_1 T_1}{2(\gamma_1 T_1 + \gamma_2 T_2)}, \quad (32)$$

$\tilde{\Psi}(p, s, t)$ satisfies the Schrödinger-like equation (in p) for a harmonic oscillator

$$\partial_t \tilde{\Psi}(p, s, t) = \left[\frac{\gamma_1 + \gamma_2}{2m} - p^2 \frac{\delta(s)}{4m^2(\tilde{T}_1 + \tilde{T}_2)} + (\tilde{T}_1 + \tilde{T}_2) \frac{\partial^2}{\partial p^2} \right] \tilde{\Psi}(p, s, t), \quad (33)$$

with $\tilde{T}_i = \gamma_i T_i$ and

$$\delta(s) = \gamma_1^2 + \gamma_2^2 + 2\gamma_1\gamma_2(1 - 2sT_1 + 2sT_2 - 2s^2T_1T_2). \quad (34)$$

In this respect it is convenient to replace

$$\hbar\omega \rightarrow \frac{1}{m} \sqrt{\delta(s)}, \quad (35)$$

so as the eigenvalues read

$$-E_n = -\frac{\gamma_1 + \gamma_2}{2m} + \frac{(n + 1/2)}{m} \sqrt{\gamma_1^2 + \gamma_2^2 + 2\gamma_1\gamma_2(1 - 2sT_1 + 2sT_2 - 2s^2T_1T_2)}, \quad (36)$$

the sign minus originating from the fact that the Schrödinger-like equation exhibits a time derivative with “complex t ”.

We finally obtain the spectral representation

$$\Psi(p, s, t) = e^{-A(s)(p^2 - p_0^2)/2m} \sum_{n=0} e^{E_n(s)t} \Psi_n(p) \Psi_n(p_0), \quad (37)$$

where $\Psi_n(p)$ are the associated normalized eigenfunctions. We have, moreover, imposed the initial condition $\Psi(p, s, t = 0) = \delta(p - p_0)$, where p_0 is the initial momentum. The rescaled cumulant generating function is thus given by the ground state energy $E_0(s)$ yielding

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{1}{t} \langle e^{sQ} \rangle &= E_0(s) \\ &= \frac{1}{2m} \left[\gamma_1 + \gamma_2 - \sqrt{\gamma_1^2 + \gamma_2^2 + 2\gamma_1\gamma_2(1 - 2sT_1 + 2sT_2 - 2s^2T_1T_2)} \right] \\ &= \frac{1}{2m} \left[\gamma_1 + \gamma_2 - \sqrt{\gamma_1\gamma_2 T_1 T_2 (s_+ - s)(s - s_-)} \right], \end{aligned} \quad (38)$$

where the branch points are given by

$$s_{\pm} = \frac{1}{2} \left[\beta_1 - \beta_2 \pm \sqrt{(\beta_1 - \beta_2)^2 + \frac{\beta_1 \beta_2 (\gamma_1 + \gamma_2)^2}{\gamma_1 \gamma_2}} \right]. \quad (39)$$

Notice the Gallavotti-Cohen symmetry $E_0(s) = E_0(-s + \beta_1 - \beta_2)$, implying the long-time fluctuation relation

$$P(Q_1, t)/P(-Q_1, t) = e^{Q_1(\beta_1 - \beta_2)} \quad (40)$$

The average heat current in the long time limit reads

$$\langle q_1 \rangle = \partial_s E_0(s)|_{s=0} = \frac{(T_1 - T_2)\gamma_1\gamma_2}{m(\gamma_1 + \gamma_2)} \quad (41)$$

In the case $\gamma_1 = \gamma_2 = \gamma$, the variance takes the simple form

$$\langle q_1^2 \rangle - \langle q_1 \rangle^2 = \partial_s^2 E_0(s)|_{s=0} = \frac{(T_1 + T_2)^2 \gamma}{4m} \quad (42)$$

Finally, the rate function reads

$$\begin{aligned} I(q) &= \max_s [sq - E_0(s)] \\ &= -\frac{1}{2m} \left[\gamma_1 + \gamma_2 - q(s_+ + s_-) - (s_+ - s_-) \sqrt{\gamma_1 \gamma_2 T_1 T_2 / m^2 + q^2} \right], \end{aligned} \quad (43)$$

In the case of a bound Brownian particle with $U(x) = \kappa x^2/2$ the Poisson bracket enters the calculation and the position of the particle comes into play. The FP-operator becomes second order in x and p and is more difficult to analyze. However it can be shown that the maximal eigenvalue is independent of κ .

Finally one can show that for the “rotated” FP operator

$$\tilde{L}(s) = e^{\beta_2 H} \mathcal{T} L(s) \mathcal{T}^{-1} e^{-\beta_2 H} \quad (44)$$

the following equality holds

$$\tilde{L}(s) = L^\dagger(-s + \beta_1 - \beta_2) \quad (45)$$

where $L^\dagger(s)$ is the adjoint operator of $L(s)$, and \mathcal{T} is the time reversal operator. Thus $L(s)$ and $L^\dagger(-s + \beta_1 - \beta_2)$ have identical spectra, and for any eigenvalue $\lambda(s)$ of $L(s)$ the following equality holds $\lambda(s) = \lambda(-s + \beta_1 - \beta_2)$. This symmetry holds for any potential $U(x)$.

References

- [1] Crispin W. Gardiner. *Handbook of Stochastic Methods for Physics, Chemistry, and the Natural Sciences*. Springer Series in Synergetics. Springer, 3 edition, 2004.