# Image Processing for Physicists

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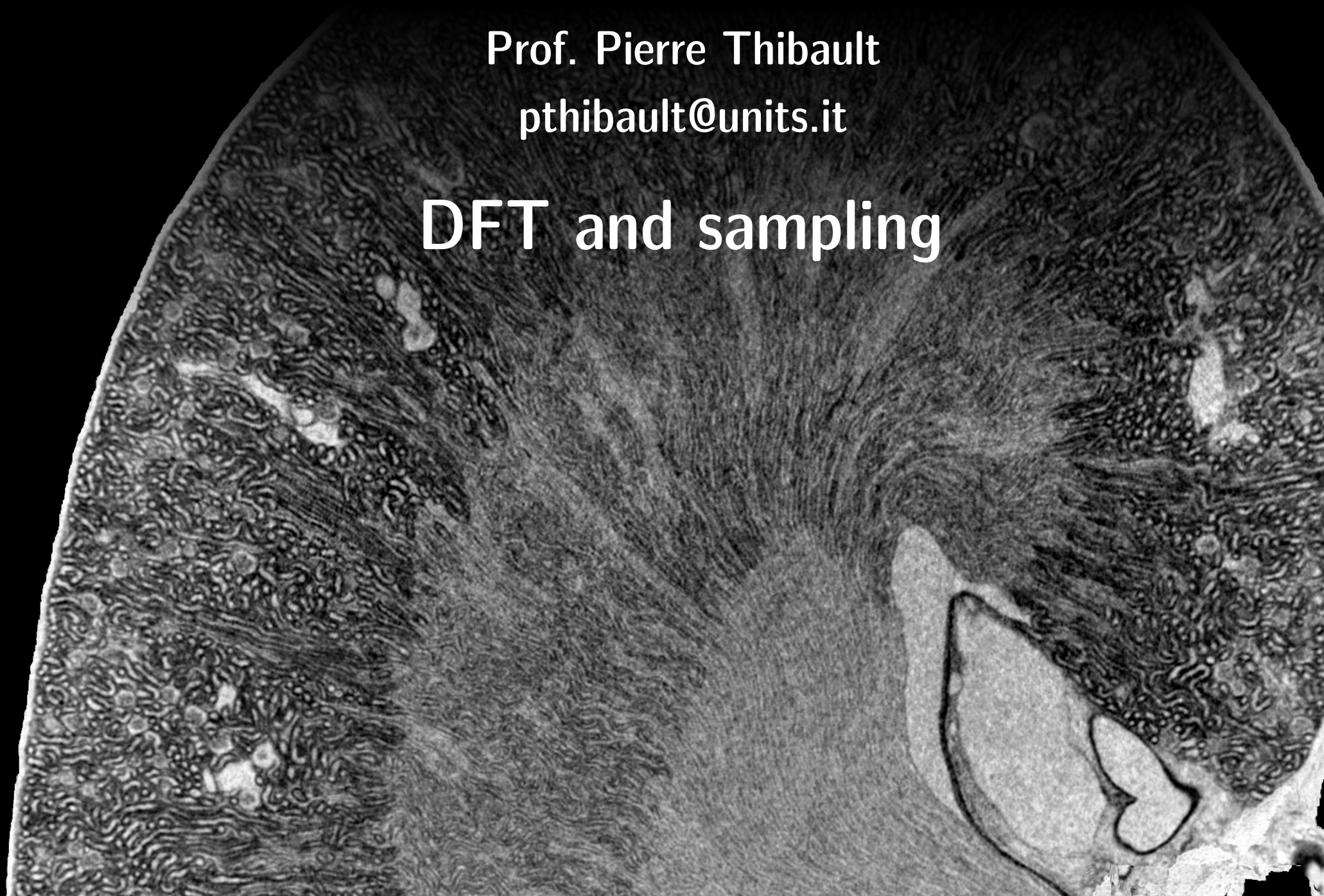
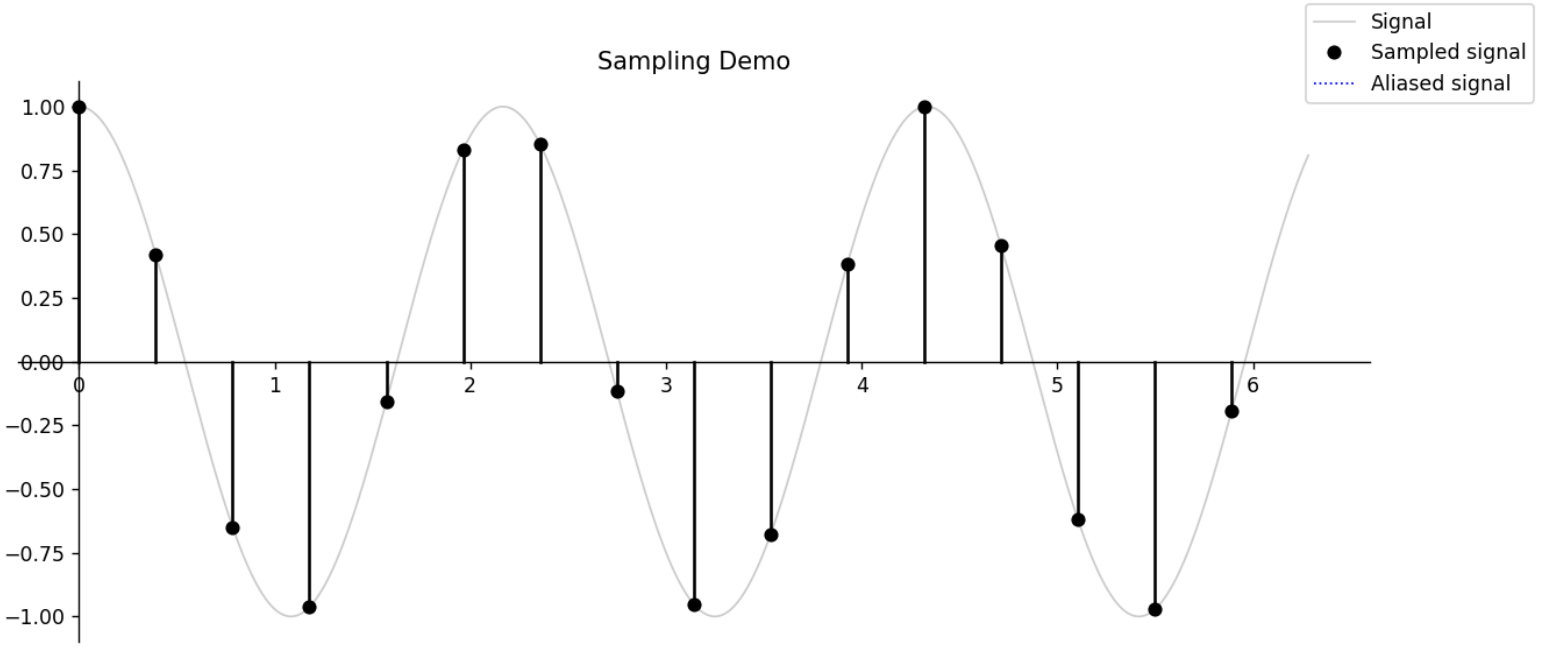
## DFT and sampling

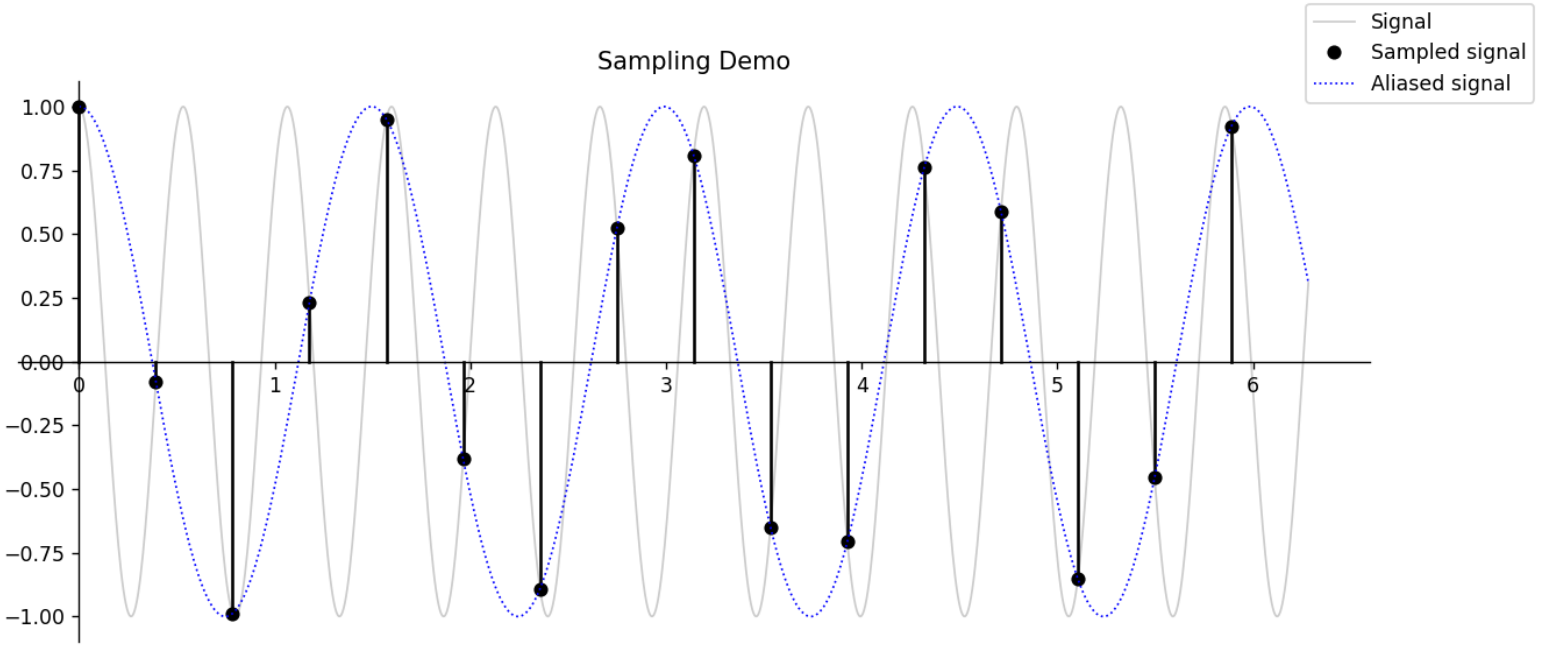


# Overview

- Sampling
  - Nyquist theorem
- Discrete Fourier transform
  - Undersampling and Aliasing
- Interpolation (resampling)

# Sampling


Sampling Demo
Signal
Sampled signal
Aliased signal
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
0
1
2
3
4
5
6


Sampling Demo
Signal
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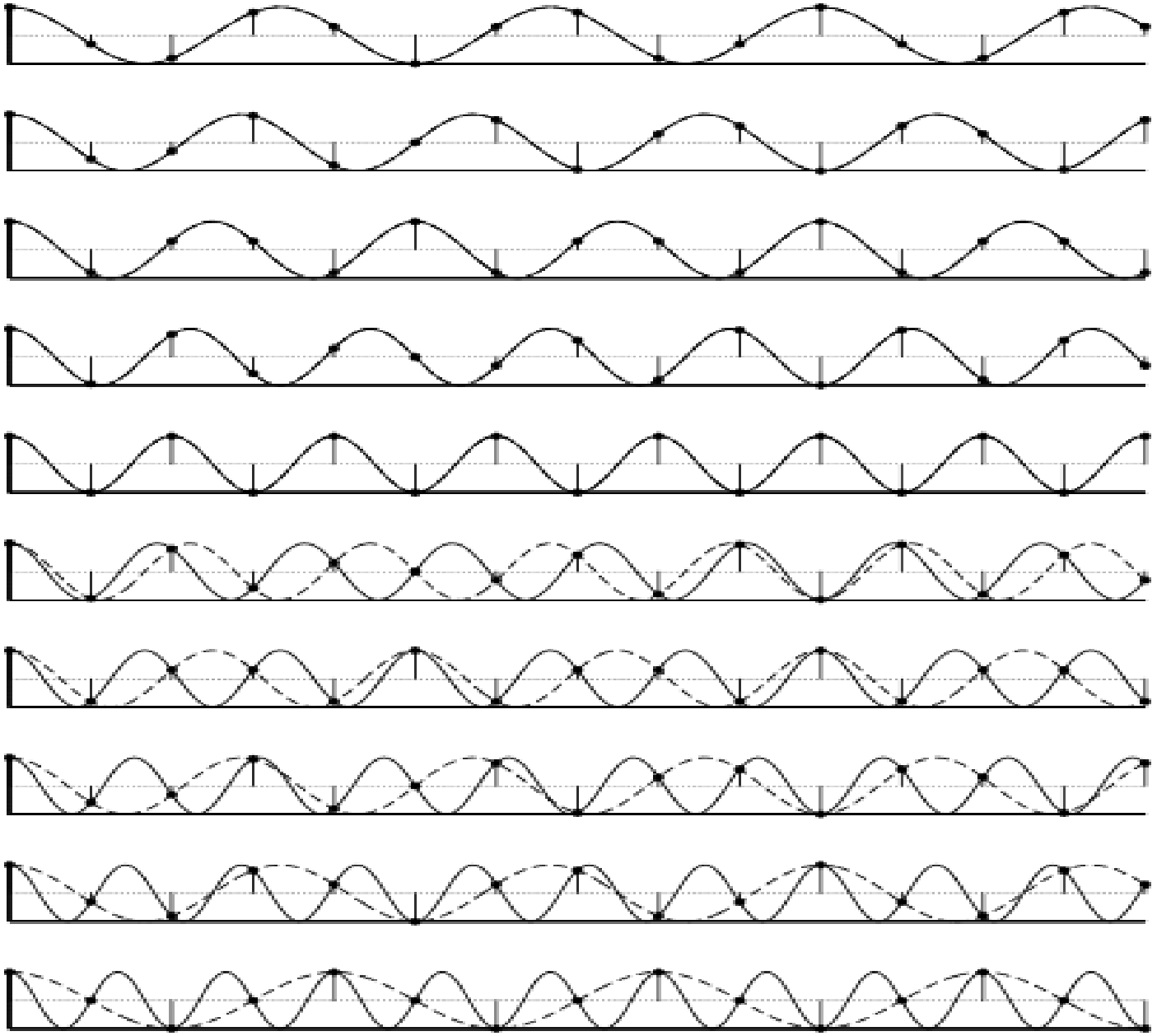
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
0
1
2
3
4
5
6

# Undersampling and aliasing



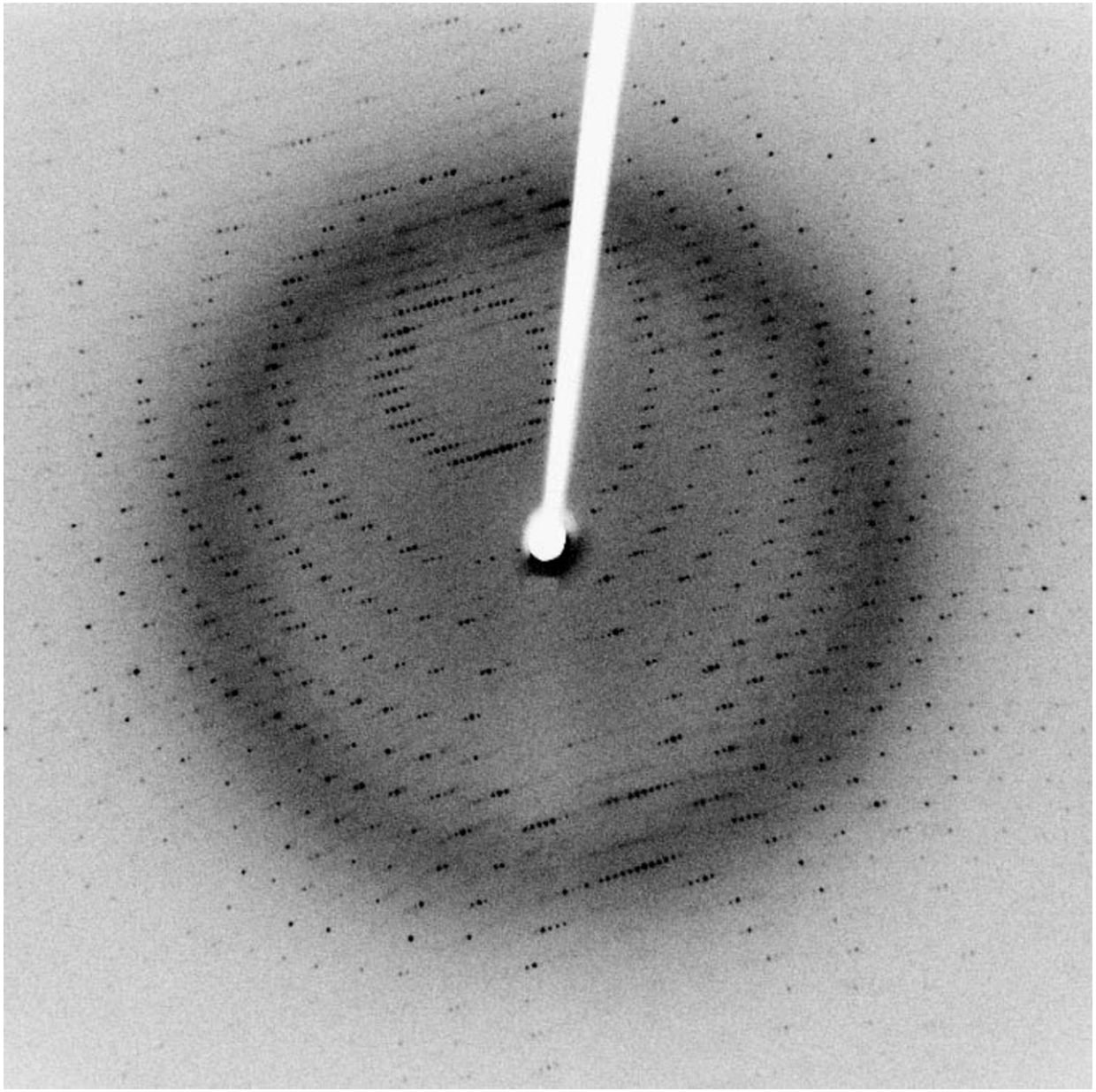
# The Nyquist-Shannon sampling theorem

"The largest frequency that can be represented in a signal sampled at intervals s is 1/2s"

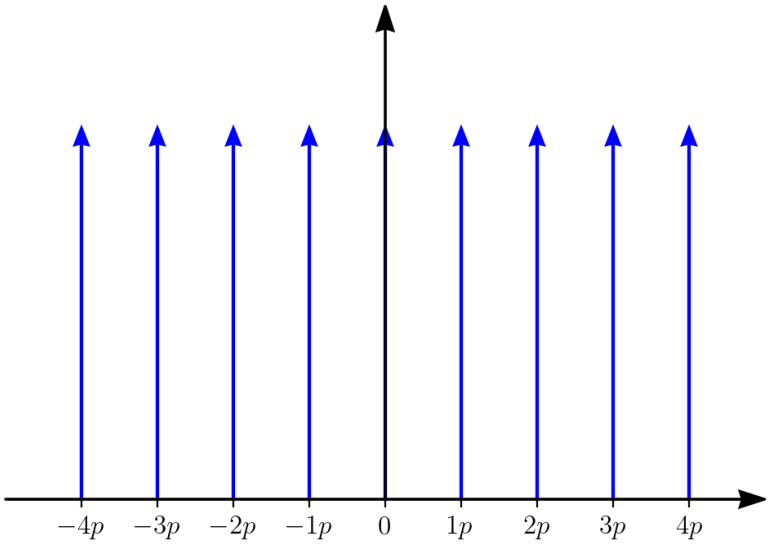
# Periodic signals

# Periodic signals

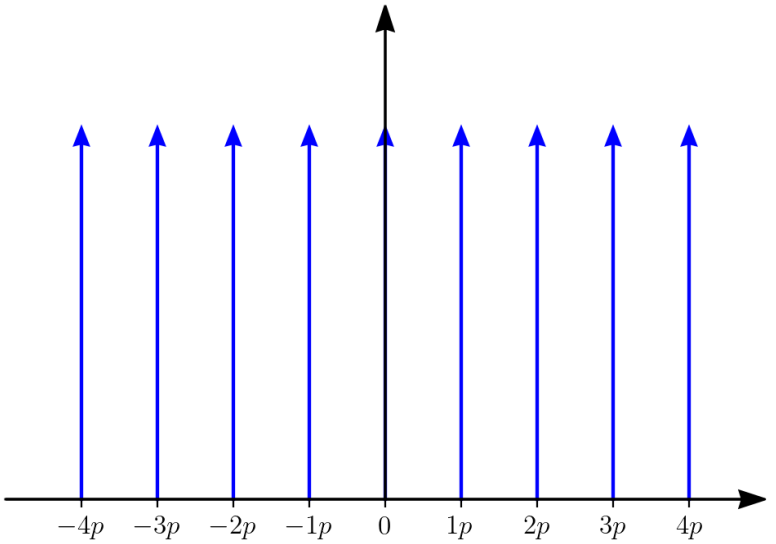
## X-ray diffraction by a crystal

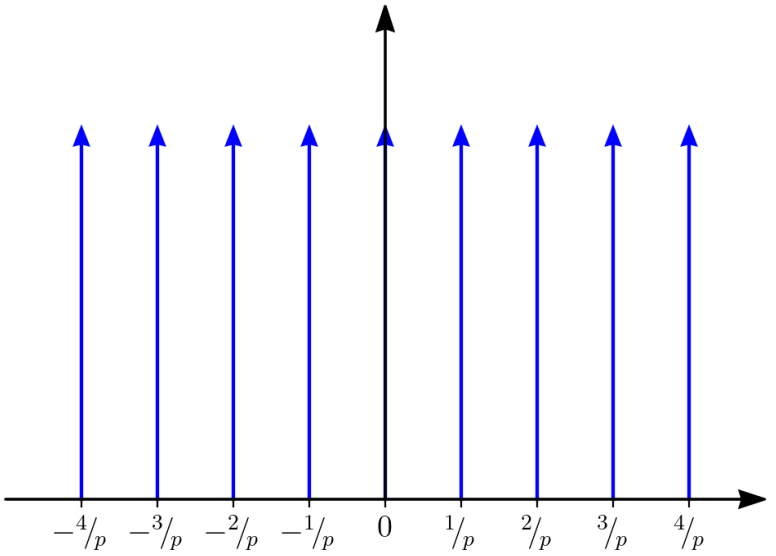


# Sampling with the Dirac comb



# Fourier transform of a Dirac comb

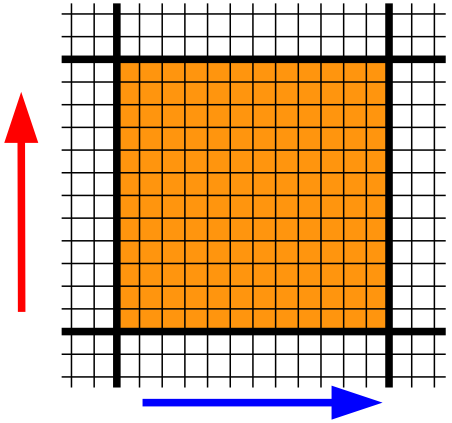


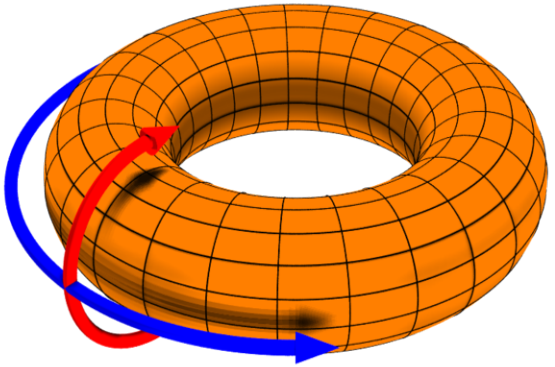



# Discrete Fourier Transform

- A **periodic** function has a **discrete** spectrum in the Fourier domain;
- A function with **discrete** values in the spatial domain is **periodic** in the Fourier domain;

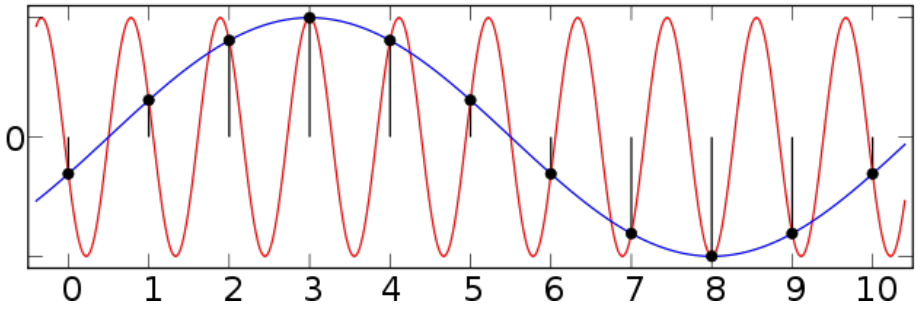
  $\Rightarrow$ A periodic and discrete function has a periodic and discrete Fourier transform.





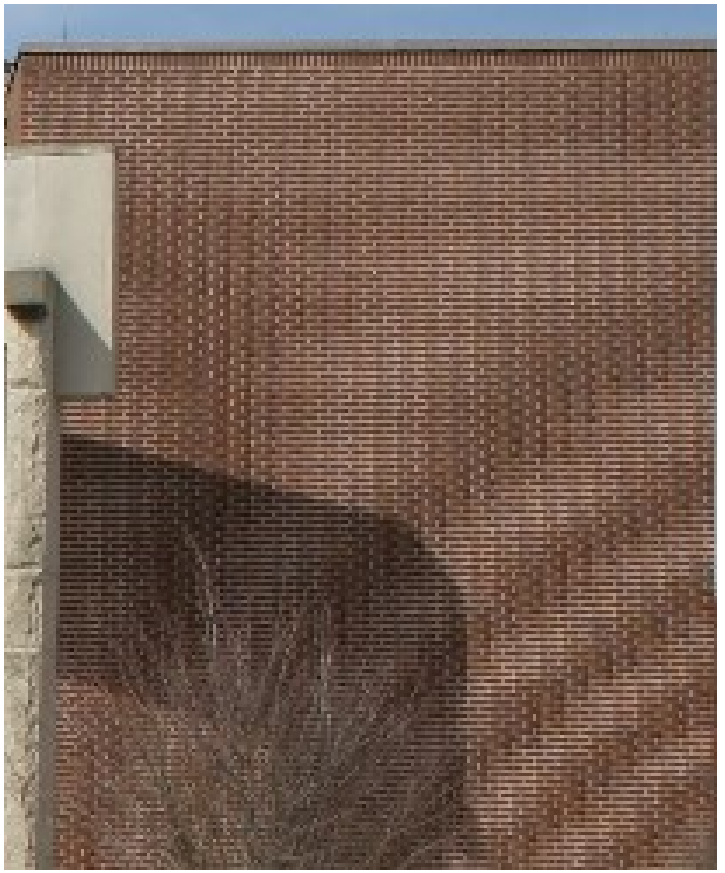
# Aliasing

Moiré: after resampling, high spatial frequencies appear as low spatial frequencies




high resolution

low resolution

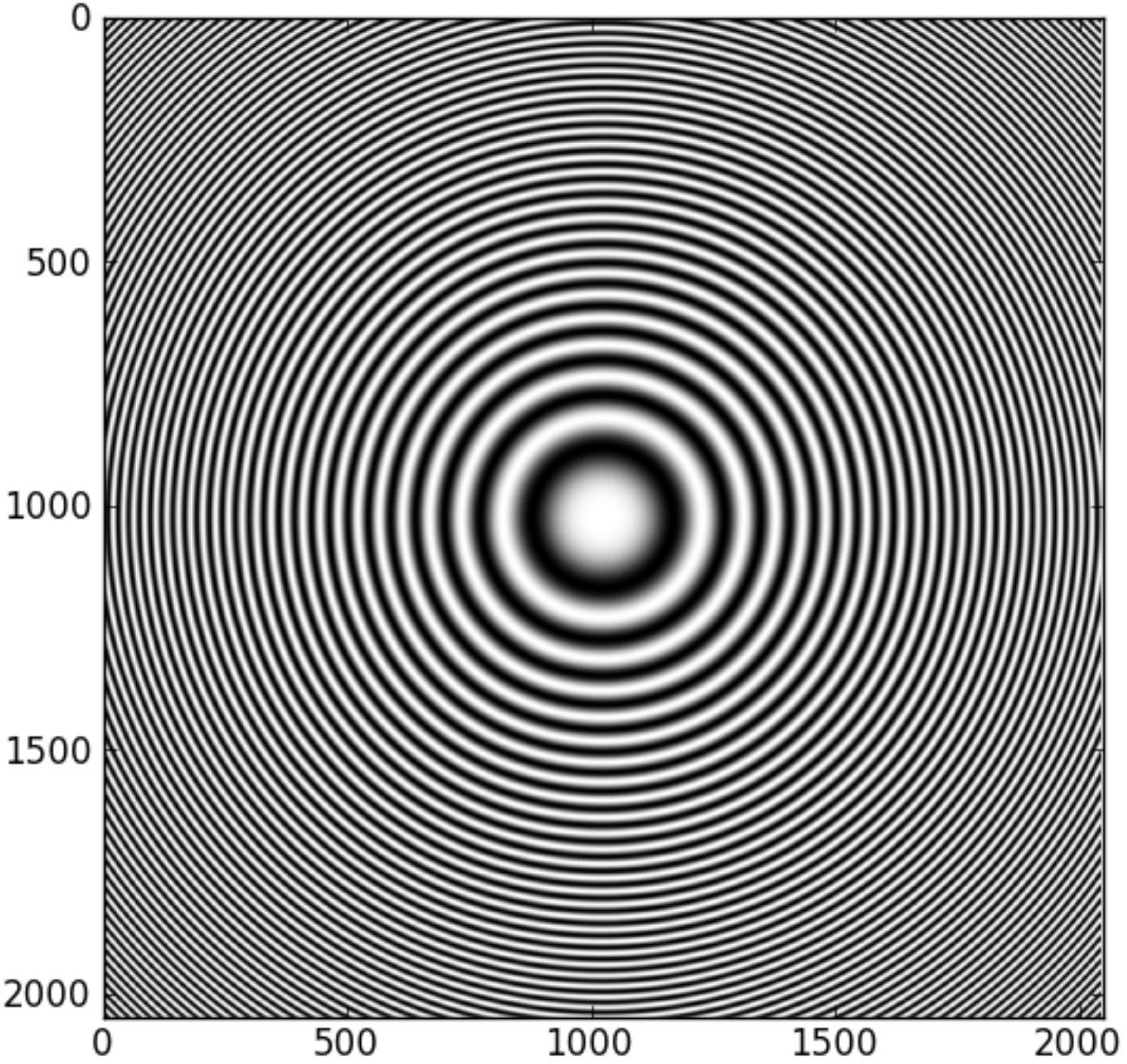


source: http://wikipedia.org

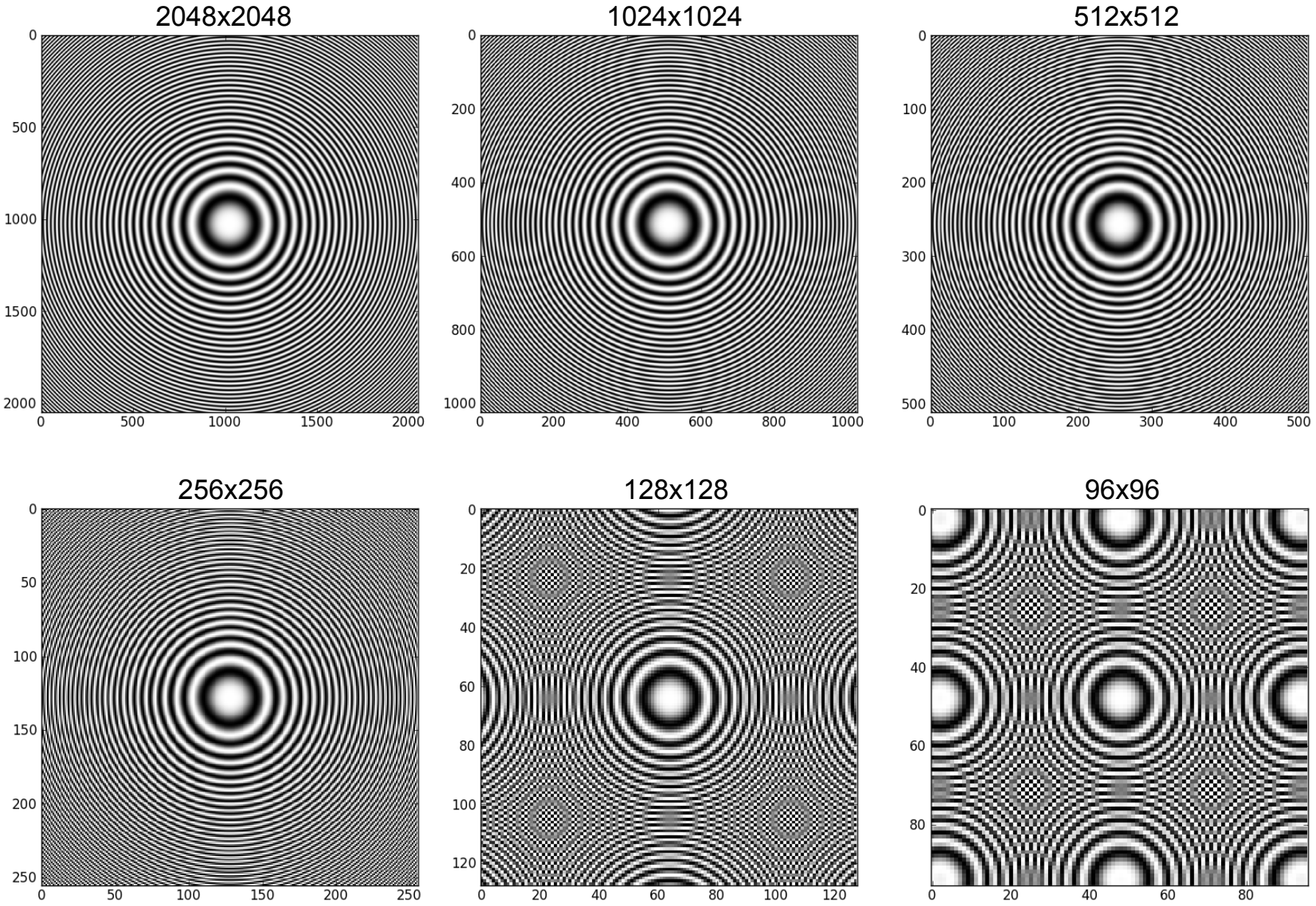
# Aliasing

# Undersampling

"Fresnel zone" test pattern: radial linear increase in spatial frequency

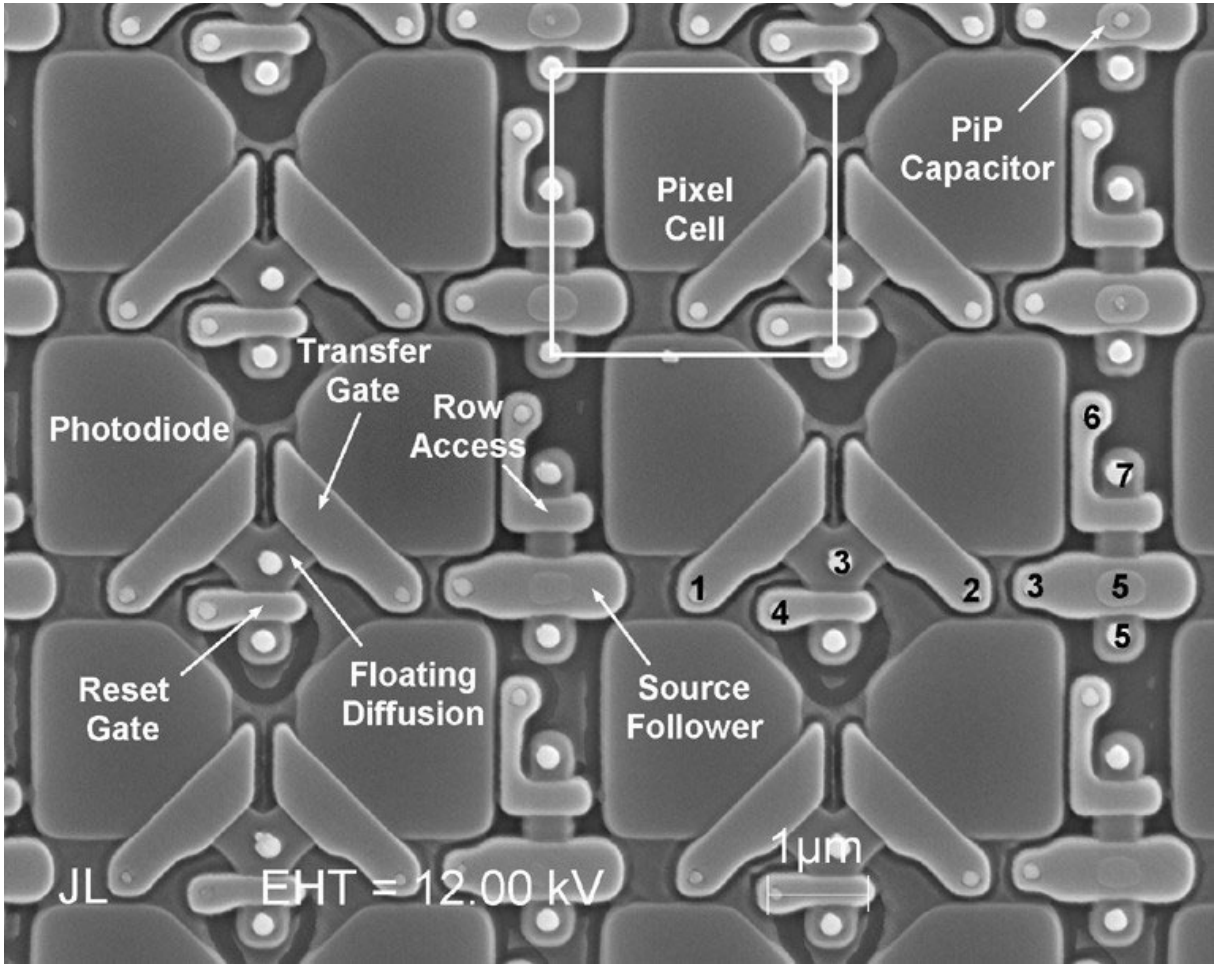


# Undersampling & aliasing



# Sampling with a pixel-array detector

- A 2D light field is sampled with a 2D pixel-array detector.

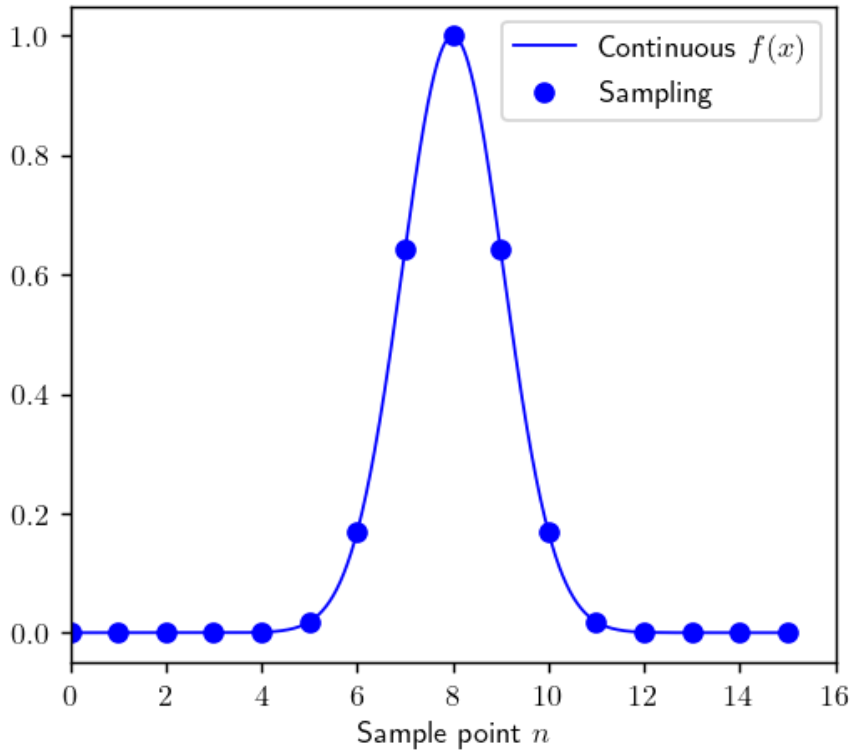


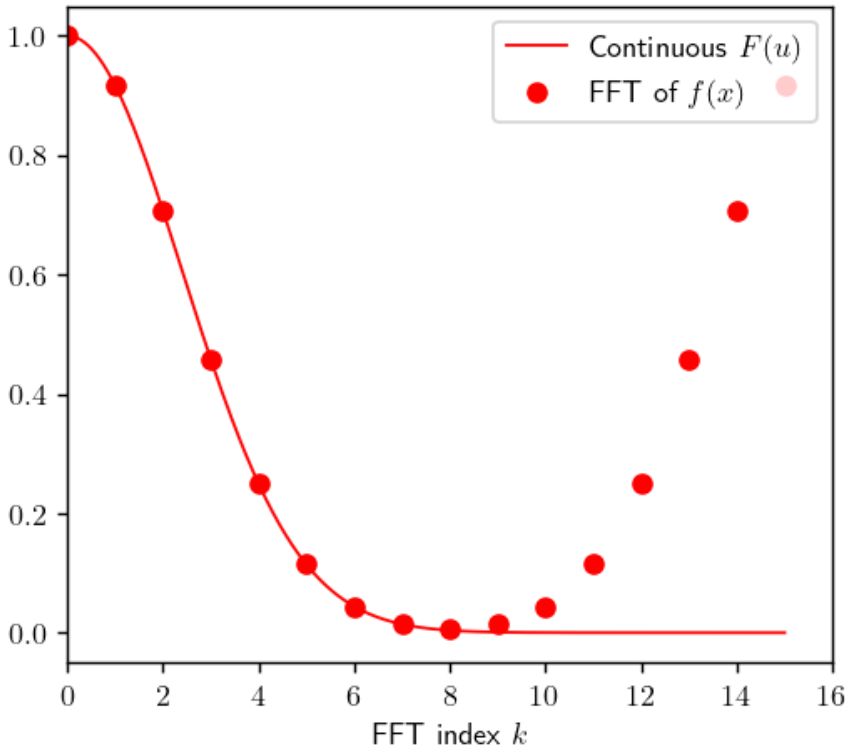
# Discrete Fourier Transform

- On computers, the signal is both sampled and of finite extent $\rightarrow$ using DFT on this signal means that is it assumed to be periodic.
- If the signal is sufficiently sampled, then the DFT can be interpreted as a sampled version of the continuous Fourier Transform.

# DFT example

- Example: relation between space, sampling and frequency



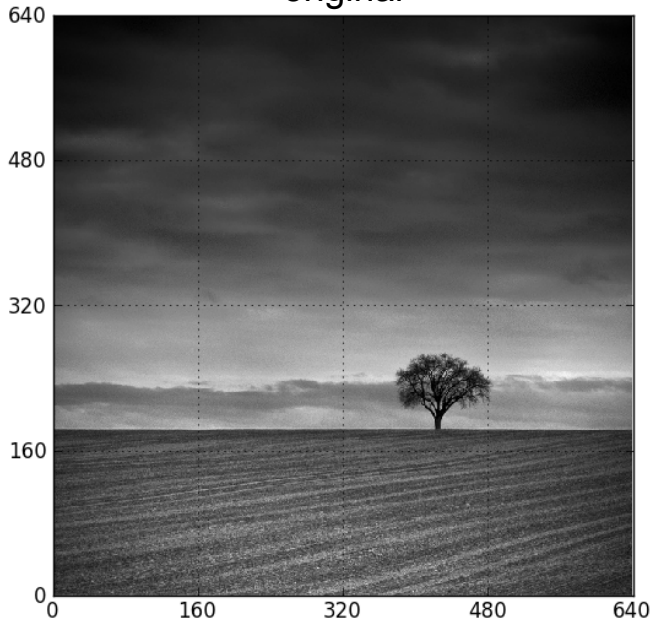




zero frequency component is in the top left corner output array.

# FT to DFT conversion

# Fourier space translation

original




amplitude of Fourier spectrum

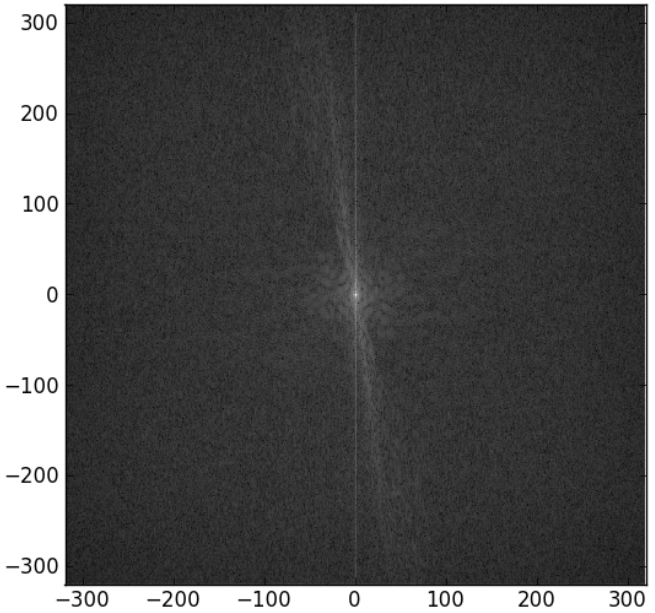
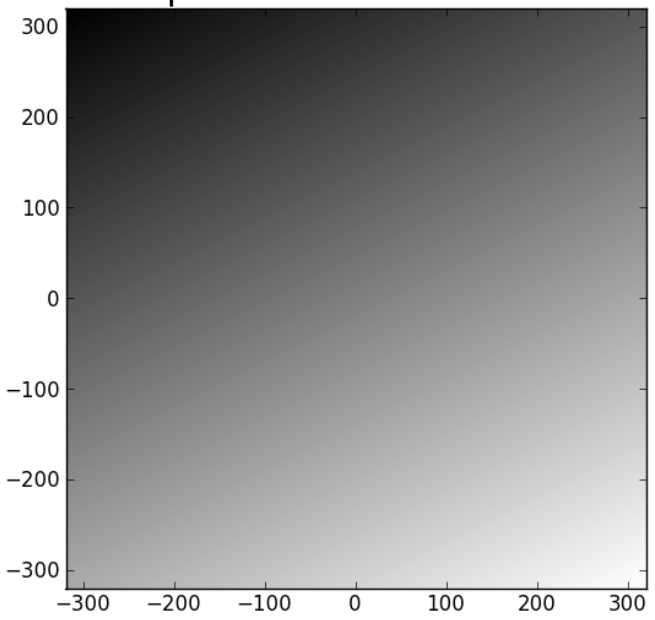



Image shifting using shifting property of FT

phase of filter function




shifted image

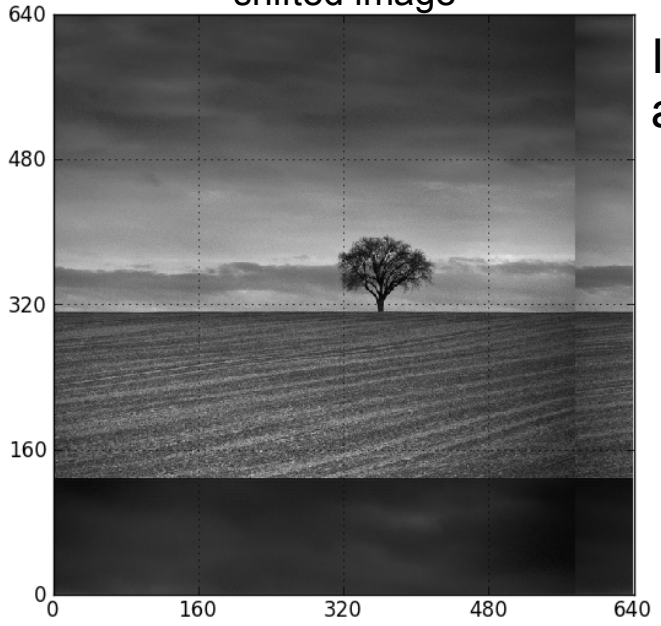
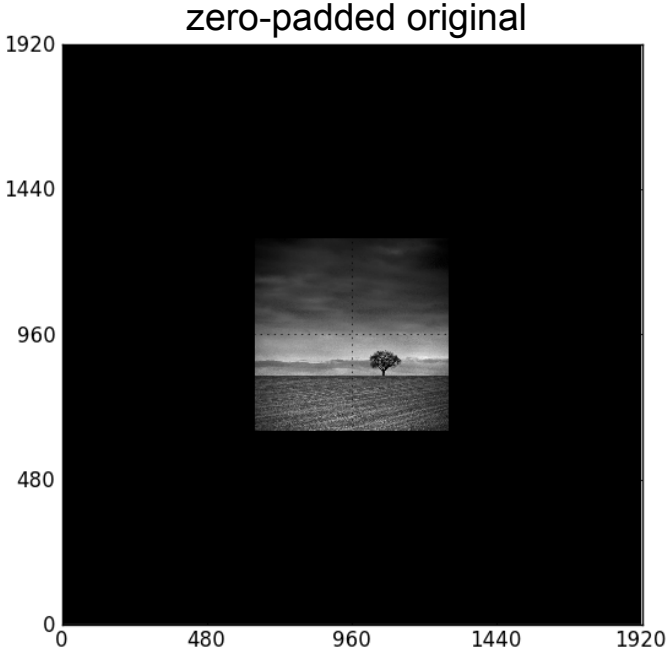
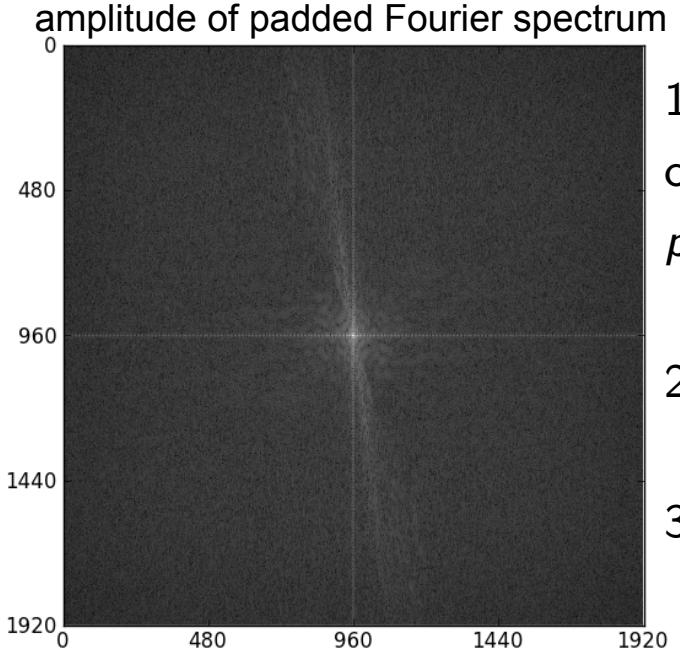



Image gets wrapped around

# Zero-padding

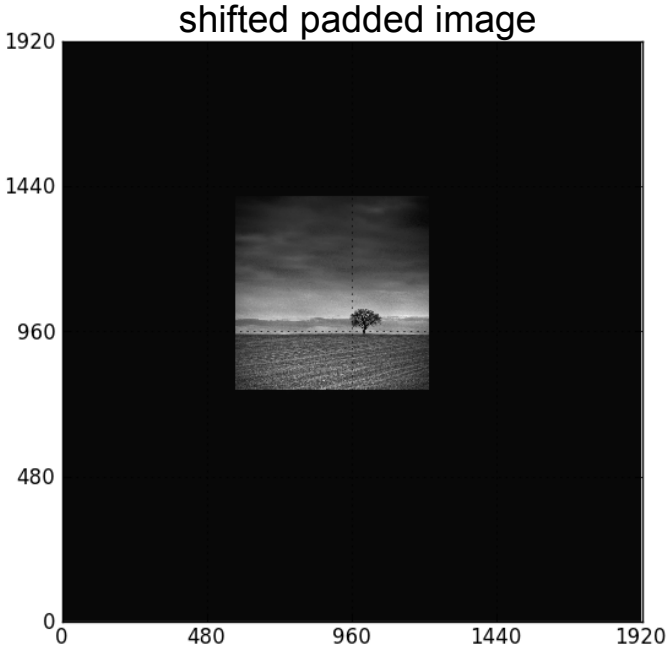


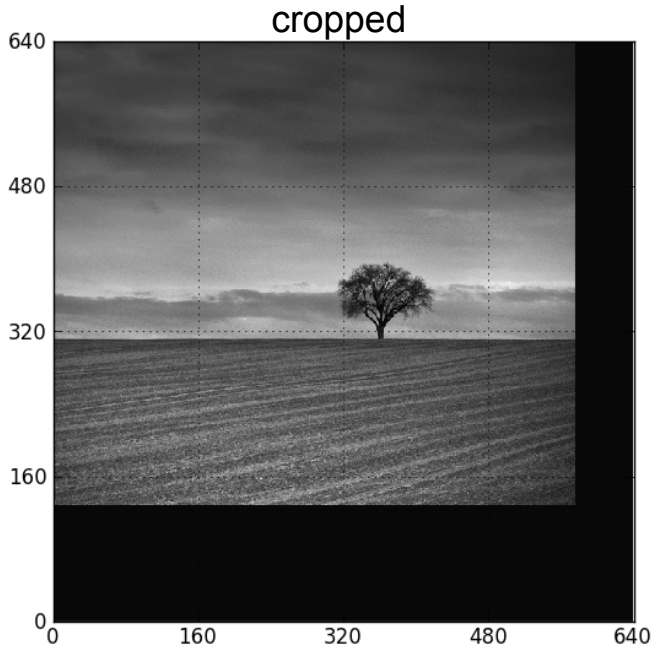




1. Add zeros around original image (*zero-padding*)

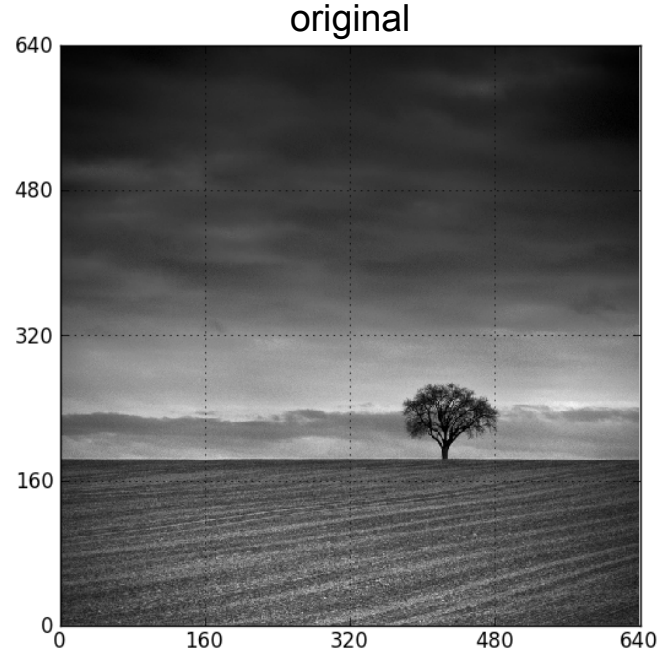
2. Shift using FT

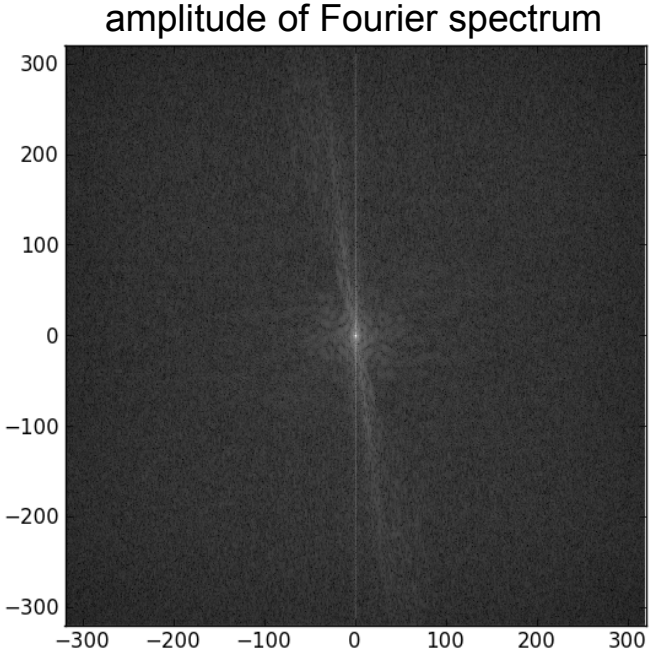
3. Crop result

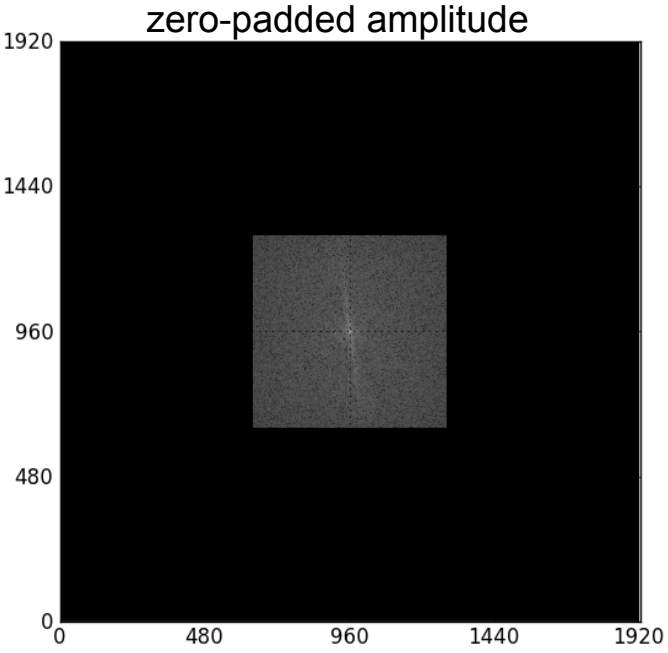


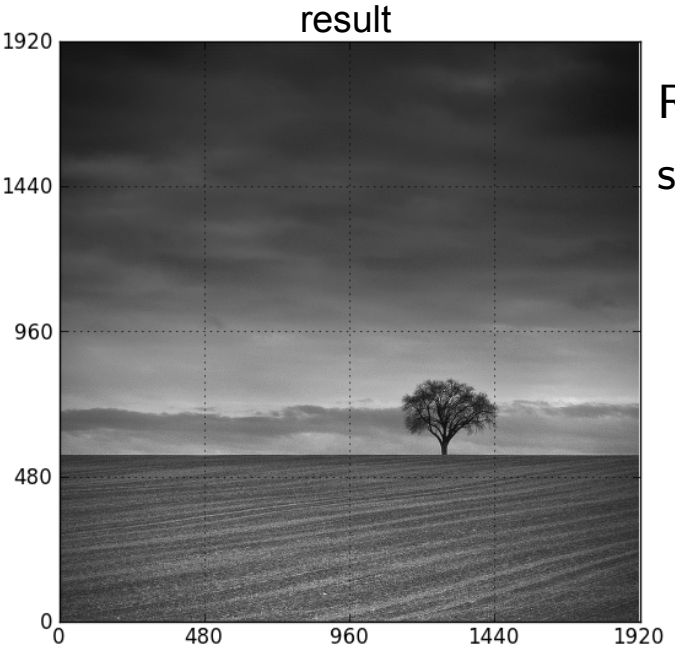



# Zero-padding in Fourier space



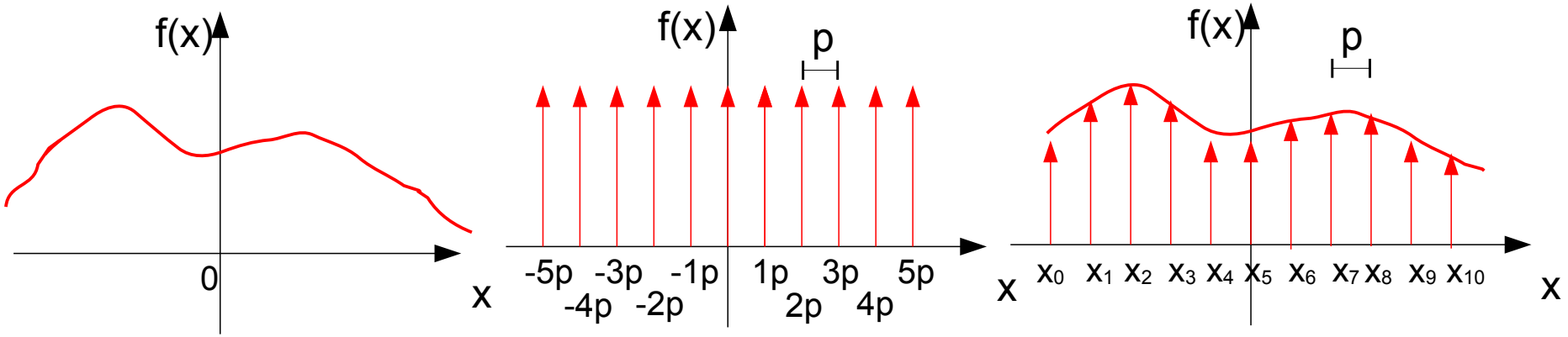



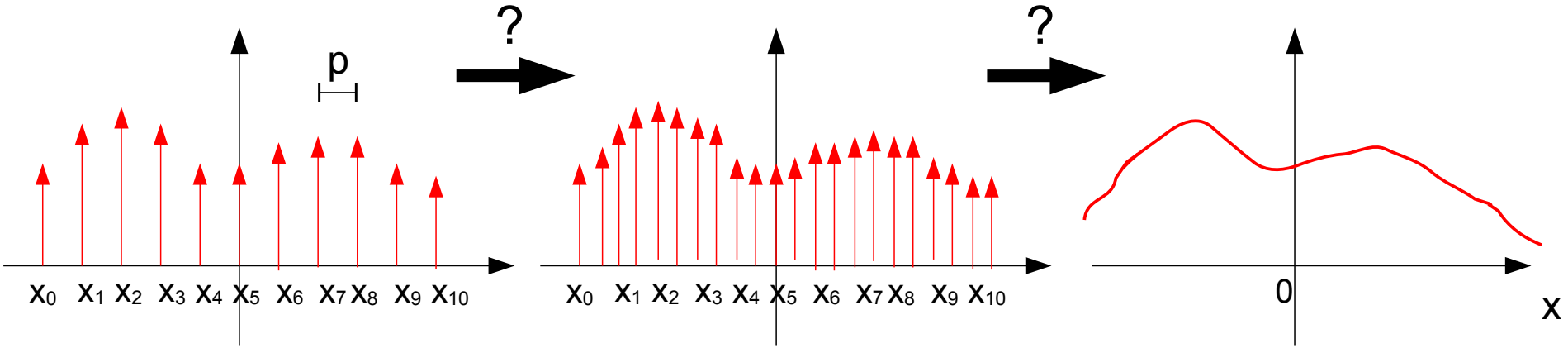

Result: increased sampling!

# Interpolation

- Discrete sampling of a continuous function



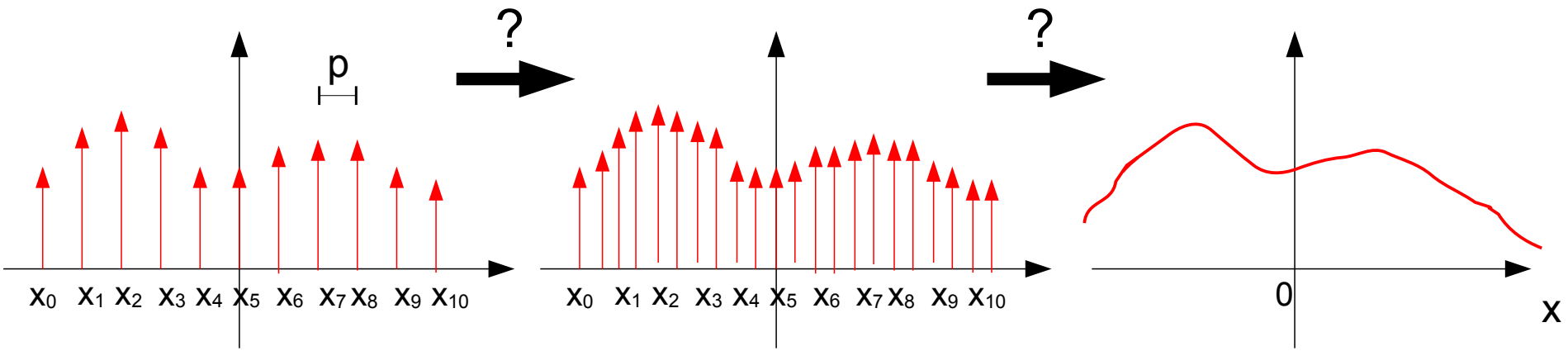

- Reconstruct original function from sampled data?



# Interpolation

**Finding unknown points between known ones**

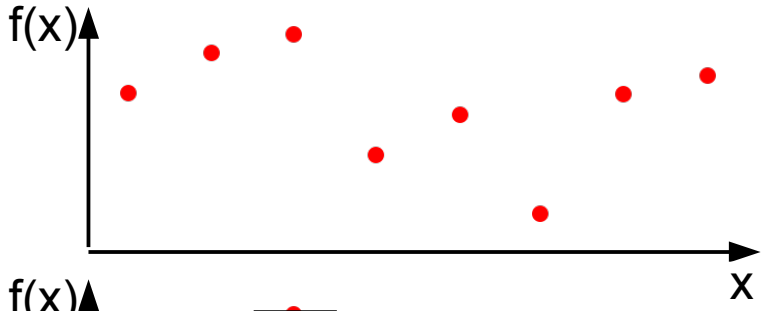
- wide field, many different approaches
- closely related to approximation theory and curve fitting



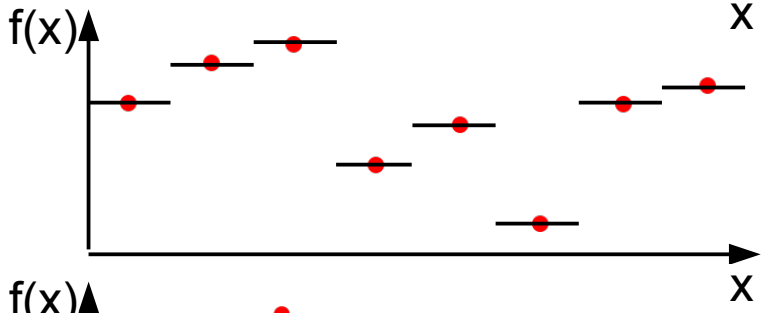
# Interpolation

## Various "classical" interpolation methods available

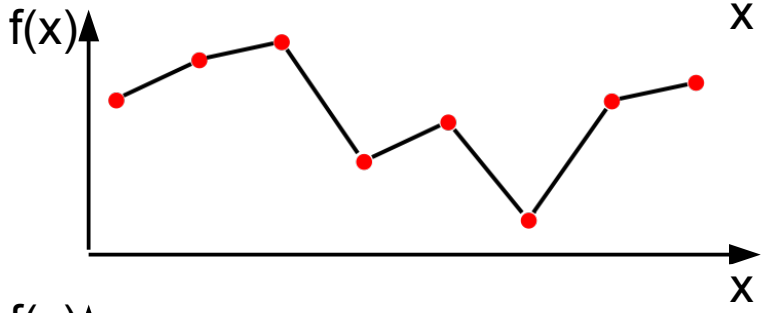
sampled points



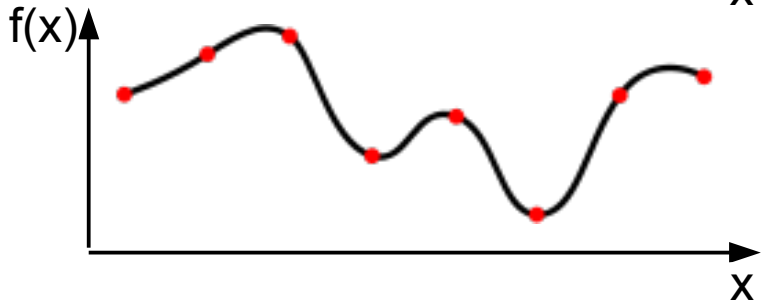

nearest neighbor




linear



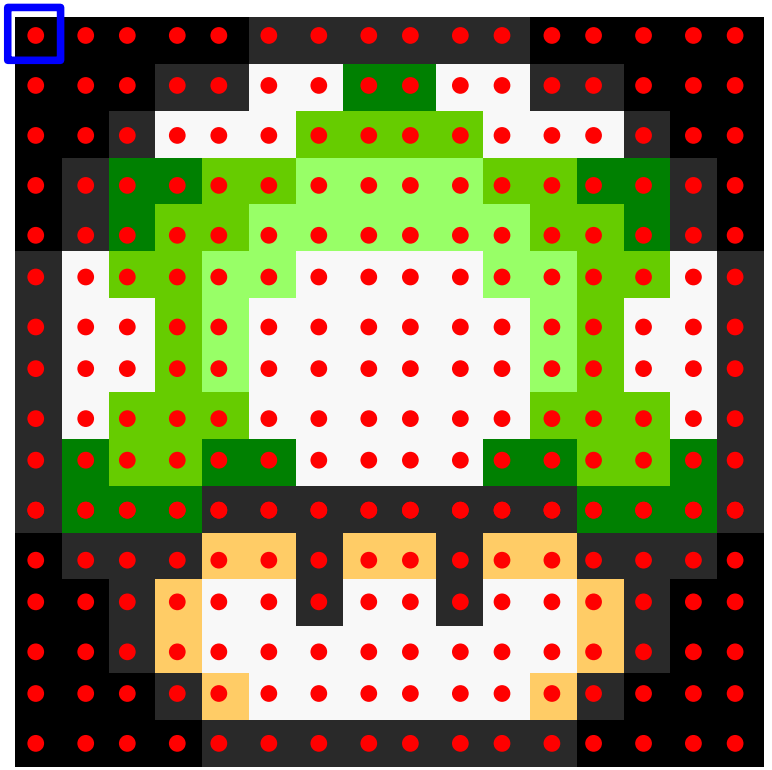

cubic spline



# Image pixels

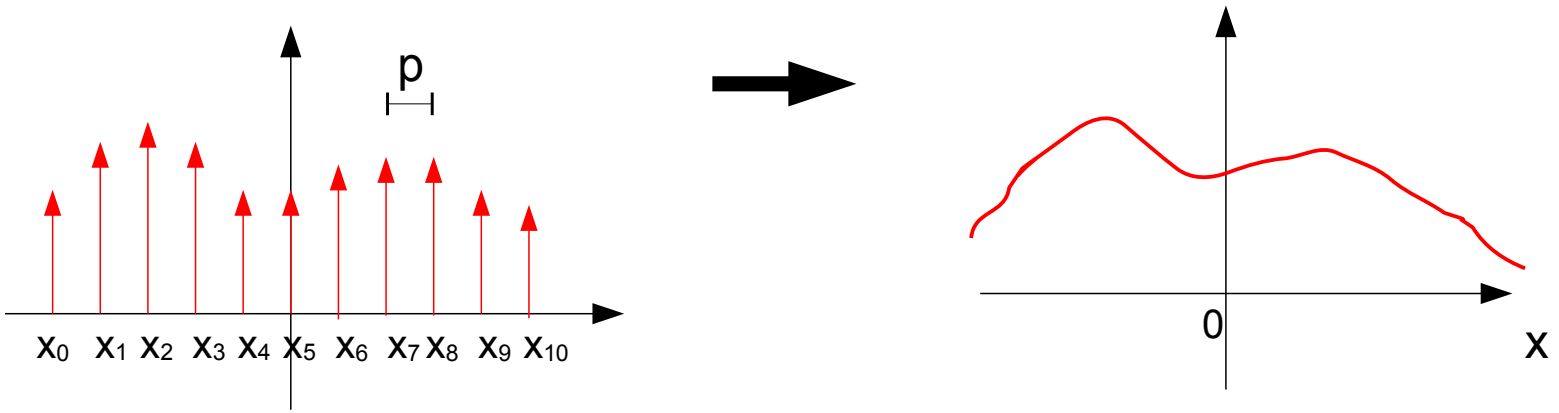
Images are discrete samples of a continuous function

- ...with coordinates
- ...and values (voltage at coordinate, integral over pixel area, ...)
- ...represented by pixel basis functions on a sampling grid



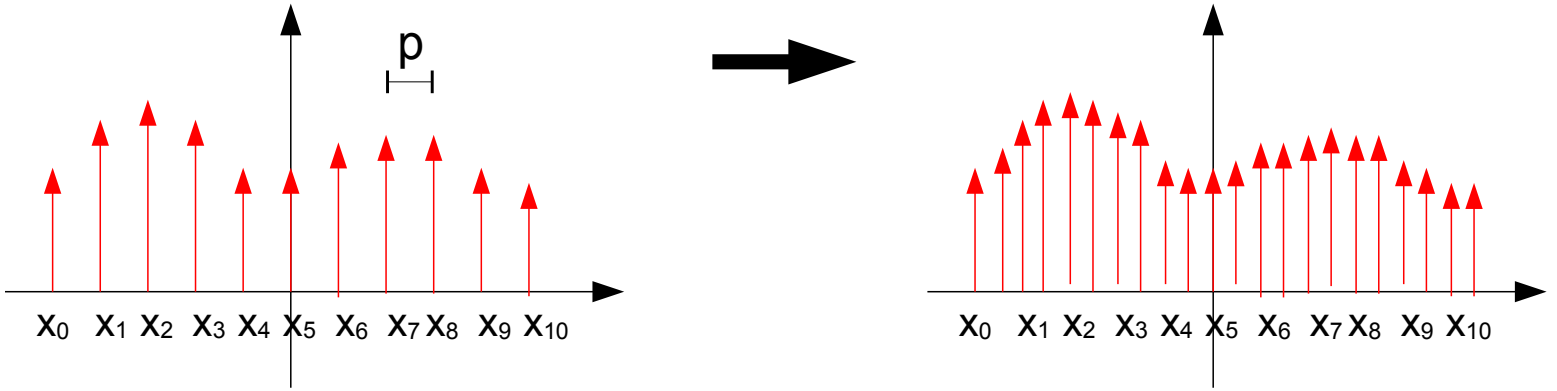
# Linear interpolation

- Interpolation as an operator
- Linear interpolation
- Shift invariance
- Kernel



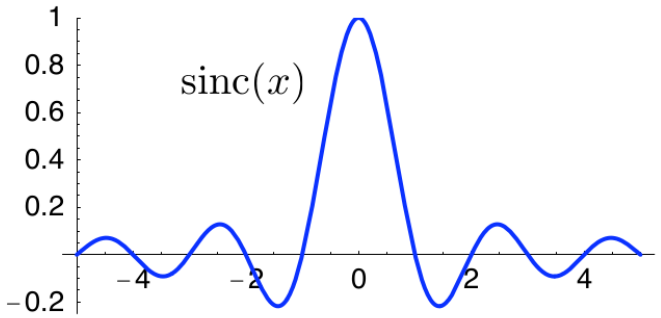
# Linear interpolation

- Linear interpolation can be written as a convolution with a kernel (e.g. a basis function)

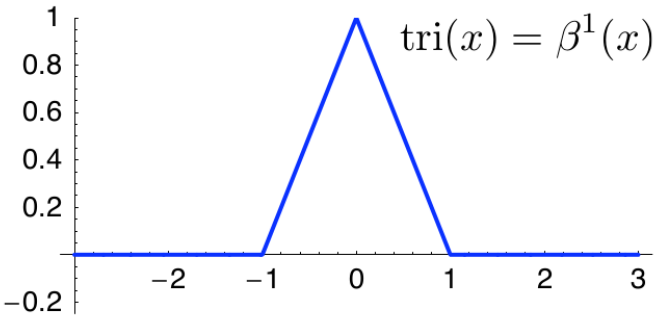


# Linear interpolation

- Bandlimited



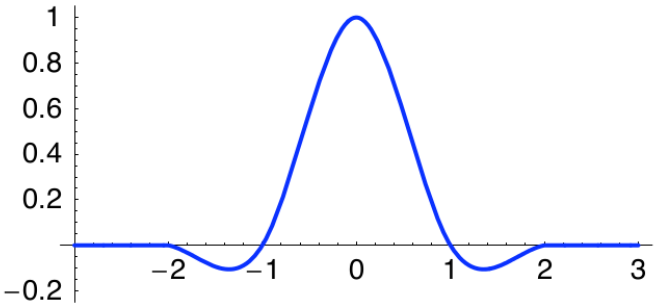

- Piecewise linear




Interpolation condition:

$$\varphi_{\mathrm{int}}(k) = \delta_k = \begin{cases} 1, & k = 0 \\ 0, & \text{otherwise} \end{cases}$$

- Cubic convolution



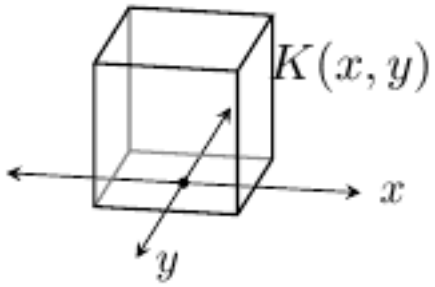

[Keys, 1981; Karup-King 1899]

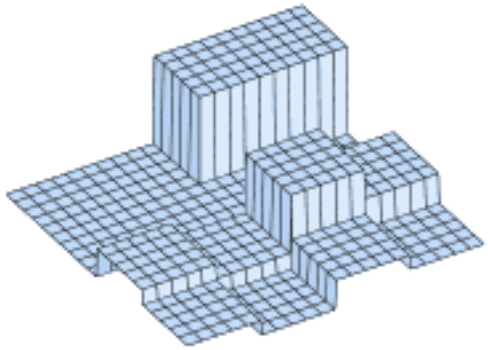
source: http://bigwww.epfl.ch/tutorials/unser_isbi_06_part1

# 2D interpolation

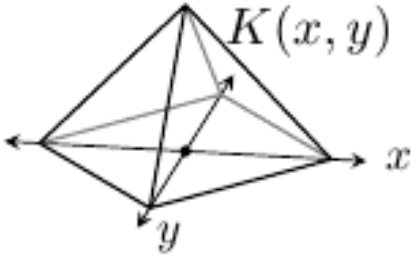
- Make 2D interpolation linear in each variable

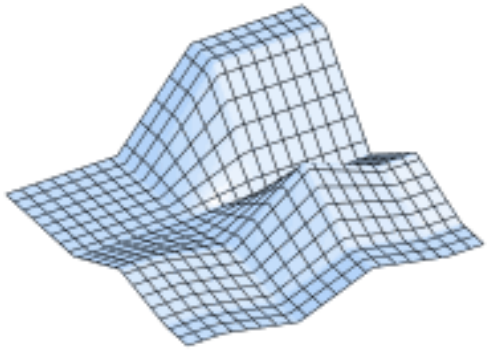
nearest neighbor



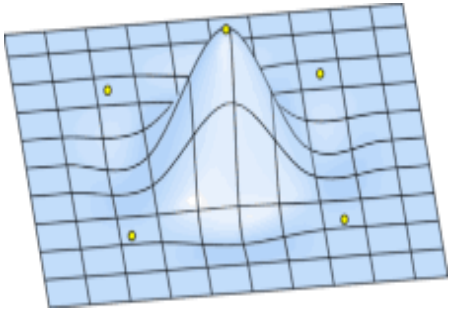



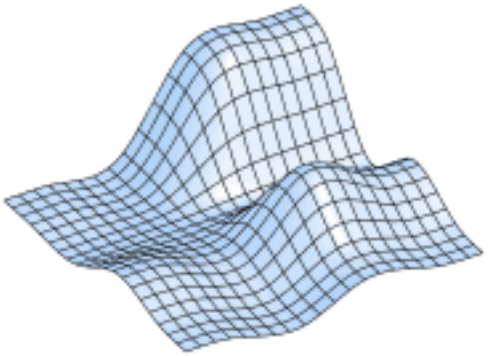
bilinear



bicubic

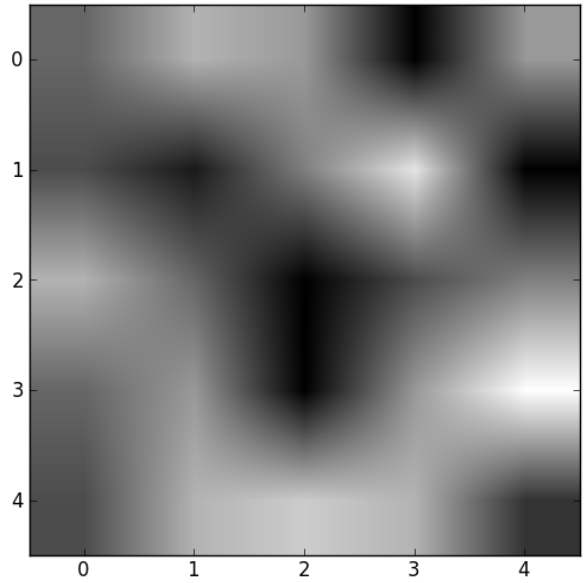




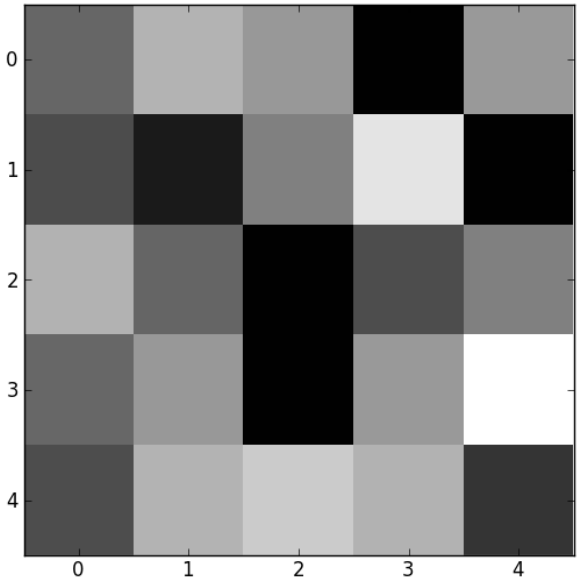
source: http://www.ipol.im/pub/art/2011/g_lmii/

# Python plotting

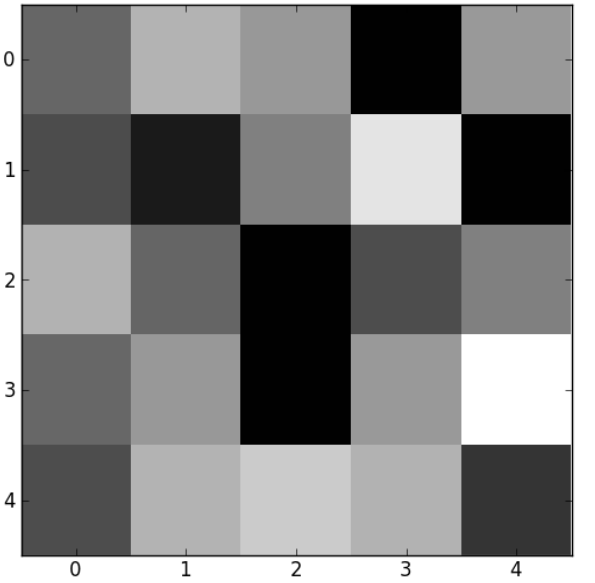
plt.imshow(im)



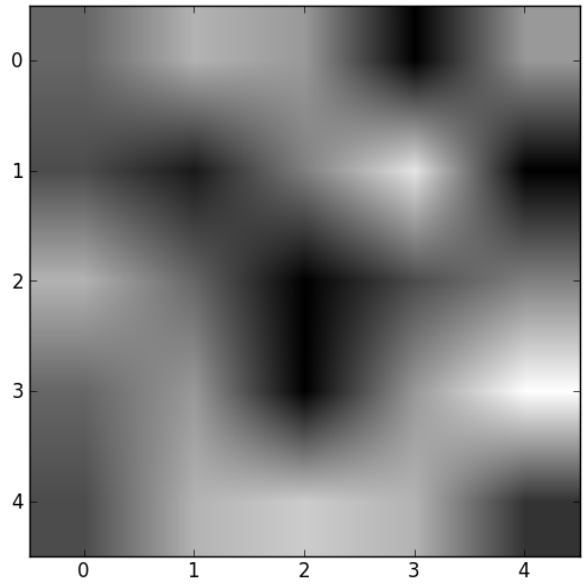

plt.imshow(im, interpolation='none')



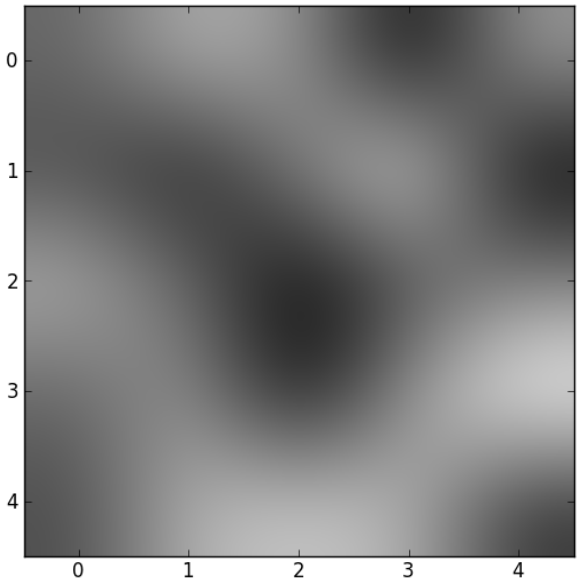

plt.imshow(im, interpolation='nearest')



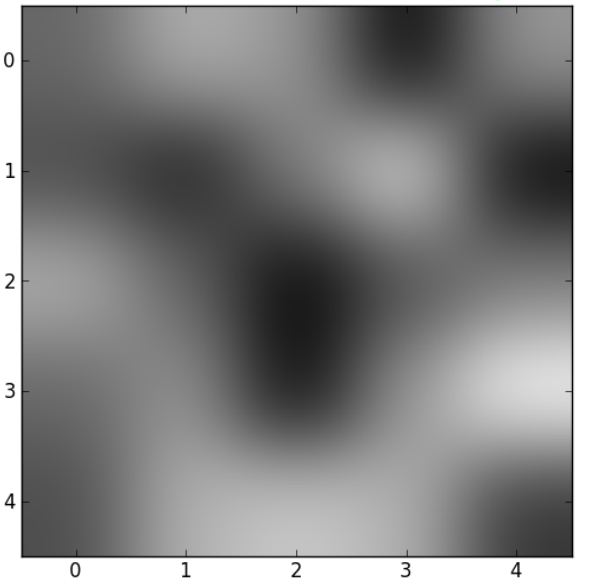

plt.imshow(im, interpolation='bilinear')




plt.imshow(im, interpolation='bicubic')

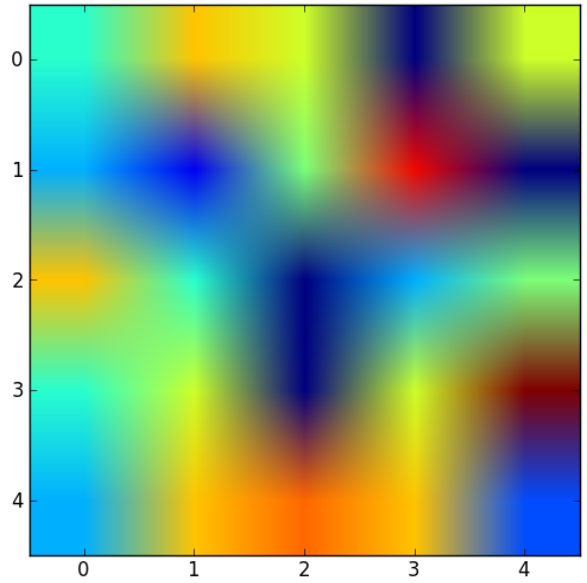



plt.imshow(im, interpolation='gaussian')

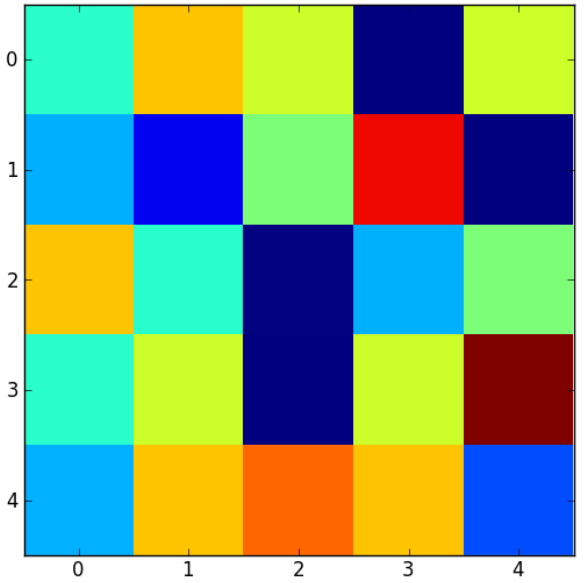


# Python plotting

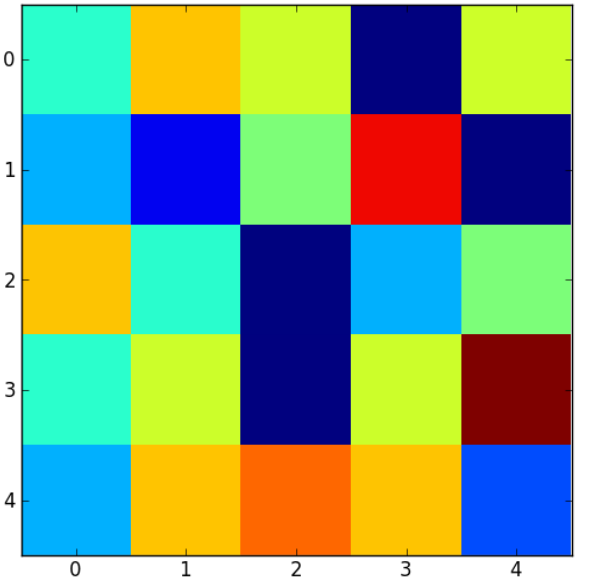
plt.imshow(im)



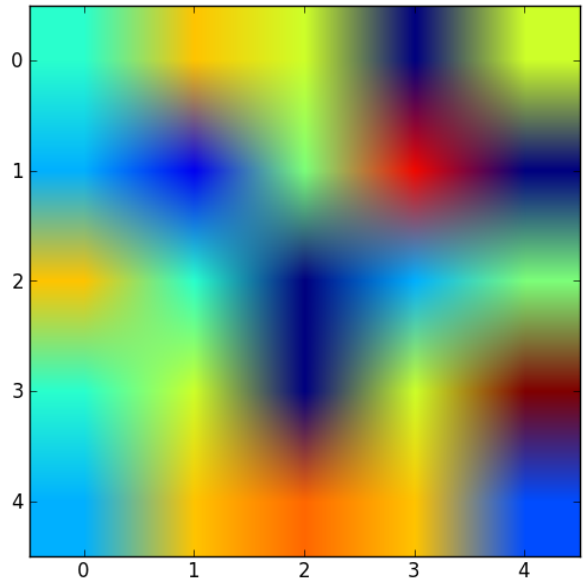

plt.imshow(im, interpolation='none')



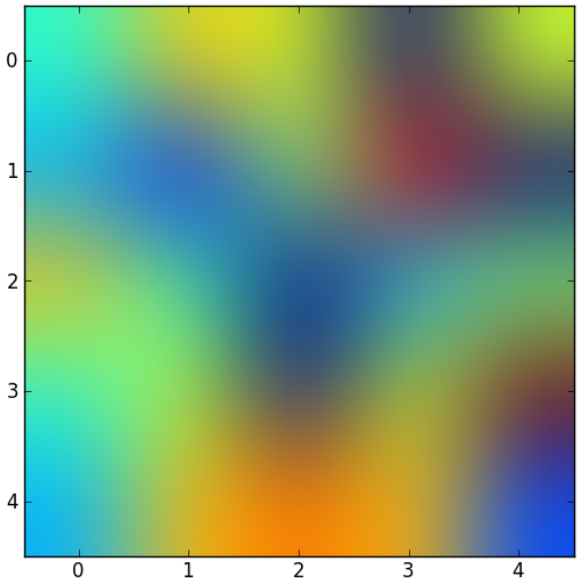

plt.imshow(im, interpolation='nearest')



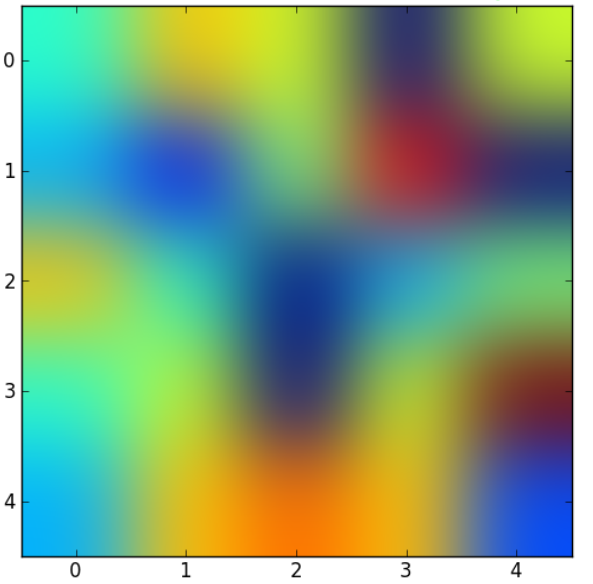

plt.imshow(im, interpolation='bilinear')




plt.imshow(im, interpolation='bicubic')

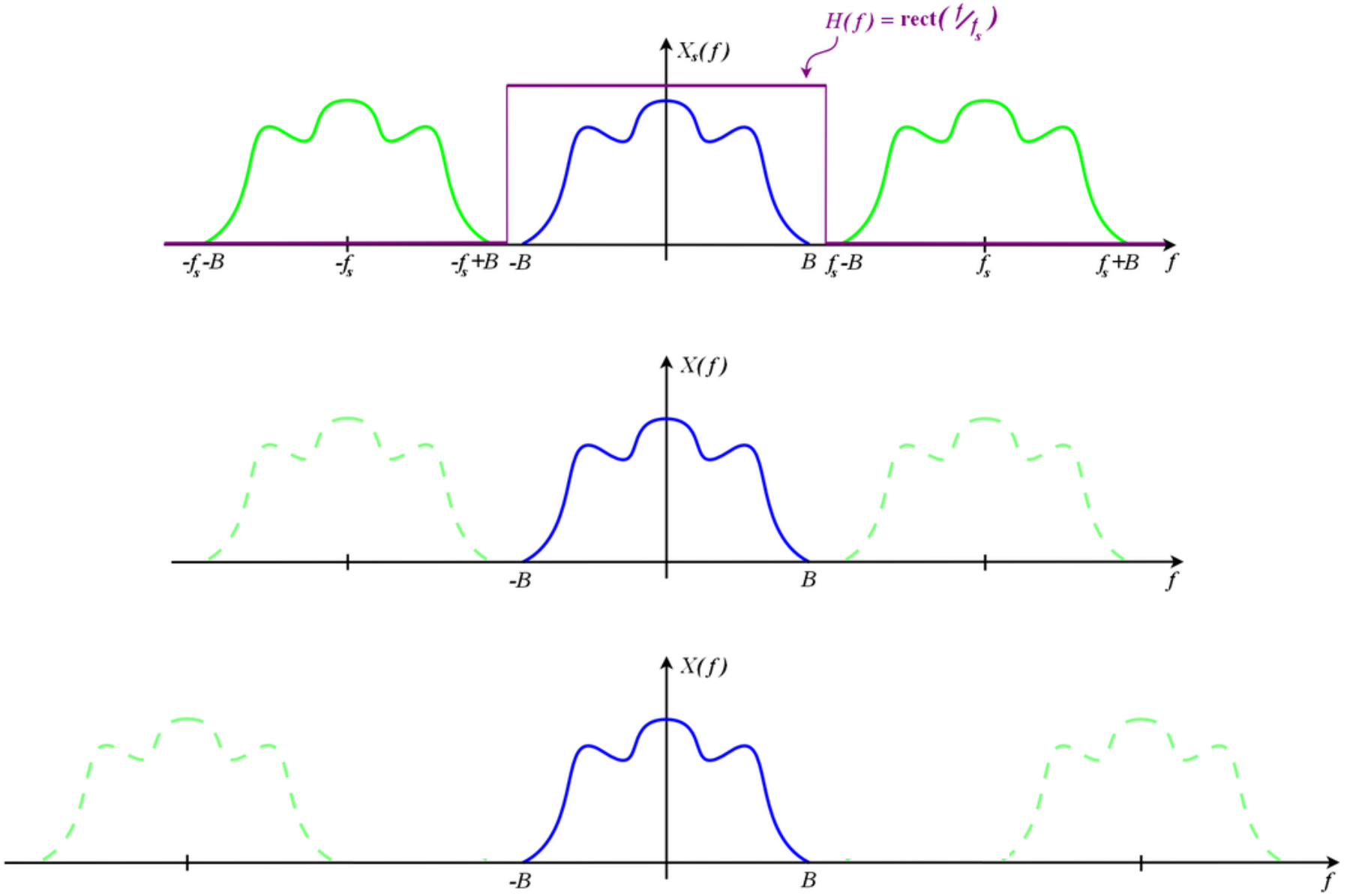



plt.imshow(im, interpolation='gaussian')



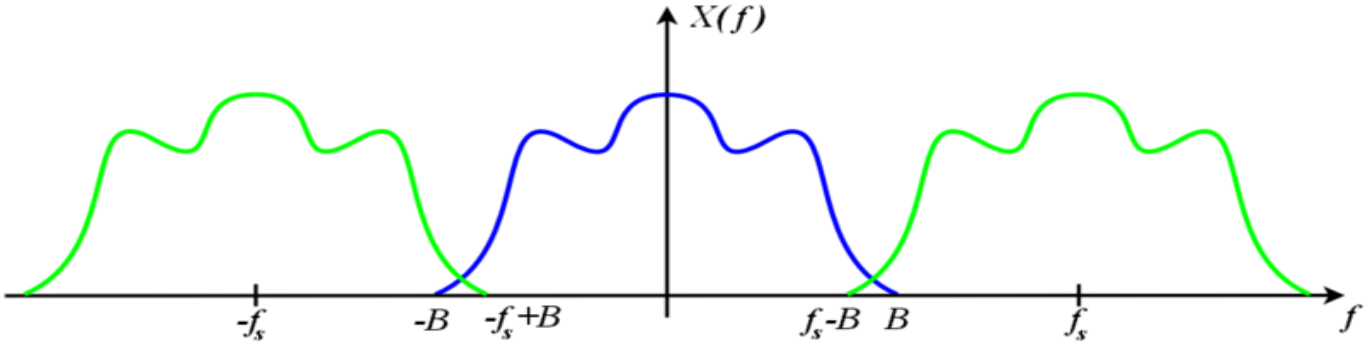
# Sinc interpolation and zero-padding

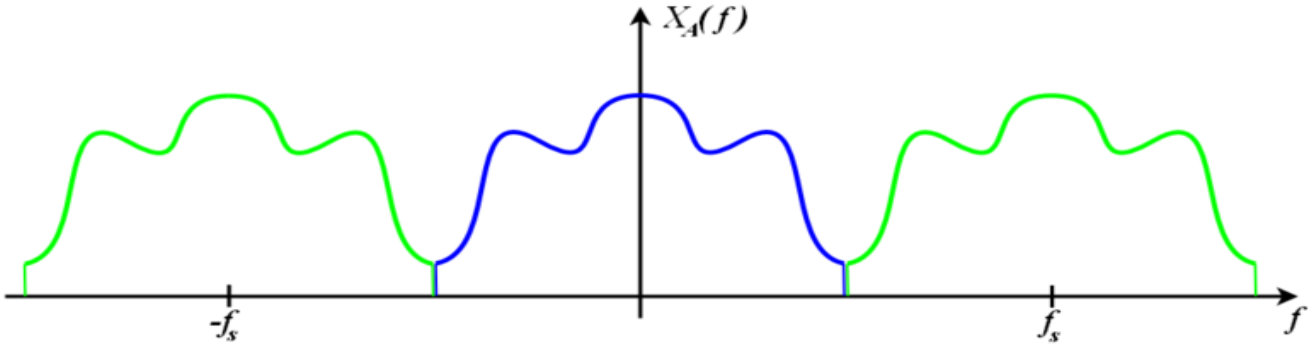
## Also known as "Whittaker–Shannon interpolation"

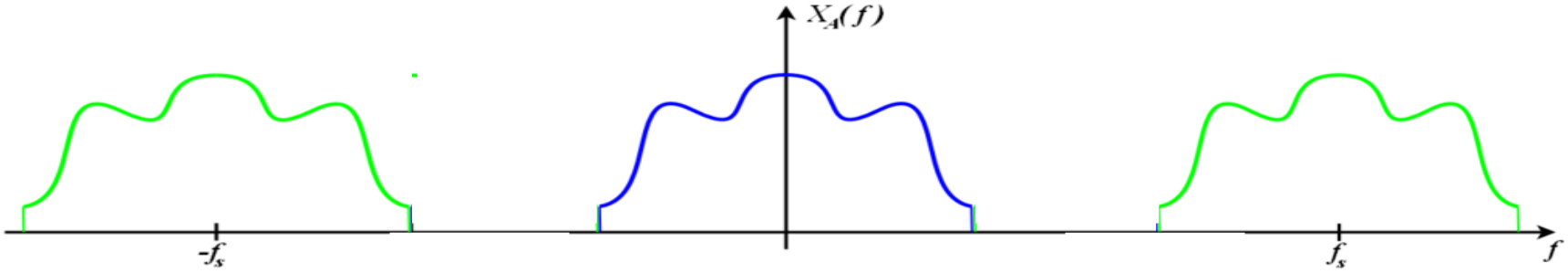


# Sinc interpolation and zero-padding

**Also known as "Whittaker–Shannon interpolation"**

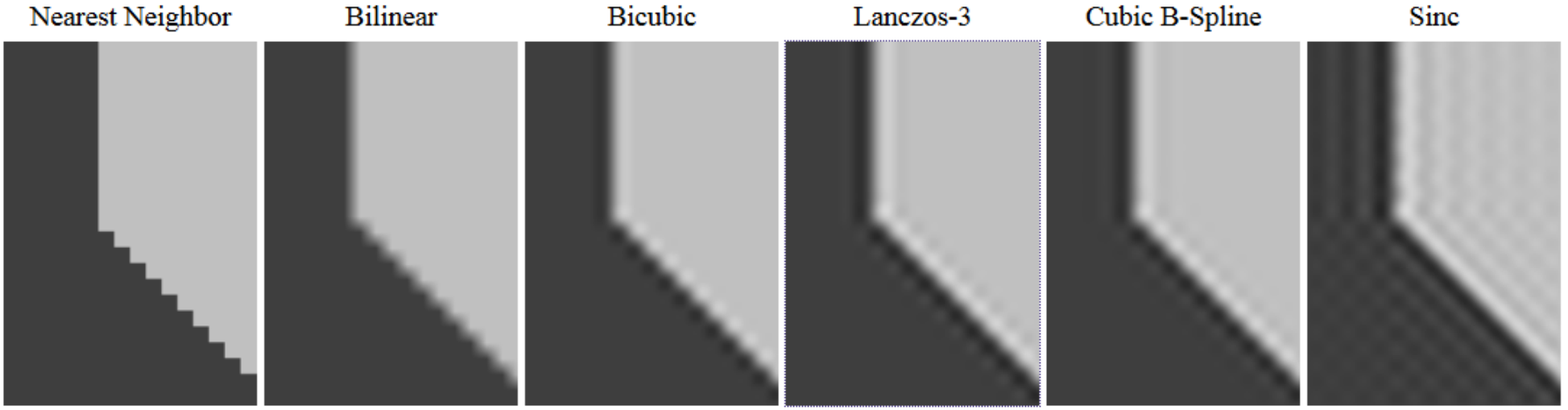


# Reconstruction from samples

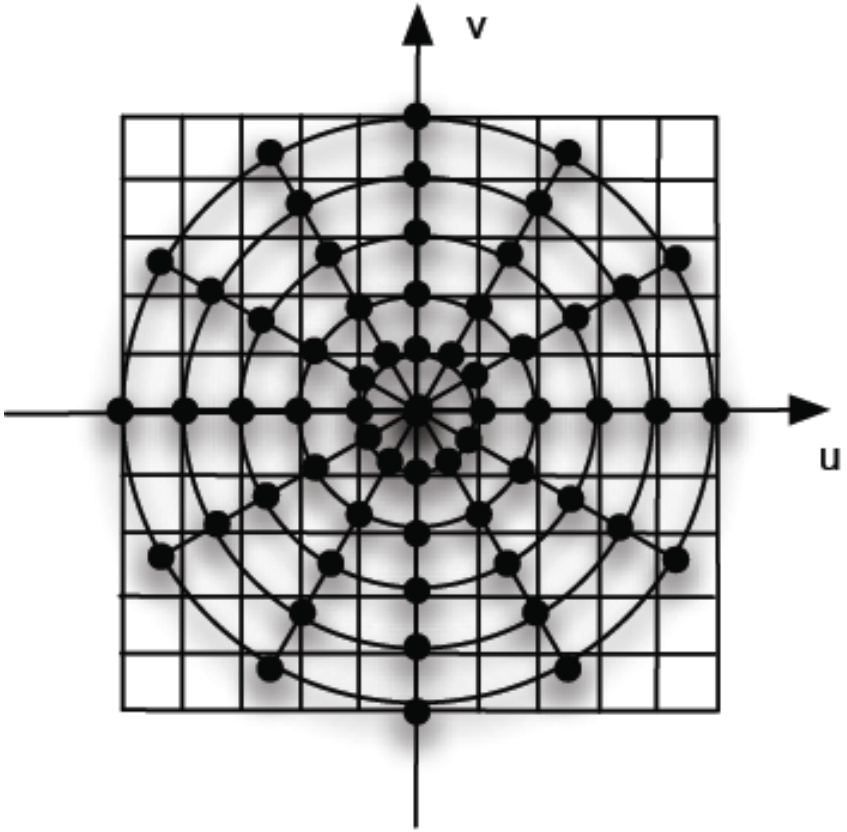
- Sinc interpolation can perfectly reconstruct a function from its samples if
  - sampled at a rate higher than Nyquist rate
  - bandlimited up to Nyquist frequency
  - no aliasing
- Sinc interpolation introduces ringing otherwise, due to leakage of aliased frequencies



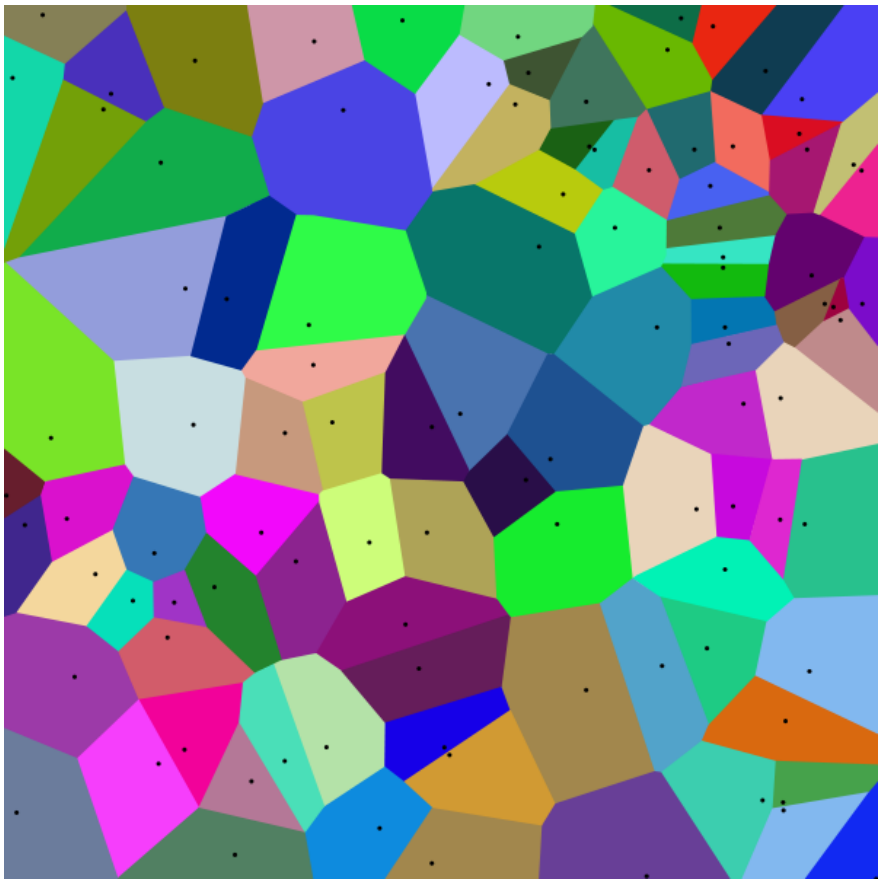

*Linear interpolation of a step edge: a balance between staircase artifacts and ripples.*

# Other Interpolation

- Change from polar to cartesian grid
- Linear, but not translation invariant




polar vs. cartesian sampling



irregular sampling

# Example: log-polar coordinates

# Summary

- Images can be represented as a sampling grid and pixel basis functions
- Need for interpolation arises when changing the grid
- Linear and translation invariant interpolation can be written as a convolution with an interpolation kernel function
- Typical interpolation kernels include nearest neighbor, linear, cubic and higher B-spline interpolation
- Zero-padding in one domain equals sinc interpolation in the other
- "ideal" sinc interpolation may lead to ringing artifacts