October 17

For $f \in C^0([a,b], X)$ $\quad X$ a Bspace

then $\int_a^b f(t)\,dt$ is well defined in the sense of Riemann integral.

$x_0 = a < x_1 < \cdots < x_n = b \quad \Delta$

$x_j^* \in [x_{j-1}, x_j]$

$S_\Delta f = \sum_{j=1}^{n} (x_j - x_{j-1}) f(x_j^*)$

$|\Delta| = \max\{ x_j - x_{j-1} : j = 1, \dots, n \}$ $\qquad \exists\ A \in X$ s.t.

$\lim_{|\Delta| \to 0} S_\Delta f = A \in X$

and this we call $A = \int_a^b f(t)\,dt$.

For $\varepsilon > 0$ arbitrary $\exists\ \delta > 0$ s.t.

$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.$ $\quad x, y \in [a,b]$

so if $|\Delta| < \delta$

and take Δ_1 and Δ_2 both with $|\Delta_j| < \delta$

$$| S_{\Delta_1} f - S_{\Delta_2} f | \le |S_{\Delta_1} f - S_{\Delta_3} f| + |S_{\Delta_3} f - S_{\Delta_2} f| < 2(b-a)\varepsilon$$

where Δ_3 is a refinement of Δ_1 and Δ_2

$\Rightarrow |\Delta_3| < \delta$

$\Delta_1 \quad x_0 = a < \cdots < x_n = b$

$\Delta_3 \quad y_0 = a < \cdots < y_N = b \qquad N \geq n$

$$\|S_{\Delta_1} f - S_{\Delta_3} f\| = \left\| \sum_{j=1}^{n} (x_j - x_{j-1}) f(x_j^*) - \sum_{\ell=1}^{N} (y_\ell - y_{\ell-1}) f(y_\ell^*) \right\|$$

$$\le \sum_{\ell=1}^{N} (y_\ell - y_{\ell-1}) \overbrace{\| f(x^*_{j(\ell)}) - f(y_\ell^*) \|}^{<\varepsilon}$$

$y_\ell^*, x^*_{j(\ell)} \in [x_{j(\ell)-1}, x_{j(\ell)}]$ with length less than δ.

$|x^*_{j(\ell)} - y^*_\ell| < \delta \Rightarrow \| f(x^*_{j(\ell)}) - f(y^*_\ell) \| < \varepsilon$

$\le (b-a)\varepsilon$

Def Suppose X and Y are B-spaces and $T\in\mathcal{L}(X,Y)$

$\|T\|_{\mathcal{L}(X,Y)}$

$X'=\mathcal{L}(X,K)$; $Y'=\mathcal{L}(Y,K)$

$X \xrightarrow{T} Y \xrightarrow{y'} K$

$Y' \xrightarrow{T'} X'$

$T'y' = y'\circ T \in X'$

$\cancel{X}\times X' \longrightarrow K$ $\quad (x,x')\to \langle x,x'\rangle_{X\times X'} =$

$Y\times Y' \longrightarrow K = \mathbb{R}$ $\quad = \langle x',x\rangle_{X'\times X} = x'(x)$

$\|T'\|_{\mathcal{L}(Y',X')} = \|T\|_{\mathcal{L}(X,Y)}$

$\|T'\|_{\mathcal{L}(Y',X')} \leq \|T\|_{\mathcal{L}(X,Y)}$

Let $\|y'\|_{Y'}=1$

$\|T'y'\|_{X'} \boxed{=} \sup\{|\langle \underbrace{T'y'}_{y'\circ T},x\rangle_{X'\times X}| : \|x\|_X=1\}$ $\quad T'=T^*$

$= \sup\{|\underbrace{\langle y',Tx\rangle_{Y'\times Y}}_{\|y'\|_{Y'}\|Tx\|_Y}| : \|x\|_X=1\}$

$\leq \underbrace{\|y'\|_{Y'}}_{=1} \underbrace{\sup\{\|Tx\|_Y : \|x\|_X=1\}}_{\|T\|_{\mathcal{L}(X,Y)}}$

$\|T'y'\|_{X'} \leq \|T\|_{\mathcal{L}(X,Y)}$ $\quad \forall\ y'\in Y'$ with $\|y'\|_{Y'}=1$

$\Downarrow$

$\|T'\|_{\mathcal{L}(Y',X')} \leq \|T\|_{\mathcal{L}(X,Y)}$

$\geq$

Let $\|x\|_X=1$

$\|Tx\|_Y = \sup\{|\langle Tx,y'\rangle_{Y\times Y'}| : \|y'\|_{Y'}=1\}$ $\quad \langle x,y'\circ T\rangle_{X\times X'}$

$\cancel{\leq \sup\{\|Tx\|_Y \underbrace{\|y'\|_{Y'}}_{1} : \|y'\|_{Y'}=1\}}$

$= \|Tx\|_Y$

$= \sup\{|\langle x,T'y'\rangle_{X\times X'}| : \|y'\|_{Y'}=1\}$

$\leq \sup\{\underbrace{\|x\|_X}_{1}\|T'y'\|_{X'} : \|y'\|_{Y'}=1\}$

$= \|T'\|_{\mathcal{L}(Y',X')}$

$\|Tx\|_Y \leq \|T'\|_{\mathcal{L}(Y',X')}$ $\quad \forall\ x$ with $\|x\|_X=1$

$\|T\|_{\mathcal{L}(X,Y)} \leq \|T'\|_{\mathcal{L}(Y',X')}$

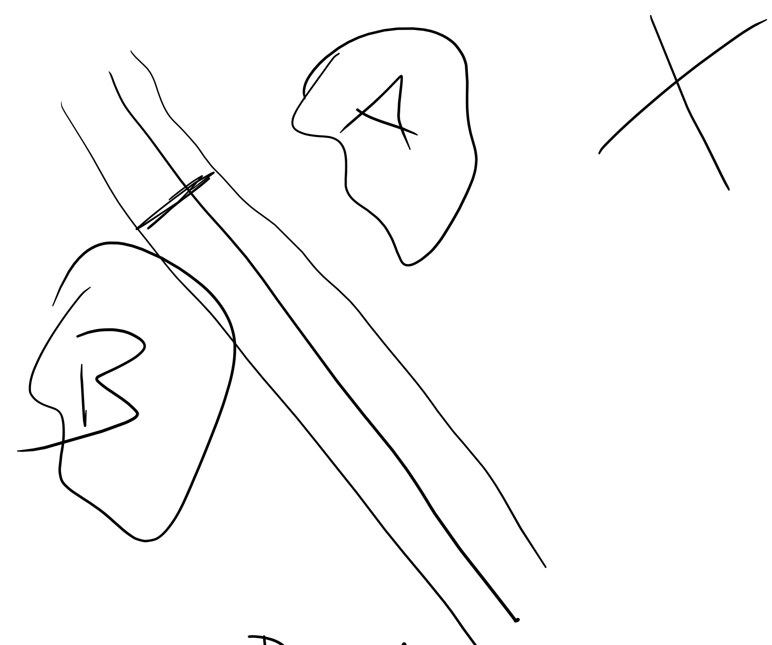
Def Let A and B be nonempty subsets of a vector space X on $\mathbb{R}$, and let $f: X \to \mathbb{R}$ be linear. Let $H = f^{-1}(a)$

$f \not\equiv 0$

1) We say that H separates A and B if

$f(A) \subseteq (-\infty, a]$

and $f(B) \subseteq [a, +\infty)$



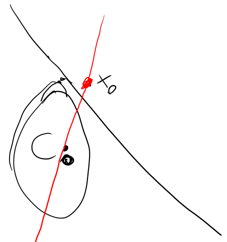
2) H separates strictly A and B if $\exists\, \varepsilon > 0$ s.t.

$f(A) \subseteq (-\infty, a-\varepsilon]$, $f(B) \subseteq [a+\varepsilon, +\infty)$

$L^p(0,1)$ $0<p<1$

Theorem X (on $\mathbb{R}$) a TVS, C an open convex non empty subspace of X, $x_0 \notin C$. Then $\exists$ $f \in X'$ s.t. $\boxed{f(x) < f(x_0)}$ $\forall x \in C$

Pf By translation we can assume $0 \in C$



p the Minkowski functional of C.

$p(x) < 1 \iff x \in C$

$\implies p(x_0) \geq 1$ because $x_0 \notin C$

$Y = \mathbb{R}x_0 \xrightarrow{g} \mathbb{R}$

$g(tx_0) = t$ $\quad g(x_0) = 1 \leq p(x_0)$

$\implies g(tx_0) \leq p(tx_0) \quad \forall t \in \mathbb{R}$

Then we know there exists a linear map

$f : X \longrightarrow \mathbb{R}$ s.t. $f|_Y = g$

and $f(x) \leq p(x)$ $\forall x \in X$.

Then $\forall x \in C$

$f(x) \leq p(x) < 1 = f(x_0) = g(x_0)$

We need to show that $f \in X'$.

We know $f^{-1}(1) \cap \underset{\text{open}}{C} = \emptyset$

$\implies f$ is continuous.

In fact, if f is not continuous, $f^{-1}(a)$ is dense in X $\forall a \in \mathbb{R}$

$f \notin X' \iff \ker f$ is dense in X

$f \in X' \iff \ker f$ is closed

$x_0 \in f^{-1}(1) = x_0 + \ker f$

Theorem (1° geom form of Hahn-Banach)

X a TVS, A and B non empty disjoint and convex, A open. Then $\exists$ a closed hyperplane H which separates them.

Pf $C = A - B = \{a-b : a \in A \text{ and } b \in B\}$

C is convex and is open $C = \bigcup_{b\in B} (A-b)$ and

$0 \notin C$ because otherwise $\exists a,b$ $0 = a-b \Rightarrow a=b$

By the lemma $\exists f \in X'$, $X \to \mathbb{R}$,

s.t. $f(c) < f(0) = 0 \quad \forall c \in C = A-B$

$f(a-b) < 0 \quad \forall a \in A, b \in B$

$f(a) < f(b) \quad \forall a \in A, b \in B$

$\alpha \in \mathbb{R}$

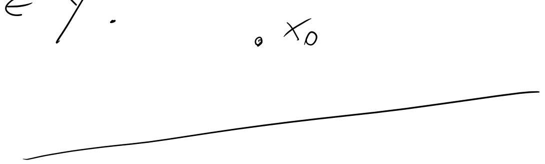
$f(a) \leq \alpha \leq f(b) \quad \forall a \in A, b \in B$

$H = f^{-1}(\alpha)$

Theorem (2nd geom form, Hahn-Banach)

X locally convex, A and B nonempty disjoint convex sets, A closed, B compact. Then $\exists$ a closed Hyperplane H separating them strictly.

Corollary Let Y be a vector subspace of a locally convex space X on $\mathbb{R}$. Assume $\overline{Y} \subsetneq X$. Then $\exists f \in X'$ $f \not\equiv 0$ s.t. $f(y) = 0 \ \forall y \in Y$.



Pf $x_0 \notin \overline{Y}$

$A = \overline{Y}$ $\quad B = \{x_0\} \Rightarrow \exists \ f \in X'$ and an $\alpha \in \mathbb{R}$ s.t. $H = f^{-1}(\alpha)$ separates strictly A and B, in particular

$$f(y) < \alpha < f(x_0) \quad \forall y \in Y$$

$$\underset{\overset{\|}{0}}{f(y)} < f(x_0) \quad \forall y \in Y$$

If $f(y_0) \neq 0$ $\quad f(\lambda y_0) = \lambda f(y_0) < f(x_0)$ $\forall \lambda \in \mathbb{R}$

$$\sup_{\lambda \in \mathbb{R}} \lambda f(y_0) = +\infty.$$

$$f(y) = 0 < f(x_0) \quad \forall y \in Y$$