Contents

Chapter 5

Wind-Driven Circulation

We will now use and integrate both Ekman theory and the geostrophic approximation to get a first solution to the wind-driven circulation. They will be the basis for the theory of the wind-driven gyres. At first, the theory will be simple, with no topography and in a steady state, but it will be able to explain many of the qualitative features of the wind-driven circulation.

The first theory presented is the steady, forced-dissipative, homogeneous model first formulated by Stommel; and different versions will be discussed.

We start with the simplest model that can capture our physical setting. We will assume (see Fig. 5.1)

- a homogeneous (or depth-integrated) model.
- Flat bottom.
- Steady state.
- The β-plane approximation.

Let's now remember the solutions for the top and bottom Ekman vertical velocities, and the momentum equations for the geostrophic flow:

$$w_E^T = \frac{1}{f_0}\left(\partial_x \tilde{\tau}^y - \partial_y \tilde{\tau}^x\right) = \frac{1}{f_0}\,\mathrm{curl}_z\,\tilde{\boldsymbol{\tau}}_T \quad = \quad \frac{1}{\rho_0 f_0}\mathrm{curl}_z\,\boldsymbol{\tau}_T \tag{5.1}$$

$$w_E^B = -\frac{1}{\rho_0}\,\nabla\cdot\boldsymbol{M}_E = \frac{1}{f_0}\,\mathrm{curl}_z\,\tilde{\boldsymbol{\tau}}_B \quad = \quad \frac{d}{2}\zeta_g, \tag{5.2}$$

where $\zeta_g = (\partial_x v_g - \partial_y u_g)$ is the vorticity of the interior geostrophic flow.

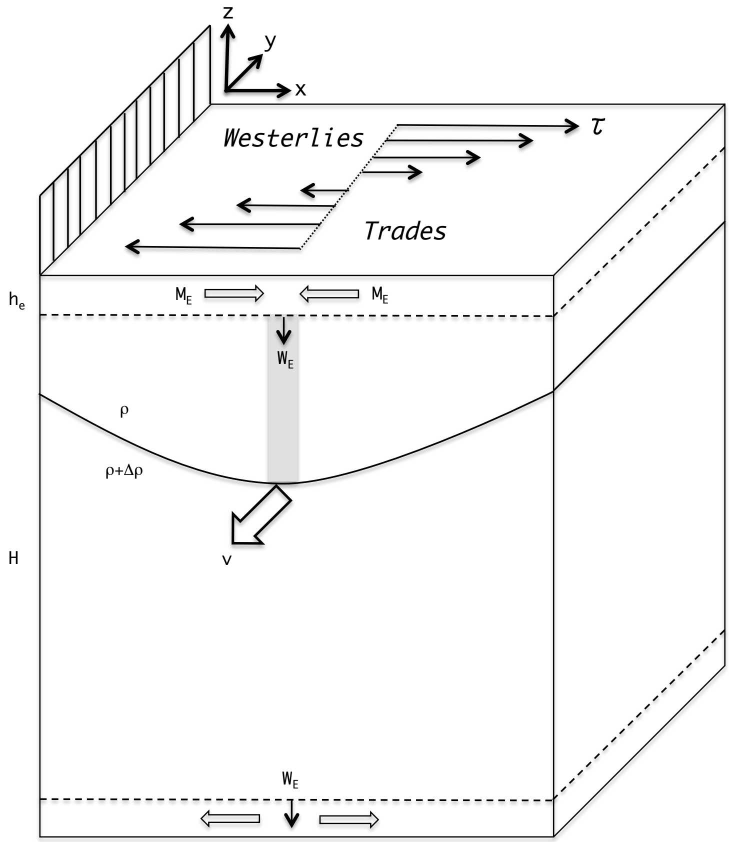

Figure 5.1: *A schematic of an idealized wind-driven Ekman pumping on a β-plane for a homogeneous ocean of depth H, resulting in a simple model for mid-latitude ocean circulation.*

The interior geostrophic flow (for a homogeneous barotropic fluid in which $\rho' = 0$) is

$$-fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \tag{5.3}$$

$$fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} \tag{5.4}$$

$$0 = \frac{1}{\rho_0}\frac{\partial p}{\partial z} \tag{5.5}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{5.6}$$

5.1 A linear geostrophic vorticity balance approach: Sverdrup Balance

Within a β-plane, the interior geostrophic flow becomes

$$-(f_0 + \beta y)v = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \tag{5.7}$$

$$(f_0 + \beta y)u = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} \tag{5.8}$$

$$0 = \frac{1}{\rho}\frac{\partial p}{\partial z} \tag{5.9}$$

$$\nabla_3 \cdot \boldsymbol{v} = 0. \tag{5.10}$$

And we will use $\boldsymbol{v} = (u, v, w)$ and $\boldsymbol{u} = (u, v)$.

Cross-differentiating the horizontal momentum equations [∂_x(5.8)-∂_y(5.7)] gives:

$$f_0\Big(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\Big) + \beta y\Big(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\Big) + \beta v = 0. \tag{5.11}$$

But since in a β-plane $\beta y \ll f_0$, we have

$$f_0\Big(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\Big) + \beta v = 0. \tag{5.12}$$

or

$$\boxed{\beta v = f_0\frac{\partial w}{\partial z}} \tag{5.13}$$

Which is a form of the linear geostrophic vorticity balance, and is known as **SVERDRUP BALANCE**.

Eq.5.13 expresses a conservation of potential vorticity. If $\frac{\partial w}{\partial z} > 0$, there will be stretching of the fluid column. *As the column stretches and shrinks it has to increase its vorticity in order to conserve angular momentum*. At large scales, the only significant vorticity is the planetary vorticity f, which in this case has to increase to balance the positive $\frac{\partial w}{\partial z}$. β is indeed a rate of vorticity change $\left(\frac{\partial f}{\partial y}\right)$. This balance is responsible for a meridional velocity v.

Geostrophy was previously studied on a f-plane, resulting in $w = 0$. We now find a vertical velocity within the geostrophic flow using the β-plane. If $\beta = 0 = \frac{\partial f}{\partial y}$, then the vertical geostrophic velocity is $w = 0$.

What is the structure of $\frac{\partial w}{\partial z}$?

Taking the vertical derivative of the horizontal momentum equations

$$-(f_0 + \beta y)\frac{\partial v}{\partial z} = -\frac{1}{\rho_0}\frac{\partial^2 p}{\partial x \partial z} \tag{5.14}$$

$$(f_0 + \beta y)\frac{\partial u}{\partial z} = -\frac{1}{\rho_0}\frac{\partial^2 p}{\partial y \partial z}. \tag{5.15}$$

But $\frac{\partial p}{\partial z} = 0$. Hence $\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0$ and the flow is barotropic and there is no vertical shear. $\frac{\partial w}{\partial z}$ is constant throughout the interior and different from zero.

Now take a vertical derivative of the vertical velocity, and remembering the Ekman solutions we find

$$\frac{\partial w}{\partial z} = \frac{w_T - w_B}{H} = \frac{1}{\rho_0 f_0 H} curl_z \boldsymbol{\tau} - \frac{d}{2H}\zeta_g, \tag{5.16}$$

where H is the depth of the interior flow.

Using the geostrophic expressions for the horizontal velocities

$$-\frac{\partial v}{\partial x} = -\frac{1}{\rho_0 f_0}\frac{\partial^2 p}{\partial x^2} \tag{5.17}$$

$$\frac{\partial u}{\partial y} = -\frac{1}{\rho_0 f_0}\frac{\partial^2 p}{\partial y^2} \tag{5.18}$$

our solution $\beta v = f_0 \frac{\partial w}{\partial z}$ becomes

$$\frac{\beta}{\rho_0 f_0^2}\frac{\partial p}{\partial x} = \frac{1}{\rho_0 f_0 H} curl_z \boldsymbol{\tau} - \frac{d}{2H\rho_0 f_0}\Big(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2}\Big), \tag{5.19}$$

since $\zeta = (\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})$. Or

$$\boxed{\underbrace{\beta\frac{\partial p}{\partial x}}_{\text{meridional velocity}} = \frac{f_0}{H}\Big(\underbrace{curl_z \boldsymbol{\tau}}_{\text{Ekman at the top}} - \underbrace{\frac{d}{2}\nabla^2 p}_{\text{Ekman at the bottom}}\Big)} \tag{5.20}$$

This is the governing equation for the ocean interior, away from the Ekman layers. It is driven by input of momentum at the surface and drag at the bottom.

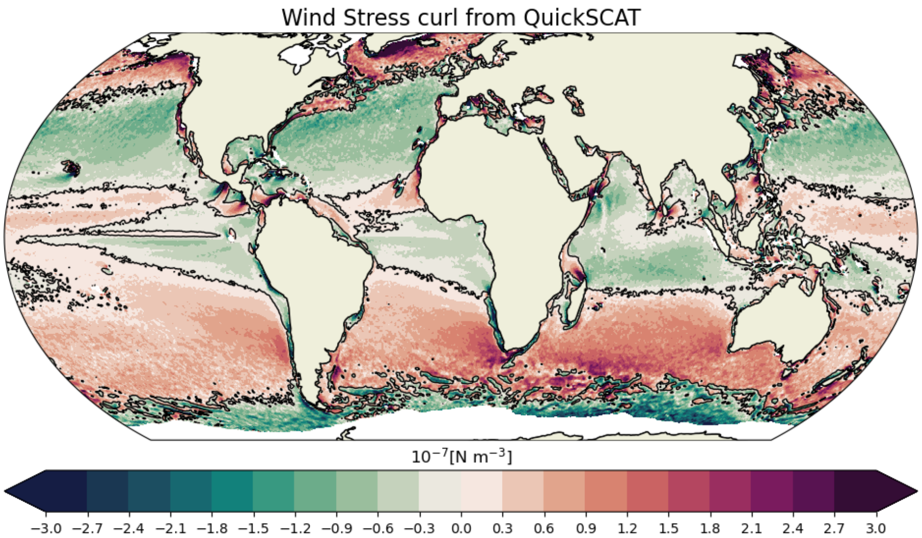

Figure 5.2: *Wind stress curl computed from QuickSCAT reanalysis [https://doi.org/10.1175/2008JPO3881.1].*

5.2 The Stommel model

We will now use the planetary-geostrophic equations. Let's define $\phi = p/\rho_0$ and $b = -g\rho'/\rho_0$. For a Boussinesq fluid, the planetary geostrophic equations are

$$\mathbf{f} \times \mathbf{u} = -\nabla\phi + \frac{1}{\rho_0}\frac{\partial \boldsymbol{\tau}}{\partial z} \tag{5.21}$$

$$\frac{\partial \phi}{\partial z} = b \tag{5.22}$$

$$\nabla_3 \cdot \mathbf{v} = 0. \tag{5.23}$$

The first equation is the horizontal momentum equation using geostrophic balance and a stress term. The second equation is the vertical momentum equation (hydrostatic balance). And the third is mass continuity.

The planetary geostrophic equations are essentially the Boussinesq primitive equations with the advection terms omitted in the horizontal momentum equation. They have been derived with a 'low Rossby number scaling', but for large scales, much larger than the deformation scale. Hence, this set of equations are composed of the geostrophic balance and the full mass continuity equations. These equations are not too useful in the atmosphere, where the deformation radius for a continuously stratified fluid, $L_d = \frac{NH}{f}$ (or $\frac{\sqrt{gH}}{f}$), is about 1000 km. Only the description of planetary waves can satisfy the PG equations. For the ocean, instead, where $L_d \simeq 100$ km, the PG equations are very useful, and used for the theory of large-scale circulation.

We now take the curl (or cross-differentiate) of (5.21) and find

$$\mathbf{f}\,\nabla \cdot \mathbf{u} + \frac{\partial f}{\partial y} v = curl_z \widetilde{\boldsymbol{\tau}} \tag{5.24}$$

where again $curl_z \boldsymbol{A} = \boldsymbol{k} \cdot \nabla \times \boldsymbol{A} = \partial_x A^y - \partial_y A^x$, and $\widetilde{\boldsymbol{\tau}} = \boldsymbol{\tau}/\rho_0$.

Now integrate over the full depth of the ocean

$$\int \mathbf{f}\,\nabla \cdot \mathbf{u}\,\mathrm{d}z + \frac{\partial f}{\partial y} \int v\,\mathrm{d}z = curl_z(\widetilde{\boldsymbol{\tau}}_T - \widetilde{\boldsymbol{\tau}}_B). \tag{5.25}$$

The first term vanishes, the divergence term, if the vertical velocities are zero at the top and bottom of the ocean. This is true for a flat-bottomed ocean but is not the case when topography will be added. We are thus left with:

$$\boxed{\beta \overline{v} = curl_z(\widetilde{\boldsymbol{\tau}}_T - \widetilde{\boldsymbol{\tau}}_B)}. \tag{5.26}$$

Where $\overline{A} = \int A \, dz$. Eq. (5.26) is equivalent to Eq. (5.13), i.e. the SVERDRUP BALANCE, *a balance between the input of vorticity from the wind-stress curl and the advection of planetary vorticity*.

We now work on the rhs of (5.26). At the top the stress is given by the wind. At the bottom, which is flat for now, we parameterize the stress with a LINEAR DRAG, or Rayleigh friction, as it would be generated by an Ekman layer, and obtain

$$\boxed{\beta \overline{v} = F_\tau(x, y) - r\overline{\zeta}}. \tag{5.27}$$

Here the meridional flow is governed by

1. $F_\tau(x, y) = curl_z \widetilde{\boldsymbol{\tau}}_T$; the curl of the wind stress at the top of the ocean.
2. $\overline{\zeta} = \frac{\partial \overline{v}}{\partial x} - \frac{\partial \overline{u}}{\partial y}$; the vorticity of the vertically integrated flow
3. r; a linear drag or Rayleigh friction.

The flow velocity is divergent-free and we can define a streamfunction

$$\overline{u} = -\frac{\partial \psi}{\partial y} \quad \overline{v} = \frac{\partial \psi}{\partial x}$$

such that

$$\boxed{\beta \frac{\partial \psi}{\partial x} = F_\tau(x, y) - r\nabla^2 \psi}. \tag{5.28}$$

This is the **STOMMEL PROBLEM or MODEL**. The contribution of Stommel is the addition of a linear bottom drag that would balance the momentum input at the surface.

5.2.1 A homogeneous model

Instead of vertically integrating our momentum equations, we can instead consider a homogeneous layer of fluid, obeying the shallow water equations. The potential vorticity equation becomes

$$\frac{D}{Dt}\left(\frac{f + \zeta}{H}\right) = \frac{F}{H}, \tag{5.29}$$

where F represents both forcing and friction. If the ocean is flat-bottomed and has a rigid lid, then

$$\frac{D\zeta}{Dt} + \beta v = F. \tag{5.30}$$

This is the barotropic PV equation. Because of the rigid lid and flat-bottom, the flow is divergent-free, and we can express it with the usual streamfunction:

$$\boxed{\frac{\mathrm{D}}{\mathrm{D}t}\nabla^2\psi + \beta\frac{\partial\psi}{\partial x} = F_\tau(x,y) - r\nabla^2\psi}. \tag{5.31}$$

The first term of the l.h.s characterizes the *time-dependent, non-linear* STOMMEL MODEL. The steady non-linear model is simply

$$J(\psi,\nabla^2\psi) + \beta\frac{\partial\psi}{\partial x} = F_\tau(x,y) - r\nabla^2\psi. \tag{5.32}$$

Where the Jacobian is

$$J(A,B) = \frac{\partial A}{\partial x}\frac{\partial B}{\partial y} - \frac{\partial A}{\partial y}\frac{\partial B}{\partial x}. \tag{5.33}$$

And so the advective term is

$$u\frac{\partial\nabla^2\psi}{\partial x} + v\frac{\partial\nabla^2\psi}{\partial y} \quad = \tag{5.34}$$

$$-\frac{\partial\psi}{\partial y}\frac{\partial\nabla^2\psi}{\partial x} + \frac{\partial\psi}{\partial x}\frac{\partial\nabla^2\psi}{\partial y} \quad = \tag{5.35}$$

$$\frac{\partial\psi}{\partial x}\frac{\partial\nabla^2\psi}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\nabla^2\psi}{\partial x} \quad = \tag{5.36}$$

$$J(\psi,\nabla^2\psi). \tag{5.37}$$

To recover the original Stommel model we need to ignore the advective derivative (the source of our non-linearities).

Take the barotropic PV equation (5.30) and perform a scale analysis of all terms:

$$\underbrace{\frac{\mathrm{D}\zeta}{\mathrm{D}t}}_{\frac{U}{L}\frac{U}{L}} + \underbrace{\beta v}_{\beta U} = F. \tag{5.38}$$

Let's define $Z = \frac{U}{L}$ a representative value for vorticity, so that in order to ignore nonlinearities the following inequality must hold: $Z \ll \beta L$, or

$$R_\beta = \frac{U}{\beta L^2} \ll 1 \tag{5.39}$$

which is called the β Rossby number[1]. Assuming a β Rossby number much smaller than unity is equivalent to the small Rossby number assumption used to obtain the PG equations.

[1]Remember that the Rossby number is $R_o = \frac{U}{fL}$

The response to an input of vorticity: relative vorticity or planetary vorticity?

Recalling the PV equation

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{f+\zeta}{H}\right) = \frac{F}{H}, \tag{5.40}$$

The ocean will respond to an input of vorticity F by either changing ζ or f. Using the above scaling approach we see that

$$\underbrace{\frac{\mathrm{D}\zeta}{\mathrm{D}t}}_{\frac{U^2}{L^2}} + \underbrace{\beta v}_{\beta U} = F. \tag{5.41}$$

The ratio of relative vorticity and advection of planetary vorticity is

$$\frac{\mathrm{D}\zeta}{\mathrm{D}t} / \frac{\mathrm{D}f}{\mathrm{D}t} \sim \frac{U}{\beta L^2} \equiv R_\beta. \tag{5.42}$$

- Consider now the basin scale (L ~1000 km, U ~0.01 m s^{-1}). The β Rossby number would be

$$R_\beta = \frac{U}{\beta L^2} = \frac{10^{-2}}{10^{-11}(10^6)^2} = 10^{-3}. \tag{5.43}$$

  Within the basin scale, the rate of change of relative vorticity is small compared to the rate of change of planetary vorticity. This means that an input of vorticity, say from the wind, does not induce the flow to increase its rotation, rather it will force the flow to move meridionally to reach a balance through f.

- Now consider a frontal zone instead (L ~10 km, U ~0.1 m s^{-1}). The β Rossby number would be

$$R_\beta = \frac{U}{\beta L^2} = \frac{10^{-1}}{10^{-11}(10^4)^2} = 10^2. \tag{5.44}$$

  Within a frontal zone, the rate of change of ζ is much larger than β. This means that the ocean will respond to F by changing ζ.

The response is thus fundamentally different, and the two regions will be governed by different dynamics: there will be a large *interior* regime and a narrow *boundary layer* regime.

5.2.2 The interior: Sverdrup balance

The Stommel model is linear, and we can obtain analytical solutions

$$\boxed{\beta\frac{\partial\psi}{\partial x} = F_\tau(x,y) - r\nabla^2\psi} \tag{5.45}$$

First, note that the Stommel model was previously derived by the β-plane approximation to the primitive equations:

$$\beta\frac{\partial p}{\partial x} = \frac{f_0}{H}curl_z\boldsymbol{\tau} - \frac{f_0 d}{2H}\nabla^2 p. \tag{5.46}$$

Now, let's have a look at the relative role of the top and bottom Ekman contributions. The ratio between the pressure gradient term and the frictional term is

$$\frac{f_0 d}{2H}P/L^2/(\beta P/L) \rightarrow \frac{f_0 d}{2H\beta L} \tag{5.47}$$

Typical values can be used for d ~15 m, f_0 ~10^{-4}, β ~10^{-11}, H ~3000 m and L ~1000 km, and the ratio is ~0.02. This implies that the frictional term can be neglected and that the Ekman pumping induced by the wind stress is much larger than the one resulting from bottom friction.

This approximation will lead us towards our first solution

$$\beta\frac{\partial p}{\partial x} = \frac{f_0}{H}curl_z\tau, \tag{5.48}$$

which implies a meridional velocity that is a function of the wind-stress curl, and is best known as *Sverdrup balance*.

Suppose, in fact, that the frictional term is small, so there is an approximate balance between the input of vorticity by the wind stress and the β-effect (or the rate of change of planetary vorticity).

Friction is small if

$$|r\zeta| \ll |\beta v|. \tag{5.49}$$

If we define $r = \frac{f_0\delta}{H}$, as suggested by (5.46), where δ is the thickness of the bottom Ekman layer, then

$$\frac{f_0\delta}{H}\frac{U}{L} \ll \beta U, \quad \text{or} \quad \frac{r}{L} \ll \beta. \tag{5.50}$$

This inequality is well satisfied in large-scale flows, where L is the horizontal scale of the motion. The vorticity equation is thus

$$\beta\frac{\partial\psi}{\partial x} = F_\tau(x,y) \tag{5.51}$$

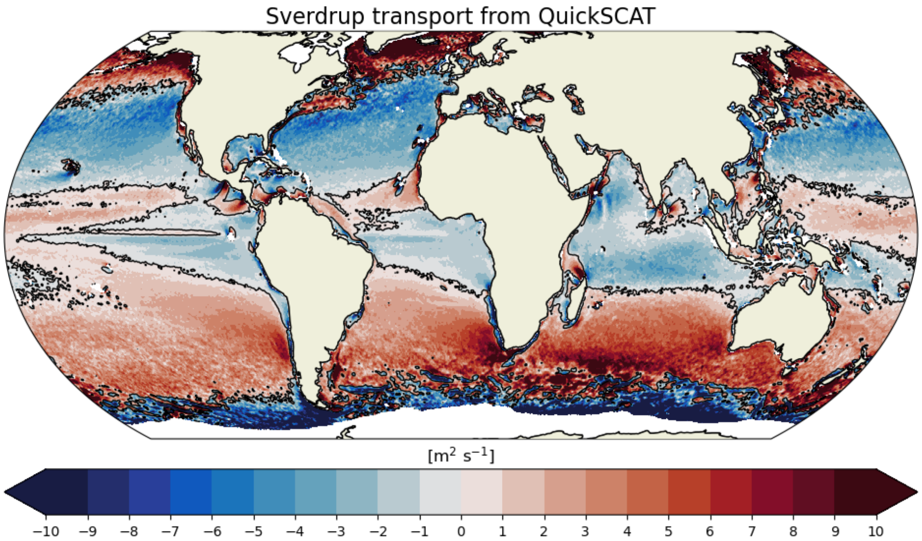

Figure 5.3: *Estimate of Sverdrup transport computed from QuickSCAT as* $\overline{v} = curl_z \widetilde{\boldsymbol{\tau}}/\beta$.

which is just an expression of Sverdrup balance

$$\boxed{\beta\overline{v} = curl_z \widetilde{\boldsymbol{\tau}}} \tag{5.52}$$

This is equivalent to the linear geostrophic vorticity balance

$$\beta v = f_0 \frac{\partial w}{\partial z} \tag{5.53}$$

where stress at the bottom is neglected. In fact, over most of the ocean, the deep flow is very weak, meaning that bottom drag is negligible.

Eq.5.52 is not a transport, rather just a balance between wind stress at the surface and the β-effect leading to a meridional velocity $\overline{v} = \frac{1}{\beta} curl_z \tau$ (Fig. 5.3).

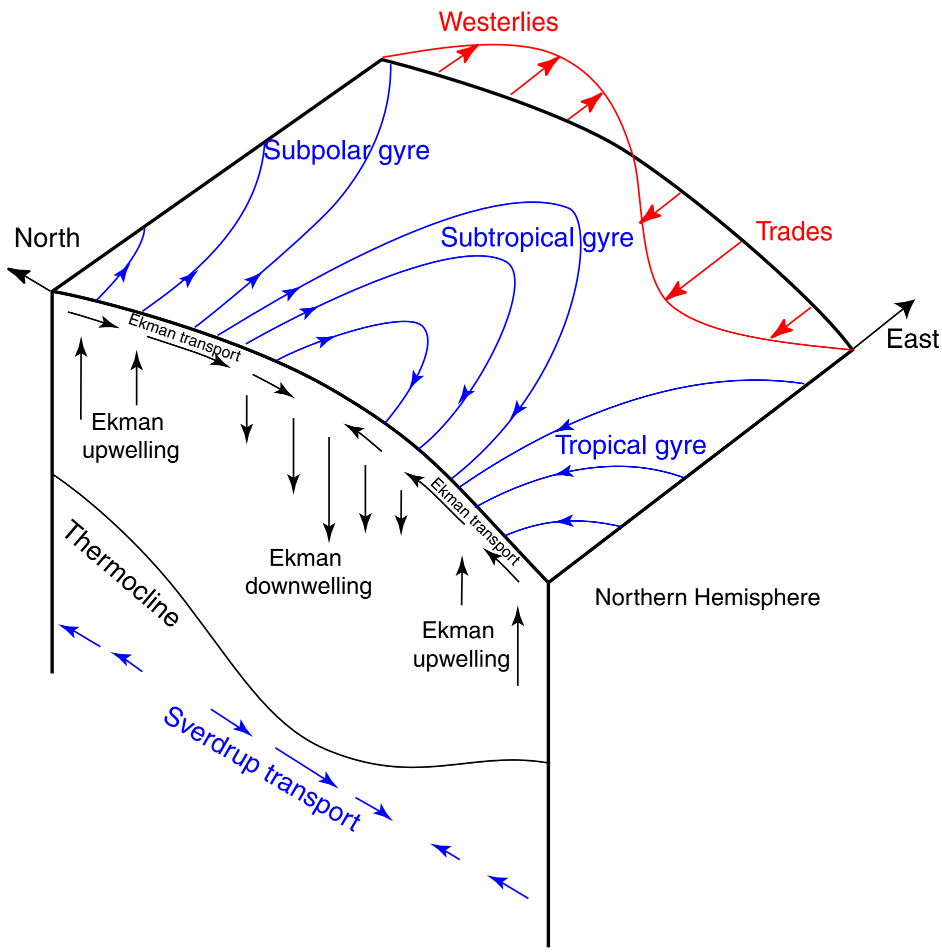

Figure 5.4: *Sverdrup balance circulation ($f > 0$). [from Talley et al. (2011)]*

Physical interpretation

Consider a schematic of the subtropical North Pacific. The winds at the sea surface are not spatially uniform. South of about 30°N, the Pacific is dominated by easterly trade winds. North of this, it is dominated by the westerlies. This causes northward Ekman transport under the trade winds, and southward Ekman transport under the westerlies. As a result, there is Ekman convergence throughout the subtropical North Pacific.

The convergent surface layer water in the subtropics must go somewhere so there is downward vertical velocity at the base of the (50 m thick) Ekman layer. At some level between the surface and ocean bottom, there is likely no vertical velocity. Therefore there is net "squashing" of the water columns in the subtropical region (*Ekman pumping*).

This squashing requires a decrease in either planetary or relative vorticity (remember potential vorticity conservation $\frac{D}{Dt}\frac{f+\zeta}{H} = 0$). In the ocean interior, relative vorticity is small, so planetary vorticity must decrease, which results in the equatorward flow that characterizes the subtropical gyre (Fig. 5.4).

The subpolar North Pacific lies north of the westerly wind maximum at about 40°N. Ekman transport is therefore southward, with a maximum at about 40°N and weaker at higher latitudes. Therefore there must be upwelling (*Ekman suction*) throughout the wide latitude band of the subpolar gyre. This upwelling stretches the water columns, which then move poleward, creating the poleward flow of the subpolar gyre.

The Sverdrup transport is the net meridional transport diagnosed in both the subtropical and subpolar gyres, resulting from *planetary vorticity changes that balance Ekman pumping or Ekman suction*. All of the meridional flow is returned in western boundary currents, for reasons described in the following sections. Therefore, subtropical gyres must be anticyclonic and subpolar gyres must be cyclonic.

Computing the transport

Assuming the ocean circulation is in Sverdrup balance, $\overline{v} = \frac{\partial \psi}{\partial x}$ gives the meridional mass transport of the vertically integrated column of fluid due to a surface wind stress. The constraint that there be no normal flow across the ocean's horizontal boundaries means that $\psi = const$ on the boundaries. We pick this constant arbitrarily to be 0. We must choose whether to choose the eastern or western boundary as the limit of integration. This cannot be determined by Sverdrup balance alone, it requires consideration of frictional boundary layers.

We choose the eastern boundary, requiring closure of the circulation in a western boundary current, and we require that the streamfunction be zero on the eastern boundary.

Integrating from east to west, and using the boundary condition $\psi = 0$ at $x = x_E(y)$, the streamfunction is (see Fig. 5.5 and Fig. 5.6)

$$\begin{aligned}
\int_x^{x_E} \frac{\partial \psi}{\partial x}\, \mathrm{d}x' &= \frac{1}{\beta} \int_x^{x_E} curl_z \widetilde{\boldsymbol{\tau}}_T \, \mathrm{d}x \\
\psi(x,y) &= -\frac{1}{\beta} \int_x^{x_E} curl_z \widetilde{\boldsymbol{\tau}}_T \, \mathrm{d}x.
\end{aligned}$$

Two examples are shown in Fig. 5.5 for the North Atlantic and the North Pacific. The Sverdrup balance gives a reasonable good estimation for the interior flow, but a western boundary current is needed to close the circulation. The Sverdrup balance integration results in a realistic large-scale gyre circulation in the tropical, subtropical and subpolar oceans (Fig. 5.6). However something is not well represented and totally missed by the Sverdrup flow. Sverdrup flow predicts an interior flow in balance with the input of vorticity by the wind stress; but the interior meridional flow must be compensated at some level somewhere to comply with mass conservation. This, we will see, is accomplished by a narrow and intense boundary current.

In the Southern Ocean, the zonal integration of the Sverdrup balance does not apply.

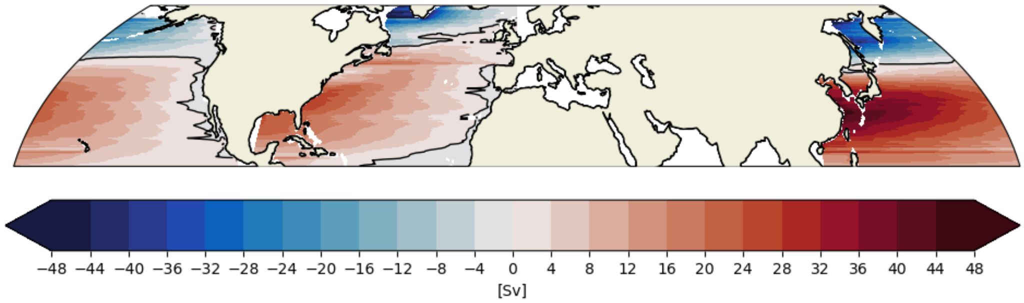

Figure 5.5: *Estimate of the depth-integrated circulation (in Sv) predicted by the Sverdrup balance in the North Atlantic and the North Pacific computed with QuickSCAT winds. The solution assumes that the depth-integrated circulation vanishes at the eastern boundary. Positive values (red) correspond to clockwise circulations and negative values (blue) to anticlockwise circulations.*

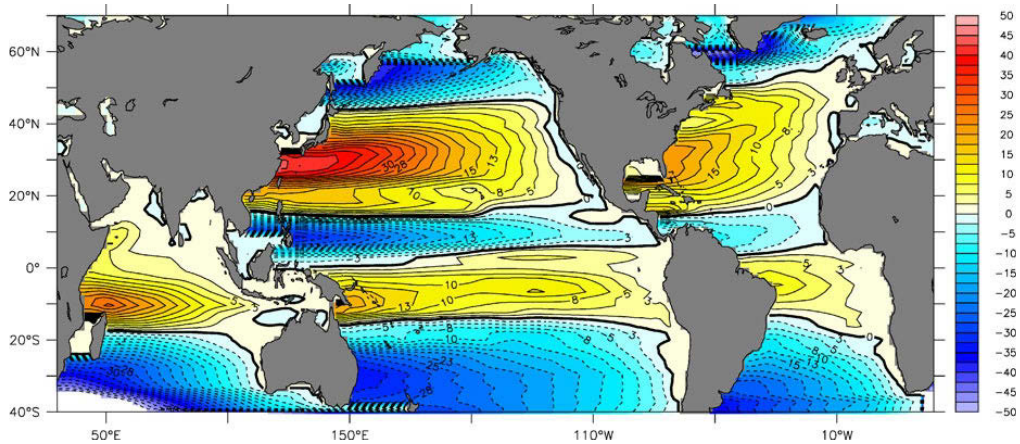

Figure 5.6: *Streamfunction ψ ($Sv \equiv 10^6\ m^3\ s^{-1}$) calculated from the Sverdrup relation and a climatological wind stress curl. Westward integration starts at 30° E with $\psi = 0$ as boundary condition. [from Olbers et al. (2012)]*

5.2.3 The boundary: Adding a return flow

We need to close the circulation induced by the interior Sverdrup flow. The interior flow was developed for the large scale. We can thus suppose that the return flow will occur in a narrow boundary layer somewhere. Where will this be? Western or eastern side of the basin?

Take the full Stommel model

$$\beta \frac{\partial \psi}{\partial x} = curl_z \boldsymbol{\tau}_T - r \nabla^2 \psi. \tag{5.54}$$

and consider a square domain of side L and rescale variables as follows

$$\begin{aligned} x = L\,\hat{x} \quad \tau = \tau_0\,\hat{\tau} \\ y = L\,\hat{y} \quad \psi = \frac{\tau_0}{\beta}\,\hat{\psi} \end{aligned}$$

Hatted variables are non-dimensional and they are $\mathcal{O}(1)$ quantities in the interior.

The Stommel model becomes

$$\begin{aligned} \beta \frac{\partial \hat{\psi}}{\partial \hat{x}} \frac{\tau_0}{\beta L} &= curl_z \hat{\boldsymbol{\tau}}_T \frac{\tau_0}{L} - r \nabla^2 \hat{\psi} \frac{\tau_0}{\beta L^2} \\ \frac{\partial \hat{\psi}}{\partial \hat{x}} &= curl_z \hat{\boldsymbol{\tau}}_T - \frac{r}{\beta L} \nabla^2 \hat{\psi} \end{aligned}$$

which is

$$\frac{\partial \hat{\psi}}{\partial \hat{x}} = curl_z \hat{\boldsymbol{\tau}}_T - \epsilon_s \nabla^2 \hat{\psi} \tag{5.55}$$

where $\epsilon_s = \frac{r}{\beta L} \ll 1$, as shown by (5.50), for the large-scale flow. We thus write a solution for the interior, where friction is small, and a solution for the boundary, where frictional effects will be large:

$$\psi(x, y) = \psi_I(x, y) + \phi(x, y)$$

where ϕ is a boundary layer correction.

The interior solution

In the interior the flow is described by $\psi_I(x, y)$ in the limit where $\epsilon_s = \frac{r}{\beta L} \ll 1$

$$\frac{\partial \psi_I}{\partial x} = curl_z \boldsymbol{\tau}_T \tag{5.56}$$

The solution of the Sverdrup interior is

$$\psi_I(x,y) = \int_0^x curl_z \boldsymbol{\tau}(x',y)\mathrm{d}x' + g(y) \tag{5.57}$$

where $g(y)$ is an arbitrary function. Given the streamfunction definition ($v_I = \partial\psi_I/\partial x$; $u_I = -\partial\psi_I/\partial y$), the corresponding velocities are

$$\begin{aligned} v_I &= curl_z \boldsymbol{\tau} \\ u_I &= -\partial_y \int_0^x curl_z \boldsymbol{\tau}(x',y)\,\mathrm{d}x' - \frac{\partial g(y)}{\partial y} \end{aligned}$$

Let's simplify our forcing and take the wind stress curl as zonally uniform, so that

$$\tau_T^y = 0, \quad \tau_T^x = -cos(\pi y) \tag{5.58}$$

so that the curl vanishes at $y = 0$ and $y = 1$ (Fig. 5.7). The curl in this case will be $curl_z \boldsymbol{\tau}_T = -\pi sin(\pi y)$. For this example, typical of subtropical latitudes, the wind stress is imparting a negative input of vorticity into the ocean everywhere.

The Sverdrup interior flow is

$$\begin{aligned} \psi_I(x,y) &= \int_0^x curl_z \boldsymbol{\tau}(x',y)\mathrm{d}x' + g(y) \\ \psi_I(x,y) &= \int_0^x \Big[-\pi sin(\pi y)\Big]\,\mathrm{d}x' + g(y) \\ \psi_I(x,y) &= x\Big[-\pi sin(\pi y)\Big] + g(y) \end{aligned}$$

We can define the arbitrary function of integration as $C(y) = -g(y)/curl_z \boldsymbol{\tau}_T$. So that our solution becomes

$$\begin{aligned} \psi_I(x,y) &= x\Big[-\pi sin(\pi y)\Big] - \Big[C(y) curl_z \boldsymbol{\tau}_T\Big] \\ \psi_I(x,y) &= x\Big[-\pi sin(\pi y)\Big] + C(y)\Big[\pi sin(\pi y)\Big] \\ \psi_I(x,y) &= \pi\Big[C(y) - x\Big] sin(\pi y) \end{aligned}$$

If C is a constant, then the zonal flow is $C\,curl_z \boldsymbol{\tau}$. Now, depending on C, we can either satisfy $\psi = 0$ at $x = 0$ or at $x = 1$

$$\psi_I(0,y) = \pi C\,sin(\pi y) = 0 \quad \text{if} \quad C = 0 \tag{5.59}$$

$$\psi_I(1,y) = \pi(C-1)\,sin(\pi y) = 0 \quad \text{if} \quad C = 1 \tag{5.60}$$

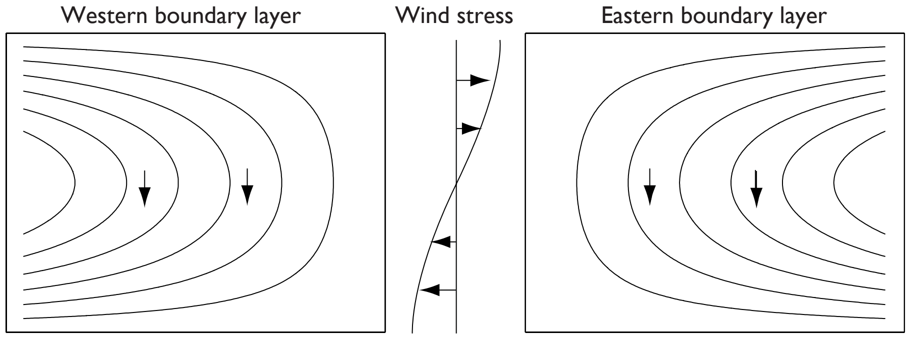

Figure 5.7: *Two possible Sverdrup flows, ψ_I, for the given wind stress. Each solution satisfies the no-flow condition at* one *boundary, either east or west. Both solutions have the same meridional interior flow. Which one is physically plausible? [from Vallis (2006)]*

We cannot satisfy both zonal boundary conditions of $\psi = 0$. And so a choice will have to be made on C, and more importantly on where the boundary layer will exist in order to satisfy the remaining boundary condition!

We could suppose the solution at the left of Fig.5.7, because the interior flow would go the same direction as the wind torque driving it. Friction should provide opposite torque in order to balance the angular momentum. An eastern boundary (solution at the right of Fig.5.7) would not be able to provide an anti-clockwise angular momentum (vorticity) capable of balancing vorticity input by the surface stress. Only the Western Boundary Current seems able to provide the required frictional force. We will expand on this 'vorticity argument' in Section 5.4

The boundary solution (asymptotic matching)

Let's now stretch the x-coordinate near the boundary, where $\phi(x, y)$ varies very rapidly in order to satisfy the boundary condition. The boundary could be at $x = 0$ or at $x = 1$:

$$x = \epsilon\,\alpha \quad \text{or} \quad x - 1 = \epsilon\,\alpha. \tag{5.61}$$

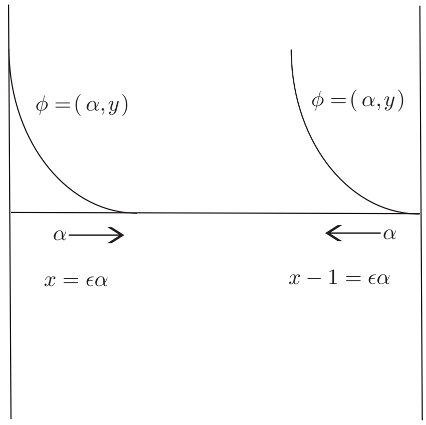

Figure 5.8: *Two possible boundary solutions. Only the one on the western side decays towards the interior and satisfies the condition that $\phi = 0$ in the interior. The solution requires that $\alpha > 0$ and $x = \epsilon\alpha$.*

α is the stretched coordinate, having values $\mathcal{O}(1)$ in the boundary and ϵ is a small parameter. We now suppose that $\phi(\alpha, y)$ and using Eq.(5.55) write:

$$\partial_x(\psi_I + \phi) + \epsilon_s \nabla^2(\psi_I + \phi) = curl_z \boldsymbol{\tau}_T \tag{5.62}$$

$$\partial_x \psi_I + \epsilon_s(\nabla^2\psi_I + \nabla^2\phi) + \frac{1}{\epsilon}\frac{\partial \phi}{\partial \alpha} = curl_z \boldsymbol{\tau}_T \tag{5.63}$$

$$\tag{5.64}$$

where $\nabla^2\phi = \frac{1}{\epsilon^2}\frac{\partial^2\phi}{\partial\alpha^2} + \frac{\partial^2\phi}{\partial y^2}$. We know that ϕ_I satisfies Sverdrup balance, so the solution becomes

$$\epsilon_s(\nabla^2\psi_I + \frac{1}{\epsilon^2}\frac{\partial^2\phi}{\partial\alpha^2} + \frac{\partial^2\phi}{\partial y^2}) + \frac{1}{\epsilon}\frac{\partial\phi}{\partial\alpha} = 0. \tag{5.65}$$

We now make the simplest choice and choose $\epsilon = \epsilon_s$, so that the leading order balance is

$$\frac{\partial^2\phi}{\partial\alpha^2} + \frac{\partial\phi}{\partial\alpha} = 0. \tag{5.66}$$

The solution of which is $\phi = A(y) + B(y)e^{-\alpha}$.

The solution grows in the negative direction of α. But the solution cannot grow towards the interior or it would violate our assumption that ϕ is small in the interior. Hence, we impose $\alpha > 0$ and $A(y) = 0$. This implies the choice of $x = \epsilon\alpha$ so that $\alpha > 0$ for $x > 0$. The boundary layer is at $x = 0$: *a western boundary layer*, and it decays eastward for increasing α, towards the interior (Fig. 5.8).

We now choose $C = 1$, so that $\psi_I = 0$ at $x = 1$, and the solution for the given wind stress is

$$\psi_I = \pi(1-x)\, sin(\pi y) \tag{5.67}$$

This satisfies the eastern boundary condition ($\psi = 0$ at $x = 1$).

$B(y)$ will now satisfy the other boundary condition in order to

$$\psi = \psi_I + \phi = 0 \quad \text{at} \quad x = 0. \tag{5.68}$$

At $x = 0$:

$$\psi = \pi\, sin(\pi y) + \phi = 0 \tag{5.69}$$

Given that $\phi = B(y)e^{-\alpha}$, we have, at $x = 0$

$$\psi = \pi\, sin(\pi y) + B(y) = 0, \tag{5.70}$$

which readily implies that $B(y) = -\pi\, sin(\pi y)$. The boundary layer correction is thus

$$\phi = -\pi\, sin(\pi y) e^{-x/\epsilon_s}. \tag{5.71}$$

The boundary layer correction is thus proportional to the interior wind stress, as it has to balance that input of vorticity.

The full solution is thus

$$\psi = \psi_I + \phi \;=\; \pi\, sin(\pi y) - x\pi\, sin(\pi y) - \pi sin(\pi y)e^{-x/\epsilon_s} \tag{5.72}$$

$$= \pi\, sin(\pi y)\Big(1 - x - e^{-x/\epsilon_s}\Big). \tag{5.73}$$

The dimensional solution is (remember that $\psi = \hat{\psi}\frac{\tau_0}{\beta}; \tau = \hat{\tau}\tau_0; y = \hat{y}L; x = \hat{x}L$):

$$\boxed{\psi = \frac{\tau_0}{\beta}\pi\Big(1 - \frac{x}{L} - e^{-x/(L\epsilon_s)}\Big) sin\frac{\pi y}{L}} \tag{5.74}$$

Given the chosen wind stress, this is a single gyre solution (Fig. 5.9), and for a realistic global wind stress the solution is shown in Fig. 5.11.

The boundary layer width

What is the width δ of the western boundary layer? In the interior, friction is small, and the balance is between wind stress and the β-effect:

$$|r\zeta| \ll |\beta v|. \tag{5.75}$$

With $r = \frac{f\delta}{H}$, this means that $\frac{f\delta}{HL} \ll \beta$. For friction to be small, we also have that

$$\epsilon_s = \frac{r}{L\beta} \ll 1 \quad \text{or} \quad \frac{r}{\beta} \ll L, \tag{5.76}$$

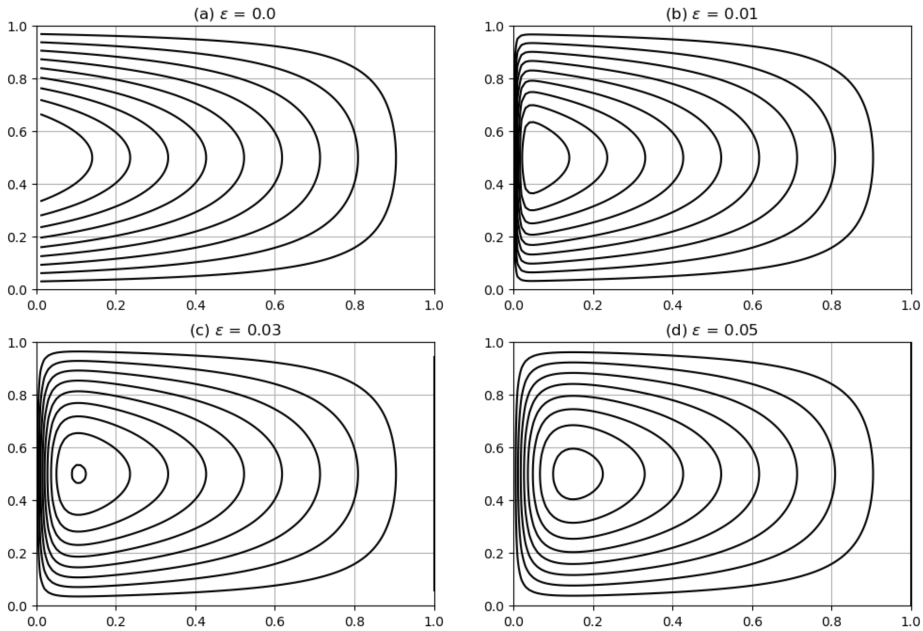

Figure 5.9: *Solutions of the Stommel model for a single-gyre wind-induced flow for different values of* ϵ*. Note that for* ϵ*=0 the model reduces to the Sverdrup balance.*

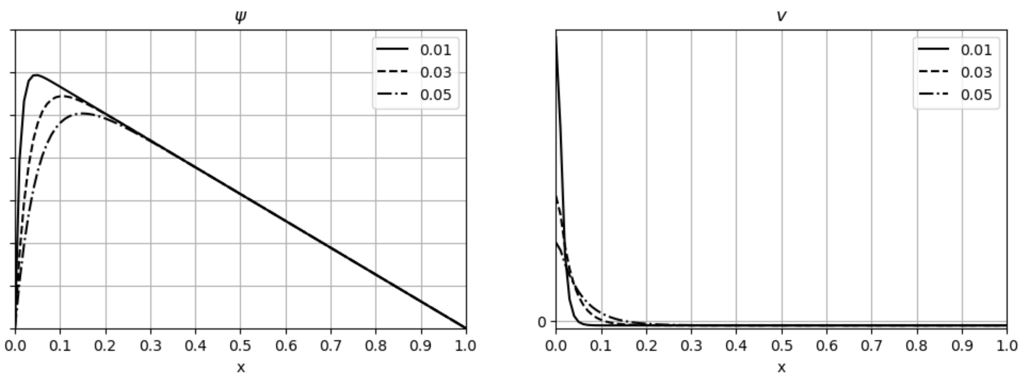

Figure 5.10: *Solutions of the Stommel model for a single-gyre wind-induced flow for different values of* ϵ*. Plotted are the streamfunction* ψ *and the meridional velocity* $v = \partial\psi/\partial x$ *at the centre of the gyre.*

where r measures bottom friction and L denotes the length scale of zonal variations of the geostrophic current.

However, when L becomes smaller, representing dynamics in the bound-

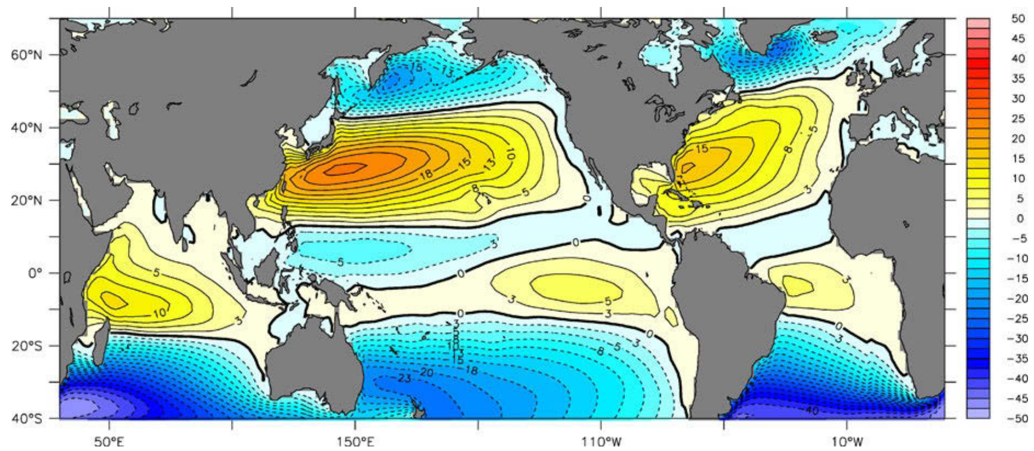

Figure 5.11: *Streamfunction ψ (in Sv) computed from the Stommel model with realistic wind stress curl and a boundary layer width δ = 100 km. [from Olbers et al. (2012)]*

ary layer, we have a different balance:

$$\frac{r}{\beta} \sim L, \tag{5.77}$$

and now $L = \mathcal{O}(\delta)$ so that the width of the Stommel boundary layer is

$$\boxed{\delta_S = \frac{r}{\beta}}. \tag{5.78}$$

Within this narrow boundary layer, $v_g > 0$ and $\overline{v} > 0$, balancing the interior Sverdrup flow. The total transport in the Sverdrup regime occurs between the eastern edge of the western boundary layer, $x = \delta_S$, and the eastern coast, $x = 1$. A corresponding transport must be compensated and returned within the boundary layer. This transport is thus prescribed by the wind outside the boundary layer, the Sverdrup regime. Because the boundary layer width is much smaller than the basin width, the currents in the boundary layer have to be much stronger than in the Sverdrup regime, as observed.

An f-plane solution

In the Stommel model, dissipation of vorticity arises from bottom frictional stresses within a bottom boundary layer.

In the case of a constant f, so that $\beta = \frac{\partial f}{\partial y} = 0$, the input of vorticity from the wind simply balances the opposing frictional dissipation everywhere. This leads to symmetric solutions, which are not realistic.

$$\underbrace{\beta \frac{\partial \psi}{\partial x}}_{\substack{0 \text{ for} \\ \text{the } f\text{-plane}}} = \underbrace{F_\tau(x,y)}_{\substack{\text{wind input} \\ \text{of vorticity}}} \quad \underbrace{-r\nabla^2\psi}_{\substack{\text{frictional dissipation} \\ \text{of vorticity}}} \quad . \tag{5.79}$$

The vertical geostrophic velocity vanishes in the f-plane, and the two Ekman induced vertical velocities have to compensate each other. This is possible if

$$curl_z \widetilde{\boldsymbol{\tau}}_T = curl_z \widetilde{\boldsymbol{\tau}}_B = \frac{d}{2}\zeta_g \tag{5.80}$$

There is no boundary layer solution, and the balance is achieved everywhere within the basin (see Fig. 5.12).

If, conversely, $\beta \neq 0$, in the interior we find a balance between change in planetary vorticity and input of vorticity. In the narrow western boundary layer, the fluid column changes again its planetary vorticity but the source of vorticity is from frictional dissipation.

But given that the return flow was found on a western boundary layer, is bottom drag realistic?

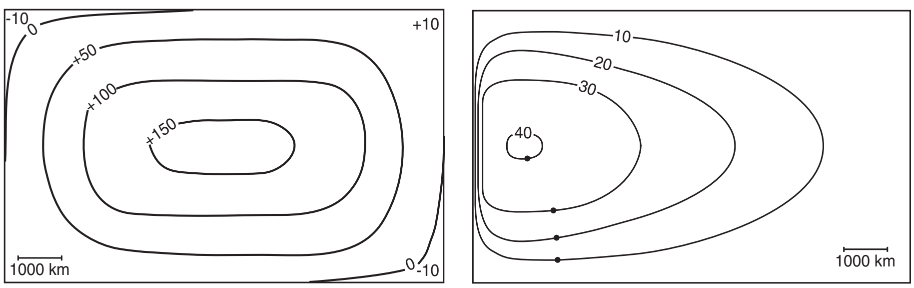

Figure 5.12: *Stommel's wind-driven circulation solution for a subtropical gyre with trades and westerlies. (a) Transport streamfunction ψ on a uniformly rotating Earth ($f = f_0$) and (b) westward intensification with the β-effect ($f = f_0 + \beta y$). [from Stommel (1948)]*

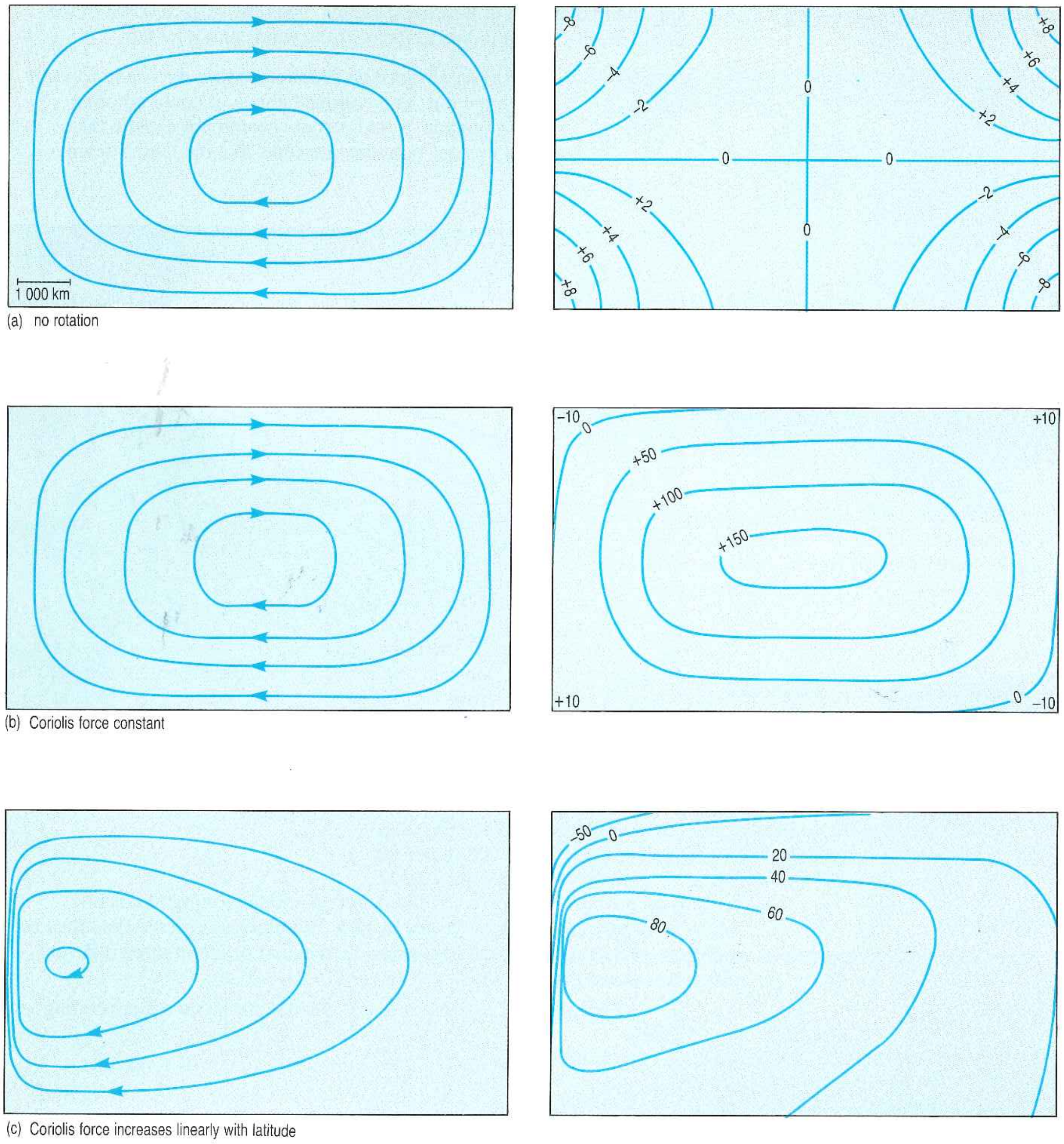

Figure 5.13: *(left panels) Streamfunction ψ and (right panels) sea-surface height η for a symmetrical gyral wind field (à la Stommel). In the case of no rotation $f = 0$ winds simply drive a symmetric circulation, just as you might expect from stirring a coffee cup. If f =const and $\beta = 0$ as in a flat Earth, there is again a symmetric solution with fluid rotating in geostrophic balance. Western intensification requires Earth to be a spinning sphere with planetary vorticity varying with latitude. [from Stommel (1948)]*

5.3 The Munk model

An Ekman bottom drag is not appropriate to balance the interior wind-driven circulation. This is because the circulation does not reach all the way down to the bottom and some other form/term is required to balance the interior transport. An extension of the Stommel problem was formulated by Munk, who introduced lateral harmonic viscosity.

Munk does not use a bottom drag and, given that the boundary layer is on a side, introduces horizontal viscosity. We can start from the set of primitive equations and our fluid is governed by

$$-fv = -\frac{\partial \phi}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\nu_h}{\rho_0}\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\nu_h}{\rho_0}\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\nu_v}{\rho_0}\frac{\partial u}{\partial z}\right) \quad (5.81)$$

$$fu = -\frac{\partial \phi}{\partial y} + \frac{\partial}{\partial x}\left(\frac{\nu_h}{\rho_0}\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\nu_h}{\rho_0}\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\nu_v}{\rho_0}\frac{\partial v}{\partial z}\right) \quad (5.82)$$

$$0 = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \quad (5.83)$$

or in a simpler form

$$\mathbf{f} \times \boldsymbol{u} = -\nabla\phi + \frac{1}{\rho_0}\nabla \cdot (\nu \nabla \boldsymbol{u}) \quad (5.84)$$

$$\nabla_3 \cdot \boldsymbol{u} = 0 \quad (5.85)$$

which are very similar to the set of equations used by Stommel (Eq. 5.21), but now we have introduced a term related to horizontal turbulent viscosity. These will be the key to introduce a frictional dissipation similar to the Stommel bottom drag.

Again, assume a vertically-integrated ocean, let's vertically integrate and pose:

$$\Phi = \int_{-H}^{z} \phi \, \mathrm{d}z; \quad \overline{u} = \int_{-H}^{z} \rho_0 u \, \mathrm{d}z; \quad \overline{v} = \int_{-H}^{z} \rho_0 v \, \mathrm{d}z \quad (5.86)$$

we find

$$-f\overline{v} = -\frac{\partial \Phi}{\partial x} + \nu_h \nabla^2 \overline{u} + \int_{-H}^{z} \frac{\partial}{\partial z}\nu_v \frac{\partial u}{\partial z} \, \mathrm{d}z \quad (5.87)$$

$$f\overline{u} = -\frac{\partial \Phi}{\partial y} + \nu_h \nabla^2 \overline{v} + \int_{-H}^{z} \frac{\partial}{\partial z}\nu_v \frac{\partial v}{\partial z} \, \mathrm{d}z \quad (5.88)$$

$$0 = \frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} \quad (5.89)$$

We have set $w = 0$ at $z = 0$ and $z = -H$, and note that the stress tensor was defined as

$$\tau^x = \left(\nu_v \frac{\partial u}{\partial z}\right)_{z=0} - \left(\nu_v \frac{\partial u}{\partial z}\right)_{z=-H} \tag{5.90}$$

$$\tau^y = \left(\nu_v \frac{\partial v}{\partial z}\right)_{z=0} - \left(\nu_v \frac{\partial v}{\partial z}\right)_{z=-H}. \tag{5.91}$$

Ignoring bottom contributions this yields

$$-f\overline{v} = -\frac{\partial \Phi}{\partial x} + \nu_h \nabla^2 \overline{u} + \tau_T^x \tag{5.92}$$

$$f\overline{u} = -\frac{\partial \Phi}{\partial y} + \nu_h \nabla^2 \overline{v} + \tau_T^y \tag{5.93}$$

$$0 = \frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y}. \tag{5.94}$$

Now, as usual, take the curl of the horizontal momentum equations and use a streamfunction for the non-divergent flow to obtain:

$$\boxed{\beta \frac{\partial \psi}{\partial x} = curl_z \boldsymbol{\tau}_T + \nu_h \nabla^4 \psi} \tag{5.95}$$

The operator $\nu_h \nabla^4$ parameterizes viscosity as a biharmonic turbulent viscosity. This simple model captures a western boundary 'return' current and an interior Sverdrup flow. The simple model points to the role of the wind stress curl, and not the wind *per se*. The strength of the return current is dictated by dynamics outside of the boundary layer itself, i.e. the interior wind stress curl. This explains why some boundary currents (the Gulf Stream) are stronger than others (the Brazil current), which are driven by weaker wind stress curl (Fig. 5.14).

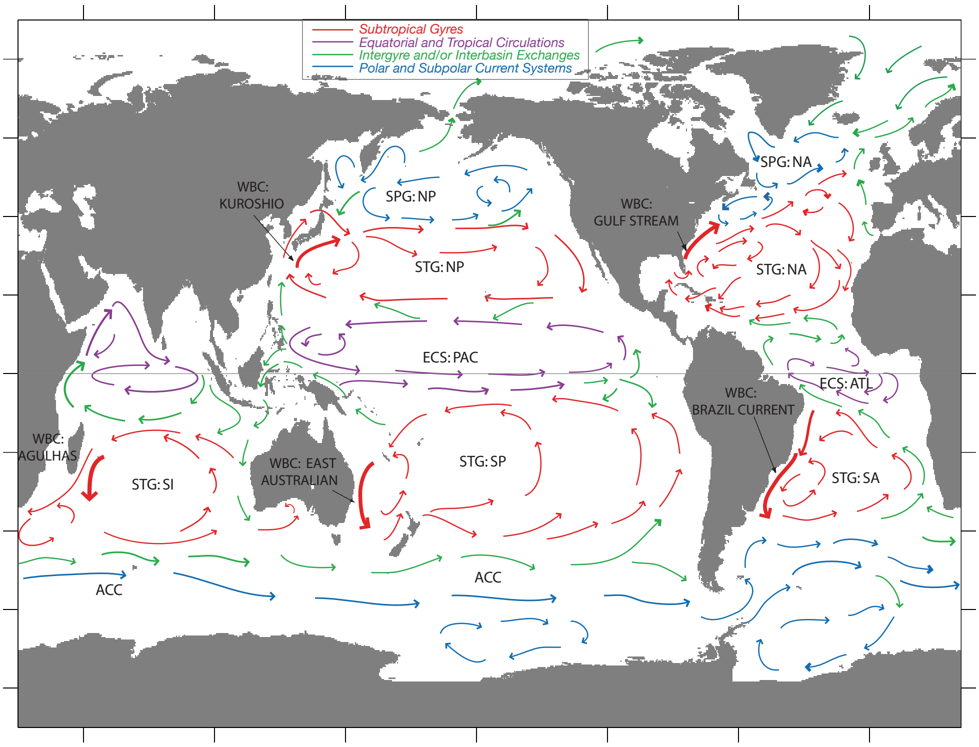

Figure 5.14: *A schema of the main currents of the global ocean [from Vallis (2006)].*

5.3.1 Interior and boundary solutions

The vorticity equation now reads

$$\boxed{\beta\frac{\partial\psi}{\partial x} = curl_z\boldsymbol{\tau}_T + \nu_h\nabla^2\zeta = curl_z\boldsymbol{\tau}_T + \nu_h\nabla^4\psi}. \tag{5.96}$$

This is the so-called MUNK MODEL. We need two boundary conditions at each wall because of the higher-order term. One is $\psi = 0$ to satisfy no-normal flow condition. The second boundary condition could be:

1. Zero vorticity ($\zeta = 0$). Since $\psi = 0$ along the boundary, this is equivalent to $\frac{\partial^2\psi}{\partial n^2} = 0$, where $\frac{\partial}{\partial_n}$ denotes a derivative normal to the boundary. At $x = 0$, this condition becomes $\frac{\partial v}{\partial x} = 0$: there is no horizontal shear at the boundary. This is called a *'free-slip'* condition.

2. No flow along the boundary. This is equivalent to $\frac{\partial\psi}{\partial n} = 0$. At $x = 0$, this condition becomes $v = 0$. This is called a *'no-slip'* condition.

Either could be used, and we will solve the *'no-slip'* problem. If we use the same wind stress

$$\tau^x = -cos(\pi y/L), \tag{5.97}$$

and non-dimensionalize (5.96) in a similar way to the Stommel problem

$$\frac{\partial\hat{\psi}}{\partial\hat{x}} - \epsilon_M\nabla^4\hat{\psi} = curl_z\widetilde{\boldsymbol{\tau}}_T. \tag{5.98}$$

Here $\epsilon_M = \nu/(\beta L^3)$. Again, the full solution will be the contribution of a western boundary layer correction and an interior Sverdrup flow

$$\hat{\psi} = \psi_I + \phi(\alpha, y). \tag{5.99}$$

The Munk problem does become

$$-\epsilon_M\Big(\nabla^4\psi_I + \frac{1}{\epsilon^4}\frac{\partial^4\phi}{\partial\alpha^4}\Big) + \frac{1}{\epsilon}\frac{\partial\phi}{\partial\alpha} = 0. \tag{5.100}$$

Of which the leading order balance is

$$-\frac{\partial^4\phi}{\partial\alpha^4} + \frac{\partial\phi}{\partial\alpha} = 0. \tag{5.101}$$

Subject to suitable boundary conditions and the interior Sverdrup solution

$$\psi_I = \pi(1-x)sin(\pi y), \tag{5.102}$$

where we have taken $C = 1$ as in Eq.(5.60) of the Stommel problem, the solution to the Munk problem is (a non-trivial algebraic exercise ...):

$$\hat{\psi} = \pi \, sin(\pi\hat{y}) \left\{ 1 - \hat{x} - e^{-\hat{x}/(2\epsilon)} \left[cos\left(\frac{\sqrt{3}\hat{x}}{2\epsilon}\right) + \frac{1-2\epsilon}{\sqrt{3}} sin\left(\frac{\sqrt{3}\hat{x}}{2\epsilon}\right) \right] + \epsilon e^{(\hat{x}-1)/\epsilon} \right\}. \tag{5.103}$$

The solution, for different values of ϵ, is shown in Fig. 5.15.

The Munk viscous boundary layer brings the tangential and the normal velocity to zero (Fig. 5.16).

The boundary layer width

What is the thickness of the Munk boundary layer? We have the following balance

$$\beta \frac{\partial \psi}{\partial x} \sim \nu \nabla^4 \psi \tag{5.104}$$

$$\beta \frac{U}{L^2} \sim \nu \frac{U}{L^5} \tag{5.105}$$

$$\beta \sim \frac{\nu}{L^3}, \tag{5.106}$$

in the boundary layer lateral diffusion of momentum will be important and will extract momentum imparted by the wind stress. If lateral viscosity is important, the length scale will be $L = \mathcal{O}(\delta)$, and so the boundary layer width is given by

$$\boxed{\delta_M \sim \left(\frac{\nu}{\beta}\right)^{1/3}} \tag{5.107}$$

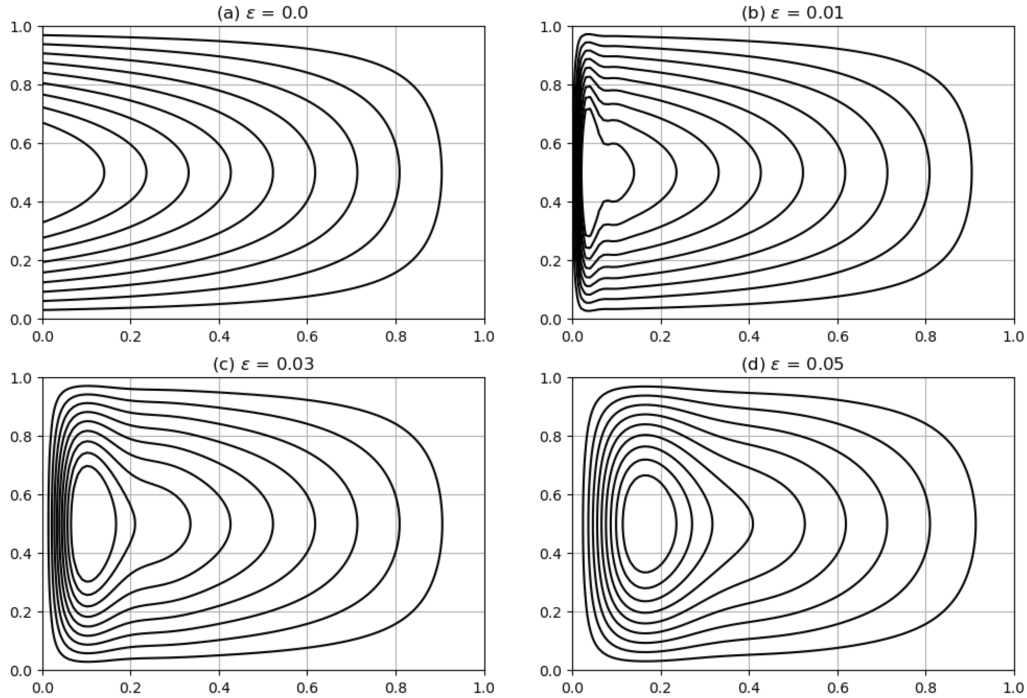

Figure 5.15: *Solutions of the Munk model for a single-gyre wind-induced flow for different values of ϵ. Note that for ϵ=0 the model reduces to the Sverdrup balance.*

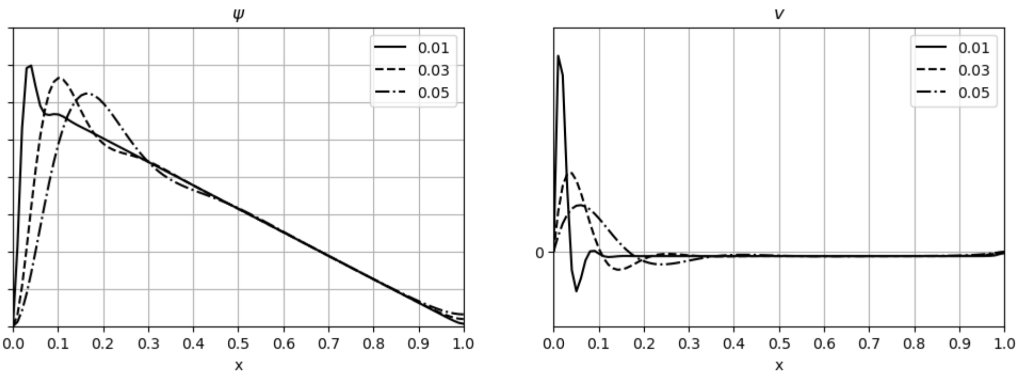

Figure 5.16: *Solutions of the Munk model for a single-gyre wind-induced flow for different values of ϵ. Plotted are the streamfunction ψ and the meridional velocity $v = \partial\psi/\partial x$ at the centre of the gyre. Note that the Munk model brings the velocity v to zero at the western boundary.*

Figure 5.17: *Franklin wondered why journeys towards the east were faster than return trips on his voyages across the Atlantic Ocean between the Colonies and Europe. His curiosity led him to be the first to chart the Gulf Stream on 1786. Franklin was talking to his cousin, Timothy Folger, who was the captain of a merchant ship. He asked why it took ships like Folger's so much less time to reach America than it took official mail ships. It struck Folger that the British mail captains must not know about the Gulf Stream, with which he had become well-acquainted in his earlier years as a Nantucket whaler. Folger told Franklin that whalers knew about the "warm, strong current" and used it to help their ships track and kill whales. But the mail ships "were too wise to be counselled by simple American fishermen" and kept sailing against the current, losing time as they did so.*

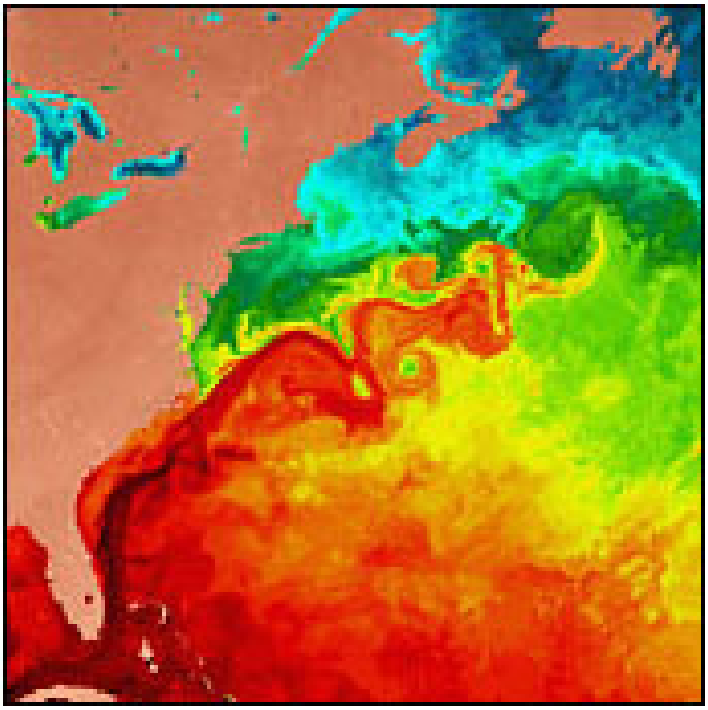

Figure 5.18: *A satellite image of the Gulf Stream.*

Neither the Stommel nor the Munk model are accurate representations of the real ocean. We need to include non-linearities and topographic effects to improve our solution.

The non-linear Stommel-Munk problem is

$$\frac{\partial \zeta}{\partial t} + J(\psi, \zeta) + \beta \frac{\partial \psi}{\partial x} = curl_z \boldsymbol{\tau}_T - r\nabla^2\psi + \nu\nabla^2\zeta. \tag{5.108}$$

And the steady non-linear Stommel-Munk problem is

$$J(\psi, \zeta) + \beta \frac{\partial \psi}{\partial x} = curl_z \boldsymbol{\tau}_T - r\nabla^2\psi + \nu\nabla^2\zeta. \tag{5.109}$$

The need for friction

Consider the steady barotropic flow

$$\frac{\mathrm{D}(f+\zeta)}{\mathrm{D}t} = F \tag{5.110}$$

satisfying

$$\boxed{\boldsymbol{u}\cdot\nabla q = curl_z \boldsymbol{\tau}_T + \text{Friction}}, \tag{5.111}$$

where $q = \nabla^2\psi + \beta y$ and the last term on the rhs represents frictional effects. $\boldsymbol{u}$ is divergent-free and we can integrate the lhs over some area A between two closed streamlines, ψ_1 and ψ_2. Using the divergence theorem[2] :

$$\int_A \nabla\cdot(\boldsymbol{u}q)\,\mathrm{d}A = \oint_{\psi_1} \boldsymbol{u}q\cdot\boldsymbol{n}\,\mathrm{d}l - \oint_{\psi_2} \boldsymbol{u}q\cdot\boldsymbol{n}\,\mathrm{d}l = 0. \tag{5.112}$$

Here $\boldsymbol{n}$ is the unit vector normal to the streamline so that $\boldsymbol{u}\cdot\boldsymbol{n} = 0$. The integral of the wind-stress curl over the area A will not be zero. This means that a balance between wind-stress curl and friction can only be achieved if every closed contour passes through a region where frictional effects are non-zero, and are important somewhere along the streamline path.

Thus, in the Stommel and Munk models, every streamline must pass through the frictional western boundary layer.

[2] Here we use the 2D divergence theorem for a vector field $F(x,y)$: $\iint_A \mathrm{div}\,F\,\mathrm{d}A = \oint_{\partial A} F\cdot\boldsymbol{n}\,\mathrm{d}l$

5.4 Westward intensification

PV balance interpretation

How does the potential vorticity balance work in Munk's model (which is combined with Sverdrup's model)?

Why do we find the boundary current on the western side rather than the eastern side, or even within the middle of the basin (if considering Stommel's bottom friction)?

In the Sverdrup interior of a subtropical gyre, when the wind causes Ekman pumping, the water columns are squashed, they move equatorward to lower planetary vorticity.

To return to a higher latitude, there must be forcing that puts the higher vorticity back into the fluid. This cannot be in the form of planetary vorticity, since this is already contained in the Sverdrup balance. Therefore, the input of vorticity must affect the relative vorticity.

Consider a western boundary current for a Northern Hemisphere subtropical gyre, with friction between the current and the side wall (Munk's model). The effect of the side wall is to reduce the boundary current velocity to zero at the wall. Therefore, the boundary current has positive relative vorticity. This vorticity is injected into the fluid by the friction at the wall, and allows the current to move northward to higher Coriolis parameter f.

On the other hand, if the narrow jet returning flow to the north were on the eastern boundary, the side wall friction would inject negative relative vorticity, which would make it even more difficult for the boundary current fluid to join the interior flow smoothly.

Therefore, vorticity arguments require that frictional boundary currents be on the western boundary. You can go through this exercise for subpolar gyres as well as for both types of gyres in the Southern Hemisphere and will find that a western boundary current is required in all cases!

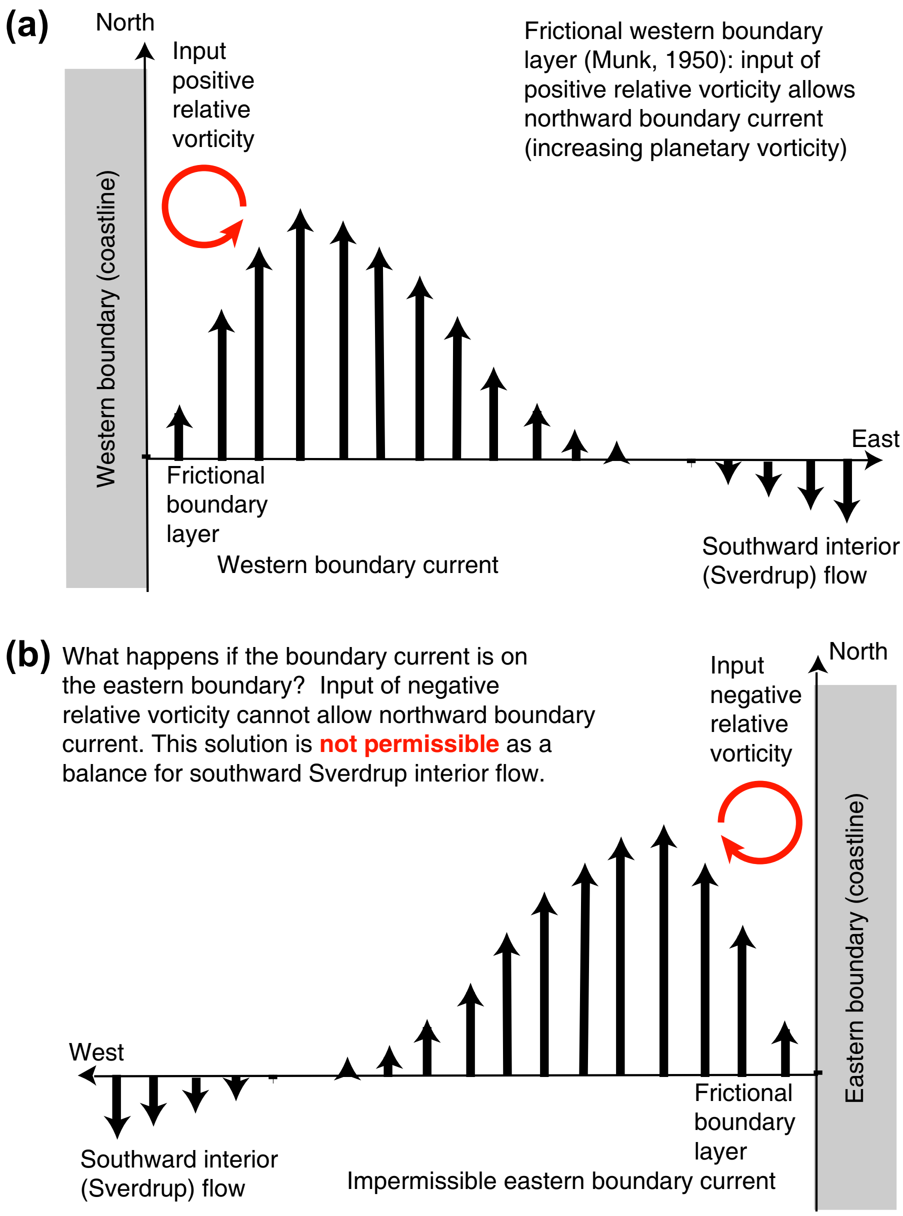

Figure 5.19: *(a) Vorticity balance at a western boundary, with side wall friction (Munk's model). (b) Hypothetical eastern boundary vorticity balance, showing that only western boundaries can input the positive relative vorticity required for the flow to move northward. [from Talley et al. (2011)]*

Western intensification understood as westward drift

Here we'll give a slightly different explanation of why the boundary current is in the west. It is not really a different explanation, because the cause is still **differential rotation**, but we'll think about it quite differently. We'll see the effect of differential rotation is to make patterns propagate to the west, and hence the response to the wind's forcing piles up in the west and produces a boundary current there.

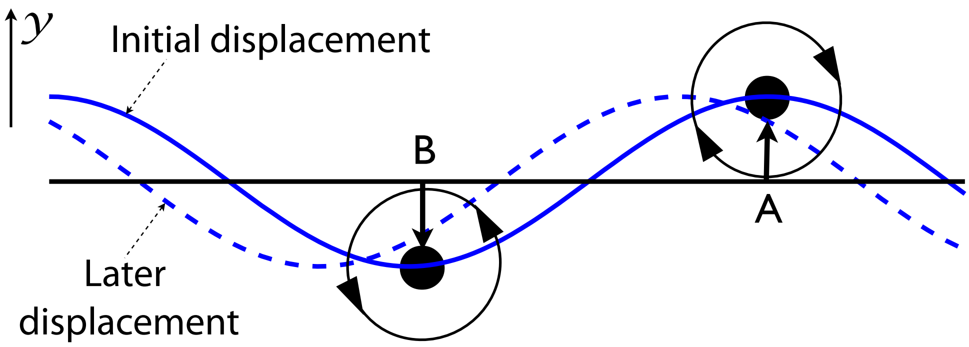

Figure 5.20: *If parcel A is displaced northwards then its clockwise spin increases, causing the northwards displacement of parcels that are to the west of A. A similar phenomena occurs if parcel B is displaced south. Thus, the initial pattern of displacement propagates westward. [from Vallis (2006)]*

Let's now imagine a line of parcels (Fig. 5.20). Suppose we displace parcel 'A' northwards. Because the Earth's spin is anti-clockwise (looking down on the North Pole) and this increases as the parcel moves northward, then the parcel must spin more in a clockwise direction in order to preserve its total vorticity

$$q = f + \zeta. \tag{5.113}$$

This spin will have the effect of moving the fluid that is just to west of the original parcel northwards, and then this will spin more clockwise, moving the fluid to its left northwards, and so on. The northwards displacement thus propagates *westward*, whereas parcels to the east of the original displacement are returned to their original position so that there is no systematic propagation to the east. Similarly, a parcel that is displaced southwards (parcel B) also causes the pattern to move westwards. We have just described the westward propagation of a simple *Rossby wave*, but the same effect occurs with more complex patterns and in particular, with the gyre as a whole.

Thus, imagine that an east-west symmetric gyre is set up, with the winds and friction in equilibrium, as in an f-plane. Differential rotation then tries to move the pattern westward, but of course the entire pattern cannot move to the west because there is a coastline in the way! The gyre thus squashes up against the western boundary creating an intense western boundary current.

This way of viewing the matter serves to emphasize that **it is not frictional effects that cause western intensification; rather, frictional effects allow the flow to come into equilibrium with an intense western boundary current, with the ultimate cause being the westward propagation due to differential rotation.**

In fact, the location of the boundary layer, on the west, does not depend on the sign of the wind-stress curl (the sign is reversed in a subpolar gyre and the flow is southward within a western boundary current) nor on the sign of the Coriolis parameter (think about what happens in the southern hemisphere where $f < 0$). The western location depends on β, which is always positive.

The Stommel & Munk models of the Wind-Driven Circulation

– The Model

1. The model uses the vertically integrated planetary-geostrophic equations (or a homogeneous fluid) with nonlinearities neglected.
2. The model uses a flat bottomed ocean.
3.
   - In the Stommel model, bottom friction is parameterized by a *linear drag*.
   - In the Munk model, lateral friction is parameterized by a *Newtonian harmonic viscosity* .

– Solution

1. The transport in the Sverdrup interior is equatorwards for an anti-cyclonic wind-stress-curl.
2. The Sverdrup transport is exactly balanced by a poleward transport in a westward boundary layer.
3. The boundary layer satisfies mass conservation, and must be a *western* boundary layer for friction to provide a force of opposite sign as the motion in the interior.

   The boundary layer is a *frictional boundary layer*.
4. The western location does not depend on the sign of the Coriolis parameter nor on the sign of the wind stress. The location does depend on the sign of β.
5.
   - In the Stommel model the balance in the western boundary layer is between $r\nabla^2\psi$ and $\beta\frac{\partial\psi}{\partial x}$. The boundary layer width is $\delta_S = \left(\frac{r}{\beta}\right)$. If r, the inverse frictional time, is $1/20$ days^{-1}, then $\delta_S \approx$60 km.
   - In the Munk model the balance in the western boundary layer is between $\nu\nabla^4\psi$ and $\beta\frac{\partial\psi}{\partial x}$. The boundary layer width is $\delta_M = \left(\frac{\nu}{\beta}\right)^{1/3}$.

5.5 Topographic effects on western boundary currents

We have so far assumed a flat ocean bottom in order to derive the equations of the Sverdrup, Stommel and Munk models. This allowed us to eliminate the depth-integrated pressure gradient force when taking the curl of the depth-integrated momentum budget. But the ocean is certianly not flat, and sloping sidewalls will actually change the behaviour of western boundary currents. They can even become inviscid if the flow is preserving its potential vorticity by flowing along f/h contours. If the ocean is flat, then a meridional flow within a boundary layer exists thanks to frictional effects permitting the flow to cross f contours. If sidewalls are sloping then the flow can move quasi-northward (along f/h contours) preserving its potential vorticity.

5.5.1 Bottom pressure stress

We now consider the effects of topography and stratification on the circulation of a wind-driven gyre. Interactions of pressure with a variable topography can generate a meridional flow. The vorticity balance of a depth-integrated flow now possesses and extra term describing the influence of topography on the flow.

Let's define $h = h(x, y)$ and let's consider a stratified ocean in which density is not a constant. The momentum equation in planetary-geostrophic approximation is

$$\boldsymbol{f} \times \boldsymbol{u} = -\nabla\phi + \boldsymbol{F} \tag{5.114}$$

where $\boldsymbol{F}$ represents both frictional and wind forcing terms. Integrating this over the entire depth of the water column

$$\boldsymbol{f} \times \overline{\boldsymbol{u}} = -\int_{\eta_B}^{\eta_T} \nabla\phi \, \mathrm{d}z + \overline{\boldsymbol{F}} \tag{5.115}$$

where $\overline{x} = \int_{\eta_B}^{\eta_T} x \, \mathrm{d}z$. Now remember the Leibnitz rule:

$$\nabla \int_{\eta_B}^{\eta_T} \phi \, \mathrm{d}z = \int_{\eta_B}^{\eta_T} \nabla\phi \, \mathrm{d}z + \phi_T \nabla\eta_T - \phi_B \nabla\eta_B, \tag{5.116}$$

where the second term on the rhs vanishes given that $\eta_T = 0$ at the top. For our purpose:

$$\int_{\eta_B}^{0} \nabla\phi \, \mathrm{d}z = \nabla \int_{\eta_B}^{0} \phi \, \mathrm{d}z + \phi_B \nabla \eta_B, \tag{5.117}$$

and so we write the vertically integrated momentum equations as

$$\boldsymbol{f} \times \overline{\boldsymbol{u}} = -\nabla \int_{\eta_B}^{0} \phi \, \mathrm{d}z - \phi_B \nabla \eta_B + \overline{\boldsymbol{F}}. \tag{5.118}$$

The second term on the rhs is the stress in the fluid due to the correlation between pressure gradient and topography. It is called bottom *form drag*.

If we rewrite the vertical integral of the pressure:

$$\int_{-h}^{0} \phi \, \mathrm{d}z = (\phi z)|_{-h}^{0} - \int_{-h}^{0} z(\partial\phi/\partial z)\mathrm{d}z = \phi_B h + \int_{-h}^{0} z\rho g \, \mathrm{d}z = \phi_B h + E, \tag{5.119}$$

where we have used hydrostasy $\partial\phi/\partial z = -\rho g$ and defined the vertically-integrated potential energy $E = g \int_{-h}^{0} z\rho \, \mathrm{d}z$.

Our vertically integrated momentum thus become

$$\begin{aligned}
\boldsymbol{f} \times \overline{\boldsymbol{u}} &= -\nabla \int_{\eta_B}^{0} \phi \, \mathrm{d}z - \phi_B \nabla \eta_B + \overline{\boldsymbol{F}} & (5.120)\\
&= -\nabla \int_{\eta_B}^{0} \phi \, \mathrm{d}z + \phi_B \nabla h + \overline{\boldsymbol{F}} & (5.121)\\
&= -\nabla \Big(\phi_B h + E\Big) + \phi_B \nabla h + \overline{\boldsymbol{F}} & (5.122)\\
&= -h\nabla\phi_B - \nabla E + \overline{\boldsymbol{F}}. & (5.123)
\end{aligned}$$

Where we have used $\nabla\eta_B = -\nabla h$, taking the top of the ocean at $z = 0$ and h the fluid column. To obtain a vorticity balance equation, and eliminating the pressure terms, we divide by h and take the curl. After using the streamfunction$(u, v) = (-\partial\psi/\partial y, \partial\psi/\partial x)$:

$$\boxed{J\Big(\psi, f/h\Big) + J\Big(h^{-1}, E\Big) = curl_z(\overline{\boldsymbol{F}}/h)} \tag{5.124}$$

Assuming a flat bottom and constant density, we see that a torque provided by the wind stress balances the torque introduced by bottom friction and a torque related to the change in planetary vorticity, just as in Stommel. However, now an extra term appears which is related to the combined effect of stratification and topographic variations (or Joint Effect of Baroclinicity And Relief - JEBAR - term): $J\left(h^{-1}, E\right)$. For a constant h, the JEBAR term vanishes and we recover the Stommel problem

$$\beta \frac{\partial \psi}{\partial x} = curl_z \overline{\boldsymbol{F}} \tag{5.125}$$

An alternative derivation accounting for the effect of topography and stratification is given by eliminating the potential energy term instead of the bottom pressure term. Going back to

$$\boldsymbol{f} \times \overline{\boldsymbol{u}} = -\nabla \int_{\eta_B}^{0} \phi \, \mathrm{d}z - \phi_B \nabla \eta_B + \overline{\boldsymbol{F}} \tag{5.126}$$

and taking the curl gives[3]

$$\beta \overline{v} = curl_z \overline{F} - curl_z(\phi_B \nabla \eta_B) = curl_z \overline{F} - J\left(\phi_B, \eta_B\right). \tag{5.127}$$

The last term on the rhs is the bottom pressure-stress curl, or form-drag curl, or bottom pressure torque. And now this equation holds for both a homogeneous and stratified fluid.

[3] $curl_z(h \nabla \phi_B) = curl_z(\phi_B \nabla \eta_B)$

f/h **contours**

For a homogeneous, frictionless and unforced gyre, the vorticity equation reduces to

$$\boxed{\beta\overline{v} = -J\Big(\phi_B, \eta_B\Big)} \tag{5.128}$$

or

$$\beta\overline{v} = -\nabla\phi_B \times \nabla\eta_B \tag{5.129}$$

and from the previous expression

$$J\Big(\psi, f/h\Big) = 0 \tag{5.130}$$

There can be a meridional flow only if pressure gradient has a component parallel to topographic contours (the isobars are not aligned with topographic contours), and the term on the rhs is non-zero. The meridional flow is driven by the curl of the form drag. In a flat-bottomed ocean, the form drag is zero, and the meridional flow must be forced or viscous.

If we consider an ocean where both forcing and friction are absent, and assuming an homogeneous gyre, the vorticity balance simplifies to

$$\boxed{J\Big(\psi, f/h\Big) = 0} \tag{5.131}$$

In an inviscid, unforced, and unstratified flow, ψ is a function of f/h, and streamlines of constant ψ and (f/h) contours coincide. In this case, the depth-integrated large-scale flow must follow f/h contours. The f/h contours form the characteristics of the differential equation above. This is called a *free mode*, driven solely by the bottom pressure-stress curl.

This is a statement about the balance between the vortex stretching by changes in topography and change in planetary vorticity of the fluid column. Consider a sloping sidewall, if a water column moves down the slope it will stretch in the vertical and increase its vorticity $(f+\zeta)$. On a basin scale this will be balanced by changes in f rather than changes in ζ, so the PV balance reduces to $q = f/h$. In order to conserve PV, the column will be displaced meridionally, moving along f/h contours. The new f will be modulated by the thickness change h_2/h_1. For a constant h, f/h contours would follow latitude circles.

This vorticity conservation principle is shown by the linear vorticity equation:

$$\beta v = f\frac{\partial w}{\partial z} + \frac{\partial}{\partial z}\Big(\tau_x^y - \tau_y^x\Big) \tag{5.132}$$

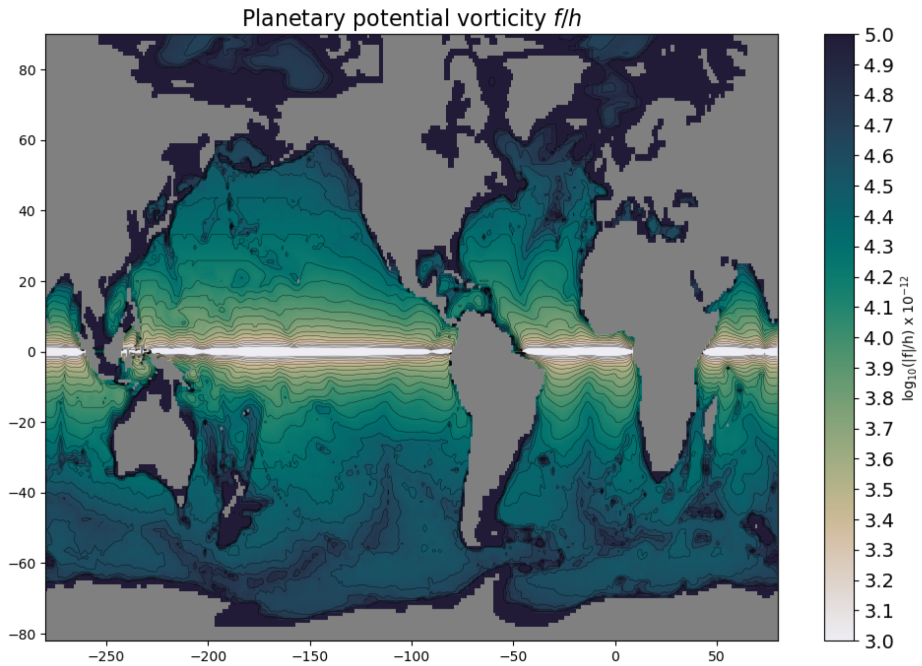

Figure 5.21: *Contours of planetary potential vorticity, f/h. Shown is $log_{10}(|f|/h\,[10^{-12}\,m^{-1}s^{-1}])$. For constant h, the f/h contours would follow latitude circles. The influence of topography on the depth-averaged flow is small in the tropics but becomes large at higher latitudes. In the Atlantic Ocean, the imprint of the mid-Atlantic ridge can be seen in the region of the subtropical gyres.*

Now, integrating vertically the vertical velocity does not vanish (assuming that $w_T = 0$):

$$\underbrace{\beta\overline{v}}_{\text{change in planetary vorticity}} = \underbrace{curl_z\boldsymbol{\tau}_T}_{\text{torque by the wind}} - \underbrace{curl_z\boldsymbol{\tau}_B}_{\text{torque by bottom friction}} - \underbrace{fw_B}_{\text{stretching of water column}} \tag{5.133}$$

Now that this is clear, we can go back to the vertically integrated vorticity balance

$$\beta\overline{v} = curl_z\overline{F} - curl_z(\phi_B\nabla\eta_B), \tag{5.134}$$

and considering both surface forcing and bottom drag we have the following vorticity budget for the vertically integrated flow

$$\underbrace{\beta\overline{v}}_{1} = \underbrace{curl_z\overline{\boldsymbol{\tau}}_T}_{2} - \underbrace{curl_z\overline{\boldsymbol{\tau}}_B}_{3} - \underbrace{curl_z(\phi_B\nabla\eta_B)}_{4}. \tag{5.135}$$

(1)+(2) is the Sverdrup balance; (1)+(2)+(3) is the Stommel/Munk problem. (4) introduces the bottom pressure torque.

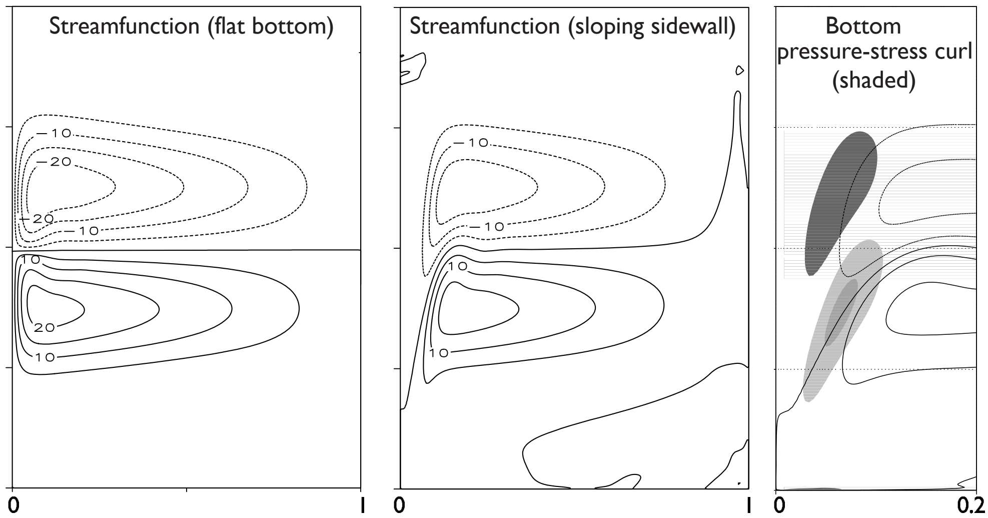

Figure 5.22: *Numerical results for a homogeneous problem, flat bottom domain and a domain with sloping western sidewall. The shaded regions in the right panels show the regions where bottom pressure-stress curl is important in the meridional flow of the western boundary current. [from Vallis (2006)]*

The torque by the wind stress drives a meridional flow across f-lines (Fig. 5.23), as in Sverdrup balance. The western boundary layer is then dominated by a balance between the meridional flow (βv) and the bottom pressure-stress curl. Only where the flow crosses f/h contours is friction needed (Fig. 5.23b). This happens where f/h contours converge and friction helps the flow move across f/h contours. In a flat-bottomed case, friction would be necessary all along the boundary layer in order to cross f contours(Fig. 5.23a).

The fact that the bottom pressure torque can play a more dominant role than frictional torque for the vorticity balance in the western boundary current questions the physical relevance of Stommel's model.

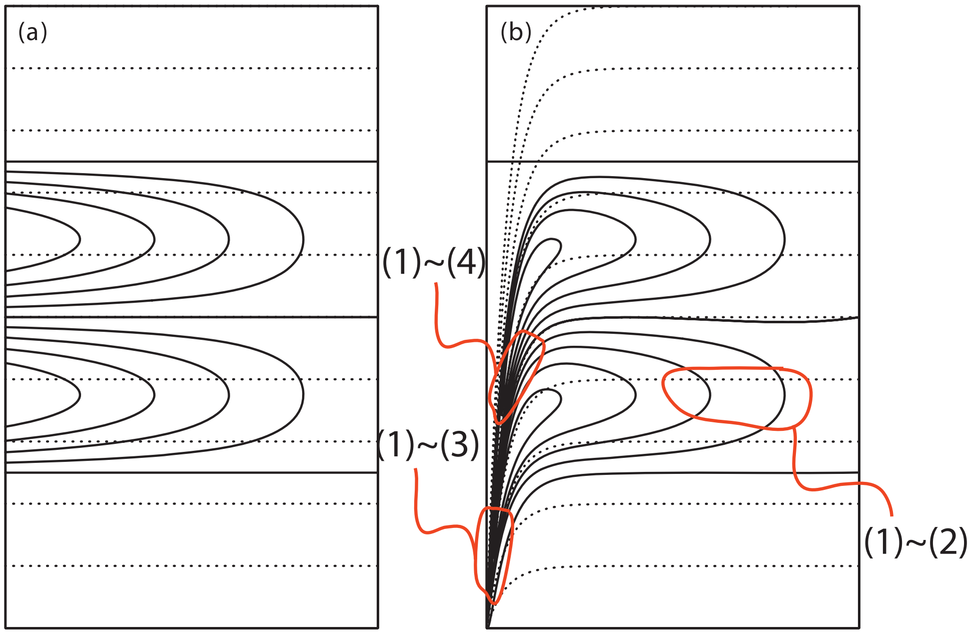

Figure 5.23: *The two-gyre Sverdrup flow for (a) a flat-bottomed domain and (b) a domain with sloping sidewalls. Dotted contours in (a) are f contours and in (b) f/h contours. [adapted from Vallis (2006)].*

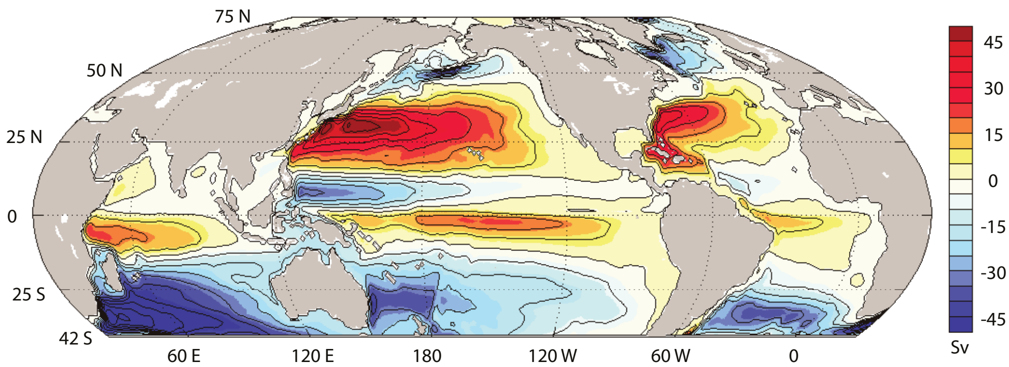

Figure 5.24: *A realistic barotropic streamfunction. [adapted from Vallis (2006)].*

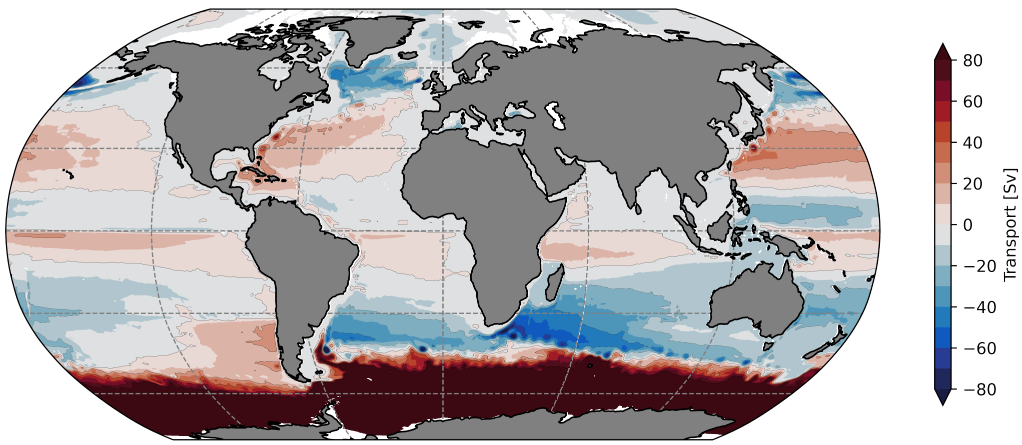

Figure 5.25: *The quasi-barotropic streamfunction from MOM at 0.25 degree resolution (time-mean for the period 2013-2017).*

Exercices

1. Compute the Sverdrup circulation in a rectangular ocean ($0 < x < L_x, 0 < y < L_y$) forced by a zonal wind stress

$$\tau_0^x(y) = -\tau_0 \, cos\frac{\pi y}{L_y}, \quad \tau_0^y = 0.$$

Take $\tau_0 > 0$ and a constant β. Show that the Sverdrup transport velocities and the streamfunction are

$$\begin{aligned} U &= -(L_x - x)\,\frac{\tau_0 \pi^2}{\beta {L_y}^2}\, cos\frac{\pi y}{L_y}, \\ V &= -\frac{\tau_0 \pi}{\beta L_y}\, sin\frac{\pi y}{L}, \\ \psi(x,y) &= (L_x - x)\frac{\tau_0 \pi}{\beta L_y}\, sin\frac{\pi y}{L_y}. \end{aligned}$$

Take the following parameters: L_x=5000 km, L_y=4000 km, τ_0=10^{-4}m^2 s^{-2} (the reference density ρ_0 is absorbed into the turbulent stress vector), $f = f_0 + \beta y$, with $f_0 = 7 \times 10^{-5}\, s^{-1}$, $\beta = 2 \times 10^{-11}\, m^{-1} s^{-1}$.

Show that the maximum transport across the basin width is $(\pi B/L)\tau_0/\beta$ and amounts to ~20 Sv.

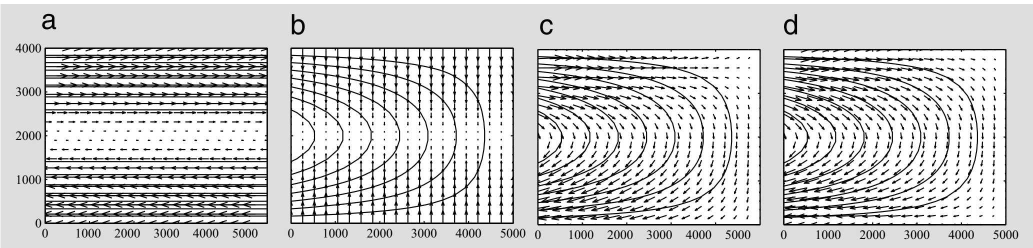

Figure 5.26: *(a) Wind stress pattern. (b) The transports due to the Ekman layer. (c) The geostrophic part with $U_g = U - U_E, V_g = V - V_E$ (note that $U_E = 0$). (d) The Sverdrup transport. The Sverdrup transport streamfunction ψ is shown in all plots.*

2. What happens, and why, to the transport at $\text{curl}_z \boldsymbol{\tau} = 0$?

3. Compute the Ekman, geostrophic and Sverdrup transports for the following parameters. What is the total flux through the basin?

- $\theta = 35°N$
- $\tau^x = 10^{-1}\ Nm^{-2}$
- $\tau^y = 0$
- L_y =1000 km; L_x=5000 km
- f=$10^{-4}\ s^{-1}$
- $\beta = 2 \times 10^{-11}\ m^{-1}s^{-1}$

4. Compute the Sverdrup subtropical meridional transport in the North Atlantic for the given parameters. What is the typical size of the interior velocity (cm s^{-1}) if the transport is carried over the upper 1 km of the ocean and the basin is 3000 km wide?

    - $\theta = 35°N$
    - $\tau^x = 0.1\ Nm^{-2}$
    - $\tau^y = -0.1\ Nm^{-2}$
    - $curl_z \boldsymbol{\tau} = -0.1 \times 10^{-6}\ N\ m^{-3}$
    - $\rho_0 = 10^3\ kgm^{-3}$
    - $\beta = 2 \times 10^{-11}\ m^{-1}s^{-1}$

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