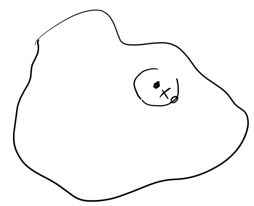
23 ottobre

<u>Def</u> Se $f : I \longrightarrow \mathbb{R}$, $x_0 \in \mathring{I}$ (parte interna di I)

allora

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$



se il limite esiste ed è finito. Se questo succede $\forall\, x_0 \in I$ diciamo che f è derivabile in I.

Osservazione
In quest'ultimo caso resta definita una nuova funzione f' e se questa è continua, cioè se $f' \in C^0(I)$, scriviamo che $f \in C^1(I)$.

<u>Lemma</u> Sia $f : I \to \mathbb{R}$ e $x_0 \in I$ t.c. $f'(x_0)$ esiste. Allora f è continua in x_0.

<u>Dim</u> Si tratta di dimostrare che $\lim_{x \to x_0} f(x) = f(x_0)$

$$\iff \lim_{x \to x_0} \big(f(x) - f(x_0)\big) = 0 \quad ✓$$

$$\lim_{x \to x_0} \big(f(x) - f(x_0)\big)\, 1 = \lim_{x \to x_0} \big(f(x) - f(x_0)\big) \frac{x - x_0}{x - x_0}$$

$$= \lim_{x \to x_0} \underbrace{\frac{f(x) - f(x_0)}{x - x_0}}_{\to f'(x_0)} \underbrace{(x - x_0)}_{\to 0} = f'(x_0) \cdot 0 = 0$$

<u>Corollario</u> Se $f \in C^1(I)$, cioè se f è derivabile in I con $f' \in C^0(I)$, risulta anche $f \in C^0(I)$.

<u>Osservazione</u> Qualche volta scriveremo $f = f^{(0)}$

Esempi di derivate.

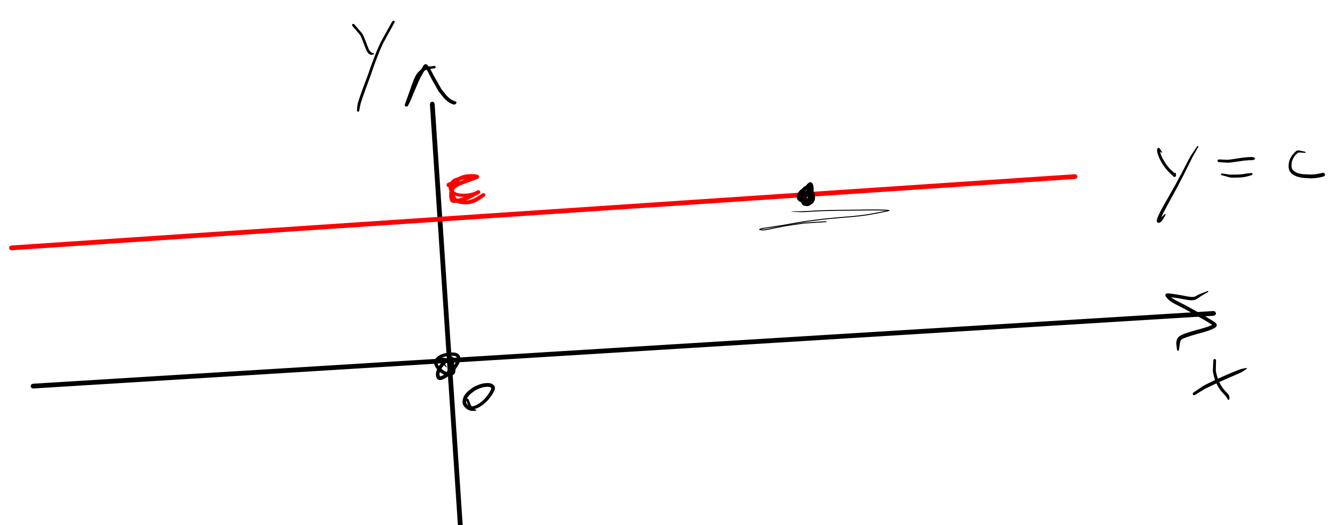
1) $\overset{c(x)}{c}$ funzione costante in $\mathbb{R}$.

$$\lim_{x \to x_0} \frac{c(x) - c(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{c - c}{x - x_0} = \lim_{x \to x_0} \frac{0}{x - x_0}$$

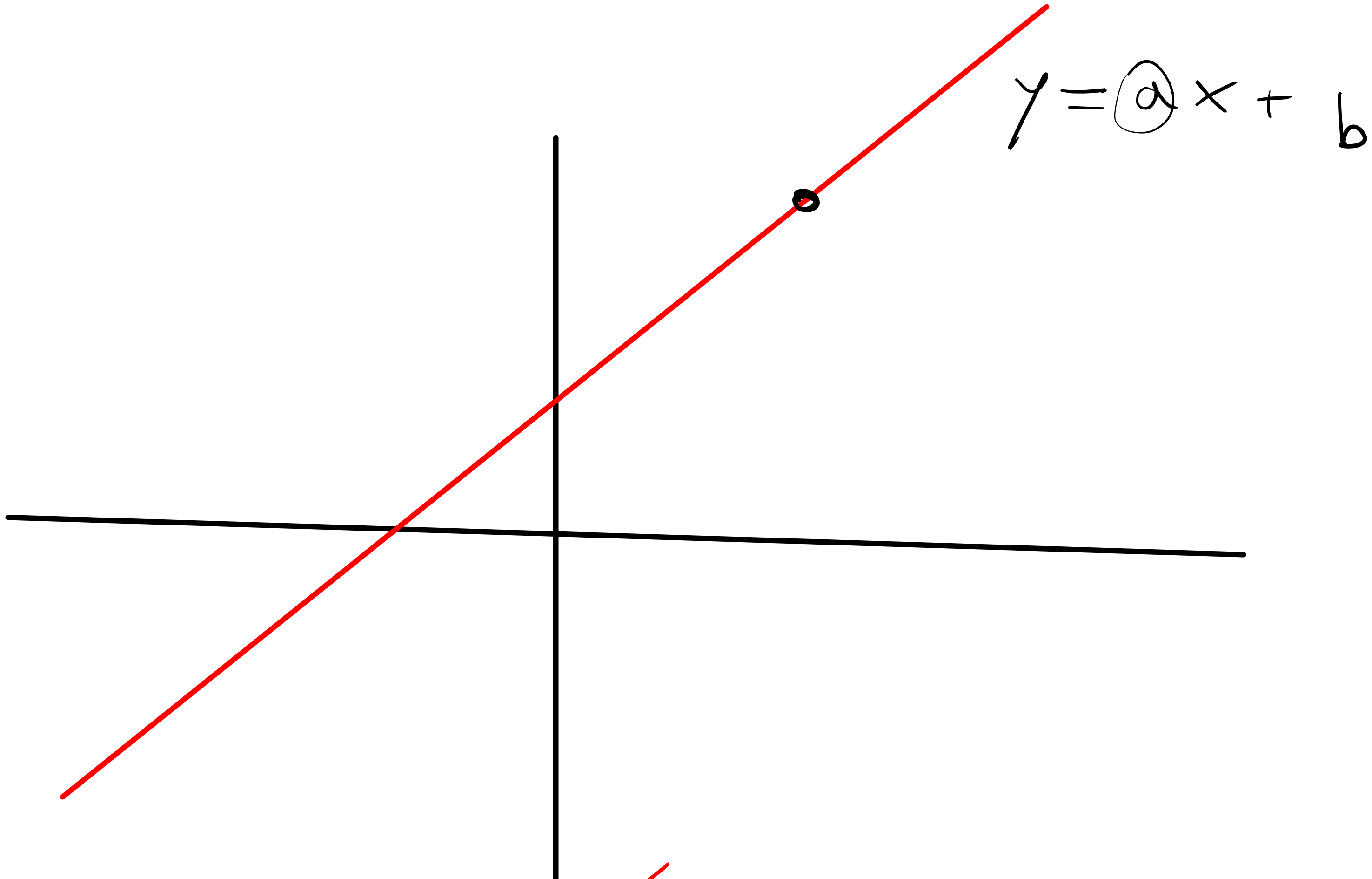
$$= \lim_{x \to x_0} 0 = 0$$

$(c)' \equiv 0$ La funzione è derivabile in $\mathbb{R}$.

$$(c)' = 0 \in C^0(\mathbb{R}) \Rightarrow c \in C^1(\mathbb{R})$$



2) $f(x) = ax + b$



$$\frac{f(x) - f(x_0)}{x - x_0} = \frac{ax + \cancel{b} - (ax_0 + \cancel{b})}{x - x_0} = a \frac{x - x_0}{x - x_0} =$$

$$= a$$

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} a = a$$

3) $(e^x)' = e^x$

cominciamo con $(e^x)'(0) = 1 = e^0$ ✓

$$(e^x)'(0) = \lim_{x\to 0} \frac{e^x - e^0}{x-0} = \lim_{x\to 0} \frac{e^x - 1}{x} = 1$$

(limite notevole)

$$(e^x)'(x_0) = \lim_{x\to x_0} \frac{e^x - e^{x_0}}{x - x_0} =$$

$x = h + x_0 \qquad h = x - x_0$

$$= \lim_{h\to 0} \frac{e^{h+x_0} - e^{x_0}}{h}$$

$$= \lim_{h\to 0} \frac{e^h e^{x_0} - e^{x_0}}{h} = \lim_{h\to 0} \underbrace{e^{x_0}}_{\to e^{x_0}} \underbrace{\frac{e^h - 1}{h}}_{\xrightarrow{h\to 0} 1}$$

$$= e^{x_0}$$

$\Rightarrow \exists\ (e^x)'(x_0)$ e che è uguale ad e^{x_0}

$$(e^x)' = e^x$$

4) (Regola della potenza)

sia $a \in \mathbb{R}$, consideriamo

x^a definito in $\mathbb{R}_+$.

$$(x^a)' = a\, x^{a-1} \quad \forall\ x > 0$$

$x_0 > 0$

$$\lim_{x \to x_0} \frac{x^a - x_0^a}{x - x_0} \qquad x = h + x_0$$

(Useremo $\displaystyle\lim_{y \to 0} \frac{(1+y)^a - 1}{y} = a$)

$$= \lim_{h \to 0} \frac{(h+x_0)^a - x_0^a}{h} =$$

$$= \lim_{h \to 0} \frac{\left(\left(1+\frac{h}{x_0}\right)x_0\right)^a - x_0^a}{h} = \lim_{h \to 0} \frac{\left(1+\frac{h}{x_0}\right)^a x_0^a - x_0^a}{h}$$

$$= \lim_{h \to 0} x_0^a \frac{\left(1+\frac{h}{x_0}\right)^a - 1}{h}$$

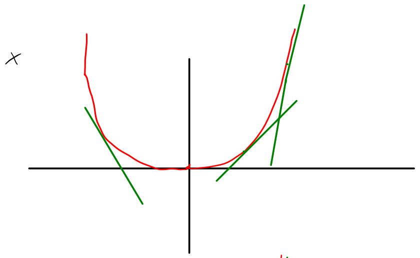
$$= x_0^a \lim_{h \to 0} \frac{\left(1+\frac{h}{x_0}\right)^a - 1}{h \frac{x_0}{x_0}} = x_0^{a-1} \underbrace{\lim_{h \to 0} \frac{\left(1+\frac{h}{x_0}\right)^a - 1}{\frac{h}{x_0}}}_{y = \frac{h}{x_0}} = a$$

$$= a\, x_0^{a-1}$$

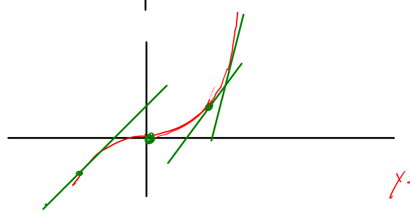
Conclusione $(x^a)' = a\, x^{a-1} \quad \forall\ x > 0$.

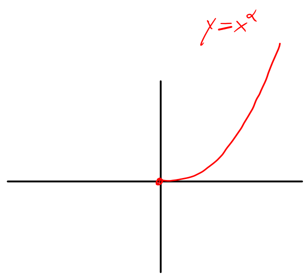
$(x^{\alpha})' = \alpha \, x^{\alpha - 1}$

$(x^2)' = 2x$



$(x^3)' = 3x^2$

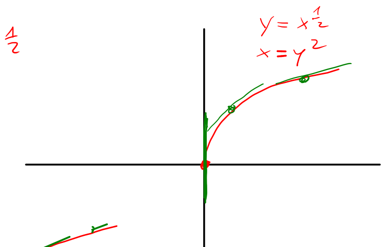




In generale se $\alpha > 1$

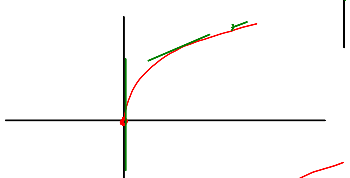
$(x^{\alpha})' = \alpha \, x^{\alpha - 1}$

$(x^{\frac{1}{2}})' = \frac{1}{2} x^{\frac{1}{2} - 1} = \frac{1}{2} x^{-\frac{1}{2}}$

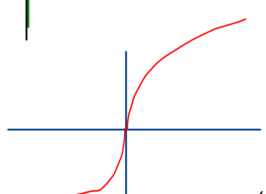


$0 < \alpha < 1$

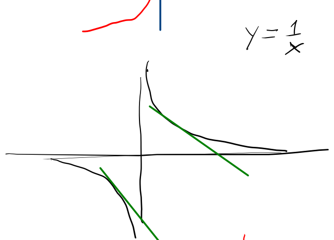
$(x^{\alpha})' = \alpha \, x^{\alpha - 1}$



$(x^{\frac{1}{3}})' = \frac{1}{3} x^{-\frac{2}{3}}$

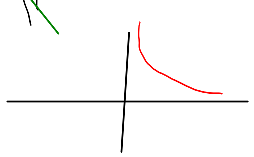


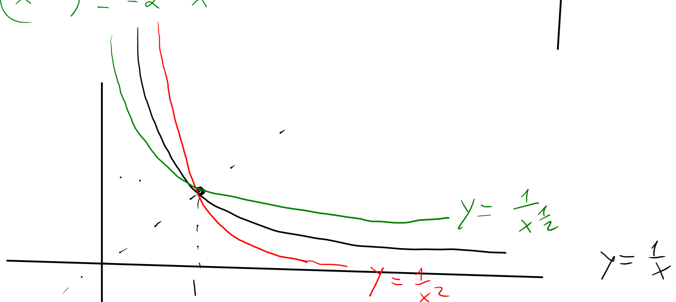
$(x^{-1})' = -x^{-2}$



$\alpha > 0$

$(x^{-\alpha})' = -\alpha \, x^{-\alpha - 1}$





Teorema Siano $f, g : I \to \mathbb{R}$, $x_0 \in I$ ed esistono $f'(x_0)$, $g'(x_0)$. Allora

1) $(f+g)'(x_0) = f'(x_0) + g'(x_0)$ (regola della somma)

2) $(f g)'(x_0) = f'(x_0) g(x_0) + f(x_0) g'(x_0)$
(regola del prodotto o di Leibniz)

2') $(cf)'(x_0) = c\, f'(x_0) \quad \forall\, c \in \mathbb{R}$

2'') $(\lambda f + \mu g)'(x_0) = \lambda f'(x_0) + \mu g'(x_0) \quad \forall\, \lambda, \mu \in \mathbb{R}$.

3) $$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0) g(x_0) - f(x_0) g'(x_0)}{g^2(x_0)}$$

3') $$\left(\frac{1}{g}\right)'(x_0) = -\frac{g'(x_0)}{g^2(x_0)} .$$

Osservazione La 3) segue dalla 2) e dalla 3')

Infatti

$$\left(\frac{f}{g}\right)' = \left(f \, \frac{1}{g}\right)' = f' \, \frac{1}{g} + f \left(\frac{1}{g}\right)'$$

$$= f' \frac{1}{g} - f \frac{g'}{g^2} = \frac{f' g - f g'}{g^2}$$

Dim di 2'' . $\lambda, \mu \in \mathbb{R}$

$$\lim_{x \to x_0} \frac{\lambda f(x) + \mu g(x) - (\lambda f(x_0) + \mu g(x_0))}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{\lambda (f(x) - f(x_0)) + \mu (g(x) - g(x_0))}{x - x_0}$$

$$= \lim_{x \to x_0} \Big[\underbrace{\lambda}_{\to \lambda} \underbrace{\frac{f(x) - f(x_0)}{x - x_0}}_{\to f'(x_0)} + \underbrace{\mu}_{\to \mu} \underbrace{\frac{g(x) - g(x_0)}{x - x_0}}_{\to g'(x_0)} \Big]$$

$$= \lambda f'(x_0) + \mu g'(x_0) .$$

Dimostrazione di 2) ma prima una osservazione

Osservazione

$$\cancel{\left(f(x)\,g(x)\right)' = f'(x)\,g'(x)}$$

Dim di 2

$$\lim_{x\to x_0} \frac{f(x)\,g(x) - f(x_0)\,g(x_0)}{x - x_0}$$

$$= \lim_{x\to x_0} \frac{\left(f(x)\,g(x) - f(x_0)\,g(x)\right) + \left(f(x_0)\,g(x) - f(x_0)\,g(x_0)\right)}{x - x_0}$$

$$= \lim_{x\to x_0} \left[\underset{\substack{\Big\downarrow\, x\to x_0 \\ g(x_0)}}{g(x)}\ \underset{\substack{\Big\downarrow\, x\to x_0 \\ f'(x_0)}}{\frac{f(x) - f(x_0)}{x - x_0}} + \underset{\substack{\Big\downarrow\, x\to x_0 \\ f(x_0)}}{f(x_0)}\ \underset{\substack{\Big\downarrow\, x\to x_0 \\ g'(x_0)}}{\frac{g(x) - g(x_0)}{x - x_0}} \right]$$

$$= f'(x_0)\,g(x_0) + f(x_0)\,g'(x_0)$$