24 ottobre

Esempio $\quad x > 0$

(1) $(\lg x)' = x^{-1}$ $\qquad ((x^{\alpha})' = \alpha\, x^{\alpha - 1})$

$x = h + x_0 \qquad x_0 > 0$

$$\lim_{x \to x_0} \frac{\lg(x) - \lg(x_0)}{x - x_0} = \lim_{h \to 0} \frac{\lg(h + x_0) - \lg(x_0)}{h}$$

(2) $\displaystyle \lim_{y \to 0} \frac{\lg(1+y)}{y} = 1$

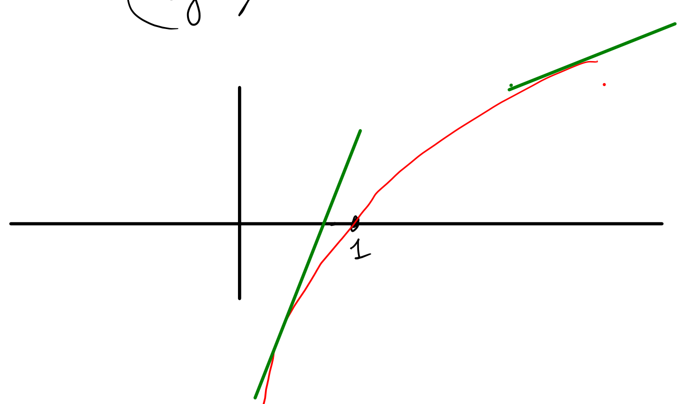
$$= \lim_{h \to 0} \frac{\lg\left(x_0\left(1 + \frac{h}{x_0}\right)\right) - \lg(x_0)}{h} =$$

$$= \lim_{h \to 0} \frac{\cancel{\lg(x_0)} + \lg\left(1 + \frac{h}{x_0}\right) - \cancel{\lg(x_0)}}{h}$$

$$= \lim_{h \to 0} \frac{\lg\left(1 + \frac{h}{x_0}\right)}{\frac{h}{x_0}\, x_0} = \lim_{h \to 0} \frac{1}{x_0} \frac{\lg\left(1 + \frac{h}{x_0}\right)}{\frac{h}{x_0}}$$

$$= x_0^{-1} \lim_{h \to 0} \frac{\lg\left(1 + \frac{h}{x_0}\right)}{\frac{h}{x_0}} \underset{y = \frac{h}{x_0}}{=} x_0^{-1} \underbrace{\lim_{y \to 0} \frac{\lg(1+y)}{y}}_{1} = x_0^{-1}$$

Conclusione $\quad (\lg x)' = \frac{1}{x} = x^{-1} \quad \forall x > 0$



Teor (derivata funzione inversa)

Sia $f : I \longrightarrow J$ ~~biettiva~~ strettamente monotona e continua, con I

J due intervalli, $x_0 \in I$, $y_0 \in J$ $\quad f(x_0) = y_0$.

Sia $g : J \longrightarrow I$ la funzione inversa. $\quad (x_0 = g(y_0))$

supponiamo che esista $f'(x_0) \neq 0$. Allora

$$g'(y_0) = \frac{1}{f'(x_0)}.$$

Dim Per definizione $f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$

$$\begin{aligned} x &= g(y) \\ y &= f(x) \end{aligned}$$

$$* \quad \lim_{y \to y_0} \frac{g(y) - g(y_0)}{y - y_0} =$$

$$\lim_{y \to y_0} g(y) = g(y_0) = x_0$$

$$= \lim_{x \to x_0} \frac{x - x_0}{f(x) - f(x_0)} = \lim_{x \to x_0} \frac{1}{\frac{f(x) - f(x_0)}{x - x_0}}$$

$$= \frac{1}{f'(x_0)}$$

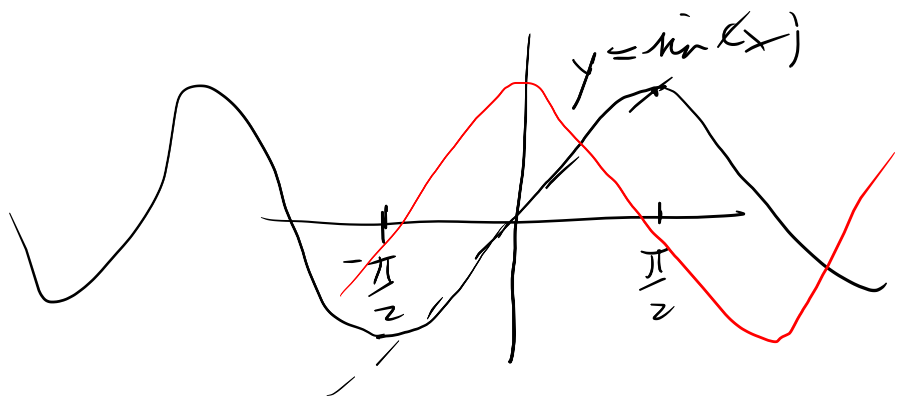
Per ogni x t.c. $f'(x) \neq 0$ avremo

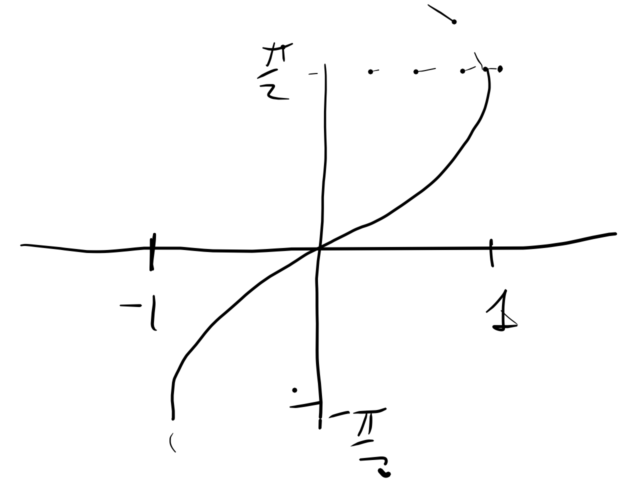
$$\boxed{g'(y) = \frac{1}{f'(x)}} \qquad \begin{aligned} y &= f(x) \\ x &= g(y) \end{aligned}$$

Se denoto con y o $y(x)$ la funzione f e con x o $x(y)$ la funzione g

$$\boxed{\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}}$$

Corollario $\quad \operatorname{arcsen} y : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$





allora per $\quad y \in (-1, 1)$

$$\operatorname{arcsen}'(y) = \frac{1}{\sqrt{1-y^2}}$$

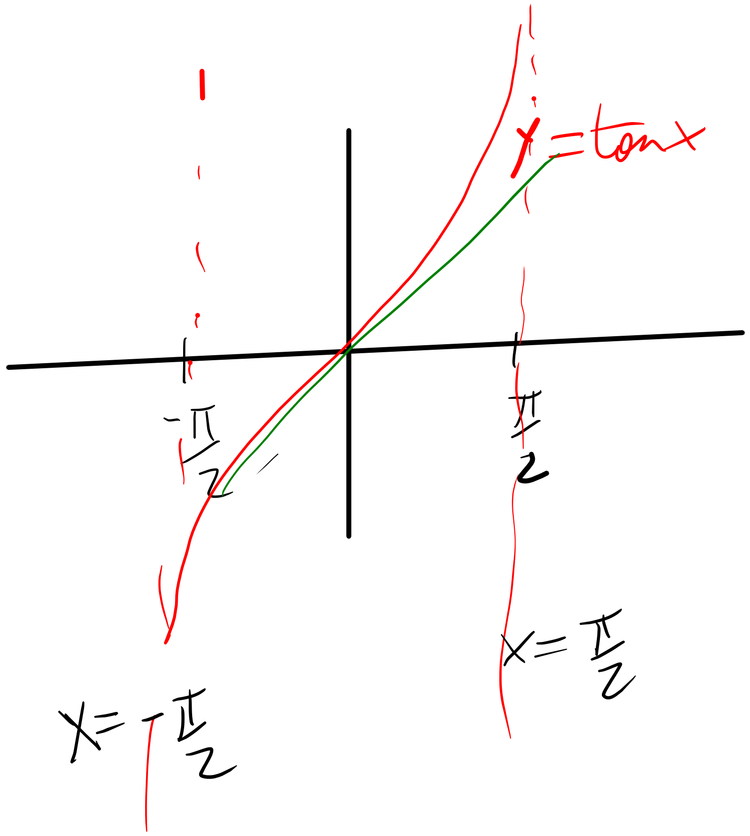
$x = \arcsin y$

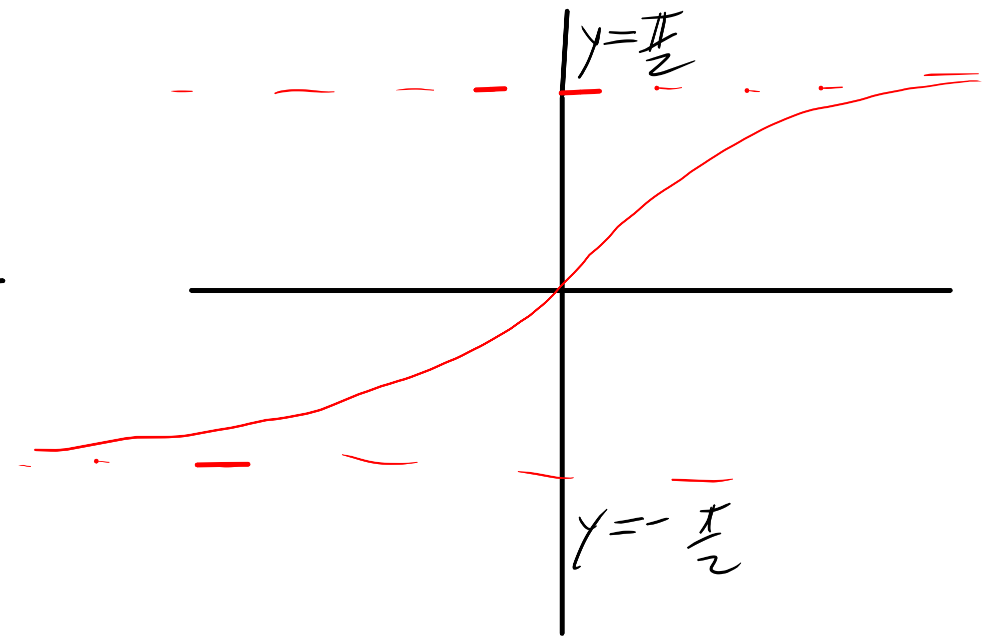
Dim $\quad$ sia $\quad y = \sin x \qquad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\arcsin'(y) = \frac{1}{\sin'(x)} = \frac{1}{\cos x} = \frac{1}{\sqrt{1-\sin^2 x}}$$

$$= \frac{1}{\sqrt{1-y^2}}$$

Corollario $\quad (\arctan y)' = \dfrac{1}{1+y^2}$





$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Dim $\quad y = \tan x \qquad \overset{\Psi}{x} = \arctan y$

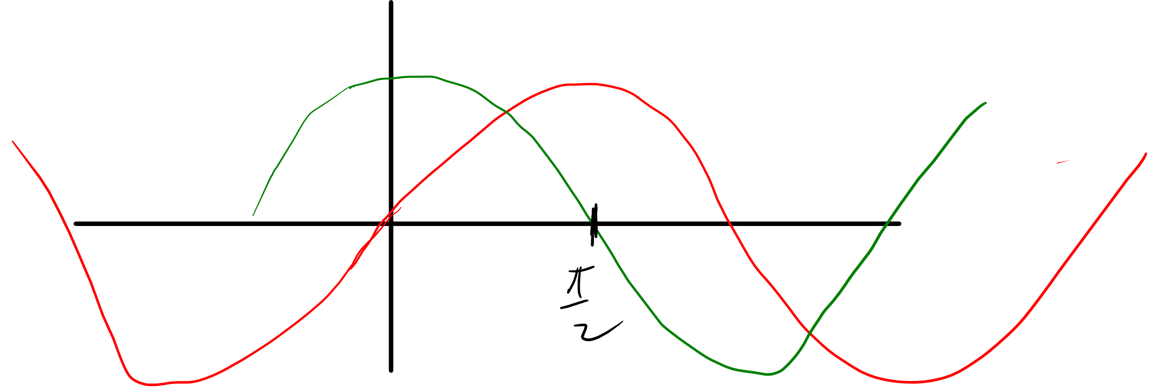
$$(\arctan y)' = \frac{1}{\tan' x} = \frac{1}{1+\tan^2 x} = \frac{1}{1+y^2}$$

(fra un attimo verifichiamo che $\tan'(x) = 1+\tan^2 x$)

Teor

$$(\sin x)' = \cos x$$

$$(\cos x)' = -\sin x$$



Dim

$$\lim_{x \to x_0} \frac{\sin x - \sin x_0}{x - x_0} \qquad x = h + x_0$$

$$= \lim_{h \to 0} \frac{\sin(h + x_0) - \sin x_0}{h} =$$

$$= \lim_{h \to 0} \left[\frac{\sin(h)\cos(x_0) + \cos(h)\sin x_0 - \sin x_0}{h} \right]$$

$$= \lim_{h \to 0} \left[\cos(x_0) \underbrace{\frac{\sin h}{h}}_{\downarrow\, 1} + \sin x_0 \underbrace{\frac{\cos(h) - 1}{h}}_{\downarrow\, h \to 0 \atop 0} \right] = \cos x_0$$

$$\lim_{h \to 0} \frac{\cos(h) - 1}{h} = - \lim_{h \to 0} \frac{1 - \cos(h)}{h \; h} \, h$$

$$= - \lim_{h \to 0} \underbrace{\frac{1 - \cos(h)}{h^2}}_{\downarrow\, \frac{1}{2}} \underbrace{h}_{\downarrow\, 0} = 0$$

$$(\sin x)' = \cos x$$

ora dimostriamo $(\cos x)' = -\sin x$

$$\lim_{x \to x_0} \frac{\cos(x) - \cos(x_0)}{x - x_0} \qquad x = h + x_0$$

$$= \lim_{h \to 0} \frac{\cos(h+x_0) - \cos(x_0)}{h} =$$

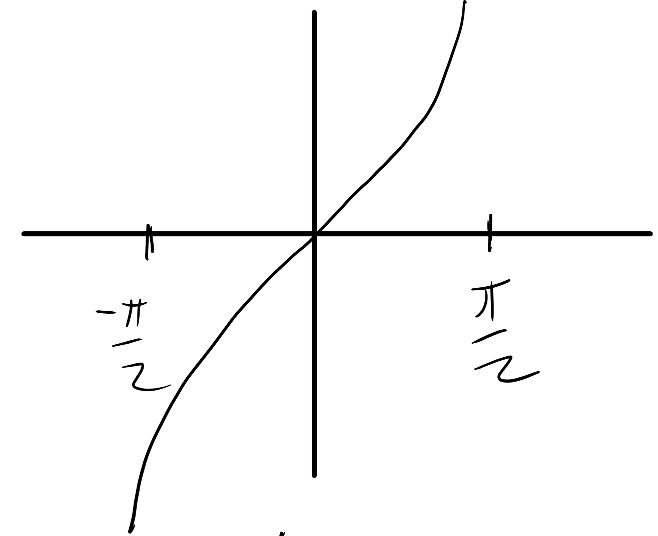
$$= \lim_{h \to 0} \frac{\cos(h)\cos(x_0) - \sin(h)\sin x_0 - \cos(x_0)}{h}$$

$$= \lim_{h \to 0} \left[\underbrace{-\frac{\sin(h)}{h}\sin x_0}_{\to -\sin x_0} + \cos(x_0) \underbrace{\frac{\cos(h) - 1}{h}}_{\to 0} \right]$$

$$= -\sin x_0$$

$$(\cos x)' = -\sin x$$

Teor $(\tan x)' = 1 + \tan^2 x = \dfrac{1}{\cos^2 x}$



Dim

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{\sin' x \cos x - \sin x \cos' x}{\cos^2 x}$$

$$\left(\frac{f}{g}\right)' = \frac{f' g - f g'}{g^2} \qquad = \frac{\cos x \cos x - \sin x (-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$= \frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} = 1 + \left(\frac{\sin x}{\cos x}\right)^2 = 1 + \tan^2 x$$

Osservazione $\sec(x) := \dfrac{1}{\cos x}$ $\qquad \tan'(x) = \dfrac{1}{\cos^2 x} = \sec^2(x)$

$\operatorname{cosec}(x) := \dfrac{1}{\sin x}$

<u>Teor</u> $(\sinh x)' = \cosh x$

$(\cosh x)' = \sinh x$

$(\mathrm{th}\, x)' = \frac{1}{\mathrm{ch}^2 x} = 1 - \mathrm{th}^2(x)$

<u>Dim</u>

$(\mathrm{sh}\, x)' = \left(\frac{e^x - e^{-x}}{2}\right)' = \left(\frac{1}{2}e^x - \frac{1}{2}e^{-x}\right)'$

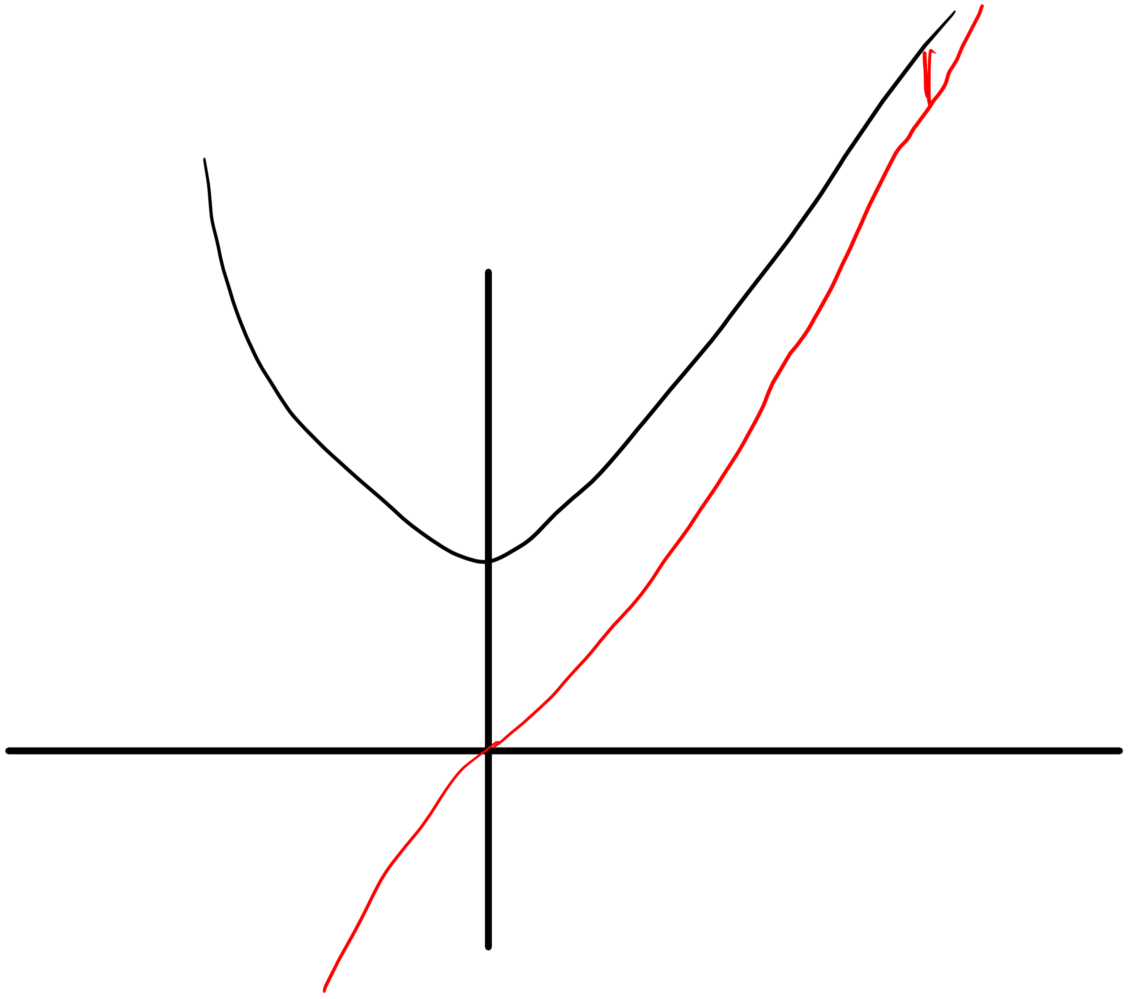
$= \frac{1}{2}(e^x)' - \frac{1}{2}(e^{-x})'$ $\boxed{(e^{-x})' = -e^{-x}}$

$= \frac{1}{2}e^x + \frac{1}{2}e^{-x} = \frac{e^x + e^{-x}}{2} = \mathrm{ch}\, x$

$$(\operatorname{ch} x)' = \left(\frac{e^{x}+e^{-x}}{2}\right)' = \left(\frac{1}{2}e^{x} + \frac{1}{2}e^{-x}\right)'$$

$$= \frac{1}{2}\left(e^{x}\right)' + \frac{1}{2}\left(e^{-x}\right)' = \frac{1}{2}e^{x} - \frac{1}{2}e^{-x} =$$

$$= \frac{e^{x}-e^{-x}}{2} = \operatorname{sh} x$$



$$(\operatorname{th} x)' = \left(\frac{\operatorname{sh} x}{\operatorname{ch} x}\right)' = \frac{\operatorname{sh}' x \operatorname{ch} x - \operatorname{sh} x \operatorname{ch}' x}{\operatorname{ch}^2 x} =$$

$$= \frac{\operatorname{ch} x \operatorname{ch} x - \operatorname{sh} x \operatorname{sh} x}{\operatorname{ch}^2 x} =$$

$$\left(\frac{f}{g}\right)' = \frac{f' g - f g'}{g^2}$$

$$= \frac{\overbrace{\operatorname{ch}^2 x - \operatorname{sh}^2 x}^{1}}{\operatorname{ch}^2 x} = 1 - \operatorname{th}^2 x$$

$$= \frac{1}{\operatorname{ch}^2 x} = \operatorname{sech}^2 x$$

<u>Osservazione</u> $\operatorname{sech} x = \dfrac{1}{\operatorname{ch} x}$

o piccolo

Def Siano f, g due funzioni $X \longrightarrow \mathbb{R}$ ($X \subseteq \mathbb{R}$)

$x_0 \in X'$ (eventualmente $x_0 = \pm\infty$)

Scriviamo che $f(x) = o(g(x))$ se

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0$$

Esempio 1) Se $\lim_{x \to x_0} f(x) = 0$ scrivo $f(x) = o(1)$

2) $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$ $\qquad 1 - \cos x = o(x)$

Esercizio $o(x) + o(x) = o(x)$

$\lim_{x \to x_0} f(x) = L \in \mathbb{R} \implies f(x) = L + o(1)$