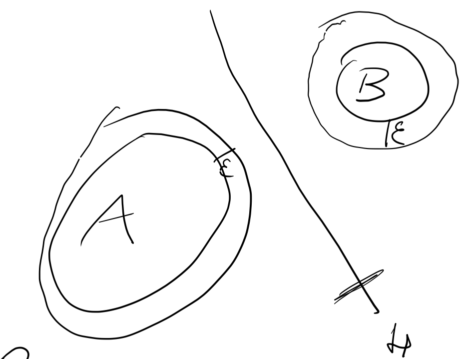
October 24

$A\cap B=\emptyset$

Theorem Let X be normed A, B convex nonempty convex A closed B is compact. Then $\exists$ a closed hyperplane H separating them strictly.



Pf

We claim that $\exists\ \varepsilon>0$ s.t.

$$(A+D_X(0,\varepsilon))\cap(B+D_X(0,\varepsilon))=\emptyset$$

~~If this is true~~ we apply the 1° version $\exists$ $f\in X'$ ($f\neq 0$) and $\alpha\in\mathbb{R}$

s.t. $H=f^{-1}(\alpha)$ separates these sets

$$f(a+z)\le\alpha\le f(b+z') \qquad \forall\ a\in A,\ b\in B,\ z,z'\in D_X(0,\varepsilon)$$

$$f(a)+f(z)\le\alpha\le f(b)+f(z')$$

$$0<\varepsilon_1=\sup\{f(z):\ z\in D_X(0,\varepsilon)\}$$

$$0>-\varepsilon_2=\inf\{f(z'):\ z'\in D_X(0,\varepsilon)\}$$

$$f(a)+\varepsilon_1\le\alpha\le f(b)-\varepsilon_2$$

$$\varepsilon_3=\min\{\varepsilon_1,\varepsilon_2\}$$

$$f(a)\le\alpha-\varepsilon_3\le\alpha+\varepsilon_3\le f(b) \qquad \forall\ a\in A, \forall\ b\in B.$$

$\exists\ \varepsilon > 0$

$$\left(A + D_X(0,\varepsilon)\right) \cap \left(B + D_X(0,\varepsilon)\right) = \phi$$

If false $\forall \varepsilon > 0$

$$z_n \in \left(A + D_X(0,\varepsilon_n)\right) \cap \left(B + D_X(0,\varepsilon_n)\right) \neq \phi$$

in A $\quad$ B

$\varepsilon_n \downarrow 0 \qquad \exists\ \{a_n\} \quad \{b_n\}$

$|z_n - a_n| < \varepsilon_n, \quad |b_n - z_n| < \varepsilon.$

$|a_n - b_n| < 2\varepsilon_n.$

$\{b_n\}$ has a convergent subsequence

It not restrive to assume $\{b_n\}$ convergente.

$b_n \xrightarrow{n \to +\infty} b \in B$

$a_n \nearrow$

$\{a_n\}$ is in A and A is closed

$\Rightarrow b \in A$

$b \in A \cap B = \phi$ $\quad \lrcorner$

Cor $Y \subseteq X$ locally convex vector space

$$\overline{Y} \subsetneq X \Rightarrow \exists\, f \in X' \setminus \{0\} \text{ s.t.}$$

$$f|_Y \equiv 0$$

Thm (Müntz-Szasz)

$I=[0,1]$, $0<\lambda_1<\lambda_2<\cdots\cdots$ $\lambda_n \longrightarrow +\infty$

$Y = \mathrm{Sp}\{1, t^{\lambda_1}, t^{\lambda_2}, \ldots\ldots\} \subseteq C^0(I)$

$\overline{Y} \subseteq C^0(I)$

1) If $\sum_{n=1}^{\infty} \frac{1}{\lambda_n} = +\infty \Rightarrow \overline{Y} = C^0(I)$

2) If $\sum_{n=1}^{\infty} \frac{1}{\lambda_n} < +\infty$ then $\overline{Y} \not\ni t^{\lambda}$

for any $\lambda \notin \{\lambda_1, \ldots\ldots\}$
$\lambda > 0$

Ex. $\lambda_j = j$, Y is the space $\mathbb{R}[t]$

which is known to be dense in $C^0(I)$

by Weierstrass approx theorem.

Algebraic basis are called Hamel bases.

$C^0(I) \xrightarrow{\Lambda} \mathbb{R}$ $\quad f \le g \Rightarrow \Lambda f \le \Lambda g \Rightarrow \exists$ positive measure

$(C^0(I))' = \{$ the set of all non necessarily positive Borel bounded measures in $I \}$ st $\Lambda f = \int_I f \, d\mu$

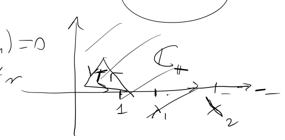
$d\mu$

$$\int_I t^{\lambda_n} \, d\mu(t) = \int_I d\mu = 0 \Rightarrow \mu \equiv 0$$

$$f(z) = \int_I t^z \, d\mu(t) \qquad \operatorname{Re} z > 0$$

$f \in H(\mathbb{C}_+)$ $\quad f(\lambda_n) = 0 \quad \forall n$

$f \in C^0(\mathbb{C}_+, \mathbb{C})$



this uses dominated convergence $\quad \int_{\partial\Delta} f(z)\, dz =$

$$= \int_I d\mu(t) \underbrace{\int_{\partial\Delta} t^z \, dz}_{0} = 0$$

Morera theorem $\Rightarrow f \in H(\mathbb{C}_+)$

$f \in L^\infty(\mathbb{C}_+)$ $\qquad H \cap L^\infty = H^\infty(\mathbb{C}_+)$

$$\left| \int_I t^z \, d\mu(t) \right| \le \int_I |t^z| \, d|\mu(t)|$$

$z = x + iy$

$0 \le |t^z| = t^x \le 1$ $\qquad |t^{iy}| = 1$

$$g(z) = f\left(\frac{1+z}{1-z} \right)$$

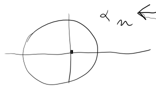
$U \ni z \longrightarrow \frac{1+z}{1-z} \in \mathbb{C}_+$

$\alpha_n \longleftrightarrow \lambda_n$

$f(\lambda_n) = 0$

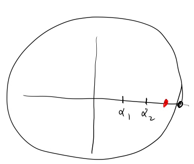
$\Updownarrow$

$g(\alpha_n) = 0$



$\lambda_n = \frac{1+\alpha_n}{1-\alpha_n}$ $\qquad g \in H^\infty(U)$

$\alpha_n = \frac{\lambda_n - 1}{1 + \lambda_n}$ $\qquad \left(\frac{t-1}{t+1} \right)' = \frac{2}{(t+1)^2} > 0$



$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n} = +\infty = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \sum_{n=1}^{\infty} \frac{1-|\alpha_n|}{1+\alpha_n} \le$$

$$\le \frac{1}{1+\alpha_1} \sum_{n=1}^{\infty} (1-|\alpha_n|) = +\infty$$

$g \equiv 0$

Recall that if $g \in H(U)$ and if $Z = \{ z \in U : g(z) = 0 \}$ has an accumulation point inside U then $g \equiv 0$.

There is a theorem which says that if $g \in H^\infty(U)$ and if Z contains a sequence $\{\alpha_n\}$ of distinct terms s.t. $\sum_{n=1}^{\infty} (1-|\alpha_n|) = +\infty \Rightarrow g \equiv 0$.

$\int t^{\lambda_n} \, d\mu(t) = \int d\mu(t) = 0 \quad \forall n \Rightarrow$

$f \equiv 0$

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$f(n) = 0 \quad \forall n$

$$0 = \int_I t^n \, d\mu(t) = \int_I d\mu(t)$$

$d\mu$ $\qquad \overline{\operatorname{Span}\{1, t, t^2, t \dots \}} \in C^0(I)$

Theorem (Krein Milman)

Let X be locally convex and K convex and compact.

Then K ~~coincides~~ with the closure of the convex hull of $\mathrm{Ext}(K)$ (= set of extreme points of K)

$x \in K$ is a point in $\mathrm{Ext}(K)$ s.t. whenever

$x = x_t = (1-t)\underset{\in K}{x_0} + t \underset{\in K}{x_1}$ with $t \in (0,1)$

$\Rightarrow x_0 = x_1 = x.$

convex hull $(\mathrm{Ext}\, K)$ = intersection of all convex sets containing $\mathrm{Ext}(K)$.

Recall if $\Omega \subseteq \mathbb{R}^d$ open then

$$\mathrm{Ext}\left(\overline{D_{L^1(\Omega)}(0,1)}\right) = \phi$$. This is affine geometry

$\{f \in L^1(\Omega) : \int_\Omega |f|\, dx \le 1\}$

T

$\forall f \in \overline{D_{L^1(\Omega)}(0,1)}$ $\exists$ $f_0, f_1 \neq f$ $\in \overline{D_{L^1(\Omega)}(0,1)}$.

$f = \frac{f_0 + f_1}{2}$

$\Rightarrow$ There is no topology which makes $L^1(\Omega)$ a locally convex TVS, so that

$\overline{D_{L^1(\Omega)}(0,1)}$ is compact

Later

$\Rightarrow$ $L^1(\Omega)$ is not the dual of Banach space

Consider that $L^p(\Omega)$ $\forall p$ ~~1<p~~ is dual space

X Banach

X' Banach

$X'' = (X')'$ is the bidual space

$J : X \hookrightarrow X''$

<u>Lemma</u> Consider $J : X \to X''$

$$\langle Jx, x' \rangle_{X'' \times X'} := \langle x, x' \rangle_{X \times X'}$$

Then J is an isometric immersion of X into $JX \subseteq X''$

$(\quad \|Jx\|_{X''} = \|x\|_X \quad \forall x \in X)$

<u>Pf</u> 1) $\|Jx\|_{X''} \leq \|x\|_X$

2) $\|Jx\|_{X''} \geq \|x\|_X$

1) Let $\|x'\|_{X'} = 1$. By the def $\|\cdot\|_{X''}$

$$|\langle Jx, x' \rangle_{X'' \times X'}| = |\langle x, x' \rangle_{X \times X'}| \leq$$

$$\leq \|x\|_X \|x'\|_{X'} = \|x\|_X$$

$$|\langle Jx, x' \rangle_{X'' \times X''}| \leq \|x\|_X \qquad \forall x' \in X' \quad \|x'\|_{X'} = 1$$

$$\|Jx\|_{X''} \leq \|x\|_X$$

$\|x\|_X \le \|Jx\|_{X''}$ ✓

$\forall x \in X \quad x \neq 0 \quad \exists\, x' \in X' \quad$ s.t. $\quad \|x'\|_{X'} = 1$

$\|x\|_X = |\langle x, x'\rangle_{X \times X'}| = |\langle Jx, x'\rangle_{X'' \times X'}|$

$\le \|Jx\|_{X''} \, \|x'\|_{X'} = \|Jx\|_{X''}$

X''

Def: Let X be a TVS and $M \subseteq X$ then

$$M^{\perp} = \{ f \in X' : \langle f, x\rangle_{X' \times X} = 0 \quad \forall x \in M \}$$

Similarly if $N \subseteq X'$ then

$$N^{\perp} = \{ x \in X : \langle x, f\rangle_{X \times X'} = 0 \quad \forall f \in N \}$$

Lemma X normed and $M \subseteq X$ a vector subspace.

Then $(M^{\perp})^{\perp} = \overline{M}$

If $N \subseteq X'$ a vector subspace

$(N^{\perp})^{\perp} \supseteq \overline{N}$.

Example $X = \ell^1(\mathbb{N}) \quad X' = \ell^\infty(\mathbb{N}) \supseteq c_0(\mathbb{N})$

$(\ell^1(\mathbb{N}))' = \ell^\infty(\mathbb{N})$

$(c_0(\mathbb{N}))' = \ell^1(\mathbb{N})$

$c_0(\mathbb{N}) \times \ell^1(\mathbb{N}) \longrightarrow \mathbb{R}$

$(\{c_n\}, \{x_n\}) \longrightarrow \sum_{n=1}^{\infty} c_n x_n$

$c_0(\mathbb{N})^{\perp} = \{0\}$

$(c_0(\mathbb{N})^{\perp})^{\perp} = (\{0\})^{\perp} = \ell^\infty(\mathbb{N})$