5. Smirnov/Lagrangian/Ambrosio representation

Consider the PDE
$$\rho_t + \mathrm{div}(\mathbf{b}\rho) = \mu, \tag{5.1}$$
with the assumptions
$$\rho \in \mathcal{M}^+_b(\mathbb{R}^{d+1}), \quad b \in L^1(\rho), \quad \mu \in \mathcal{M}(\mathbb{R}^{d+1}). \tag{5.2}$$
Because of
$$\rho(\mathbb{R}^{d+1}) < \infty,$$
it follows that
$$\mu = \mu^+ - \mu^-, \quad \mu^+(\mathbb{R}^{d+1}) = \mu^-(\mathbb{R}^{d+1}).$$
By scaling we assume that
$$\mu^\pm(\mathbb{R}^{d+1}) = 1.$$

Remark 5.1. For the transport equation for $t > 0$
$$\rho_t + \mathrm{div}_x(\rho\mathbf{b}) = 0, \quad \rho(t = 0) = \rho_0,$$
we can arrive to the same formulation by setting
$$\tilde{\rho} = \begin{cases} 0 & t \leq 0, \\ \rho & t > 0, \end{cases}$$
and for all smooth function ϕ
$$\mathrm{div}_{t,x}(\rho\phi(1, \mathbf{b})) = \rho(1, \mathbf{b}) \cdot \nabla_{t,x}\phi + \phi\rho_0 \times \delta_{t=0}.$$
In this way we consider only the trajectories inside the support of ϕ in $\{t > 0\}$.

Lemma 5.2. *The space*
$$\tilde{C} = \big\{\gamma \in C(I_\gamma, \mathbb{R}^d)\big\}, \quad \tilde{d}(\gamma, \gamma') = |t^-_\gamma - t^-_{\gamma'}| + |t^+_\gamma - t^+_{\gamma'}| + \|\gamma - \gamma'\|_{C^0(I_\gamma \cap I_{\gamma'})},$$
is Polish.

Proof. Exercise. □

Definition 5.3 (Superposition solution). If $\mathbf{b}$ is a representative in the $L^1(\rho)$-class, a *trajectory of* $\mathbf{b}$ is an a.c. function $\mathbb{R} \supset I \ni t \mapsto \gamma(t) \in \mathbb{R}^d$, I open interval, such that
$$\frac{d\gamma}{dt} = \mathbf{b}(t, \gamma(t)) \quad t \text{ a.e.}.$$

If η is a probability measure on $\tilde{C}$ concentrated on trajectories γ of the Borel vector field $\mathbf{b}$ such that
$$\int \bigg[\int_{I_\gamma} |\dot{\gamma}(t)| dt\bigg] \eta(d\gamma) < \infty$$
then the measure defined by
$$\rho(t, dx) = \int_{t \in I_\gamma} \delta_{\gamma(t)}(dx)\eta(d\gamma), \quad \int \phi(x)\rho(t, dx) = \int \mathbf{1}_{t \in I_\gamma}\phi(t, \gamma(t))\eta(d\gamma),$$
is a *superposition solution.*

Proposition 5.4. *A superposition solution is a solution to* (5.1)*.*

Proof. First of all, for η-a.e. γ it holds
$$\int_{I_\gamma} |\dot{\gamma}(t)| < \infty,$$
so that there are η-measurable maps (exercise)
$$\gamma \mapsto \gamma(t^\pm_\gamma).$$

Also,

$$\int |\mathbf{b}(t,x)|\rho(t,dx)\mathscr{L}^1(dt) = \int \bigg[\int_{I_\gamma} |\mathbf{b}(t,\gamma(t)|dt \bigg] \eta(d\gamma)$$
$$= \int \bigg[\int_{I_\gamma} |\dot\gamma|\mathscr{L}^1 \bigg] \eta(d\gamma) < \infty,$$

so that the PDE is meaningful.

By computation for every test function ϕ

$$\int \int (1,\mathbf{b}) \cdot \nabla_{t,x}\phi(t,x)\rho(t,dx)dt = \int \bigg(\int (1,\mathbf{b}) \cdot \nabla_{t,x}\phi(t,\gamma(t))\eta(d\gamma) \bigg) dt$$
$$= \int \bigg[\int_{I_\gamma} \frac{d}{dt}\phi(t,\gamma(t))dt \bigg] \eta(d\gamma)$$
$$= \int \big[-\phi(t^-_\gamma,\gamma(t^-_\gamma)) + \phi(t^+_\gamma,\gamma(t^+_\gamma)) \big] \eta(d\gamma)$$
$$= -\int \phi\tilde\mu^- + \int\int \phi\tilde\mu^+,$$

where we have defined

$$\tilde\mu^\pm = \gamma(t^-_\gamma)_\sharp \big(\mathbb{1}_{t^\pm_\gamma \in \mathbb{R}} \eta(d\gamma) \big).$$

Note that being η a probability, then $\tilde\mu^\pm(\mathbb{R}^{d+1}) \leq 1$. The PDE is satisfied with $\mu = \tilde\mu^- - \tilde\mu^+$. □

Remark 5.5. Note that there maybe trajectories which goes to ∞ in finite time or such that I_γ is unbounded. These trajectories will not contribute to the measure μ, and are not present if the measure is locally finite. This is the reason why we assume (5.2).

Note moreover that in this construction it may happen that $\tilde\mu^+ \perp \tilde\mu^-$ is false, so that some trajectories may stop in the points where another one starts. In this case $\mu^+(\mathbb{R}^{d+1}) = \mu^-(\mathbb{R}^{d+1}) < 1$.

Also the set of trajectories depends on the representative $\mathbf{b}$ in this construction. We will see below that any representative of $\mathbf{b}$ gives the same η, which means that the trajectories differ on an η-negligible set.

Now we prove the converse: we are given a solution to the PDE (5.1) with the localization assumptions (5.2) and construct a decomposition.

We start with the smooth case.

Proposition 5.6. *If $\mathbf{b}$ is smooth and ρ is smooth positive in $L^1(\mathbb{R}^{d+1})$, then there exists a Smirnov decomposition.*

Proof. Assume first that ρ is compactly supported. By disintegrating the weak formulation of the PDE

$$\rho_t + \operatorname{div}(\mathbf{b}\rho) = \mu, \quad \mu = f\mathscr{L}^{d+1},$$

using the trajectories of the flow $X(\cdot,y)$

$$\mathscr{L}^{d+1} \llcorner_{\operatorname{supp}\rho} = \int \bigg[\int \frac{\delta_X(t,y)}{J(t,y)} dt \bigg] dy,$$

$$0 = \int \rho(1,\mathbf{b}) \cdot \nabla_{t,x}\psi\mathscr{L}^{d+1} + \int \psi f \mathscr{L}^{d+1}$$
$$= \int \bigg[\int \bigg(\frac{\rho(t,X(t,y))}{J(t,y)} \frac{d\psi}{dt} + \frac{f(t,X(t,y))}{J(t,y)} \bigg) dt \bigg] dy,$$

which gives that

$$\frac{d}{dt} \frac{\rho(t,X(t,y))}{J(t,y)} = \frac{f(t,X(t,y))}{J(t,y)}.$$

Hence

$$u(t,y) = \frac{\rho(t,X(t,y))}{J(t,y)}$$

is a smooth positive BV function, with

$$\int \text{Tot.Var.}(u\cdot, y, \mathbb{R})dy = \int \bigg[\int \frac{|f(t, X(t,y))|}{J(t,y)} dt \bigg] dy = \int |f(t,x)| dt dx.$$

A smooth positive BV function can be written as (exercise)

$$u(t,y) = \int_0^{+\infty} \mathbf{1}_{\{u(\cdot,y)>\beta\}}(t) d\beta = \int_0^{+\infty} \sum_n \mathbf{1}_{I_{y,\beta,n}}(t) d\beta, \quad \{u(\cdot, y) > \beta\} = \bigcup_n I_{n,\beta},$$

so that we obtain

$$\begin{aligned} \int \psi \rho \mathscr{L}^{d+1} &= \int \psi(t, X(t,y)) \frac{\rho(t, X(t,y))}{J(t,y)} dt dy \\ &= \int \int \sum_n \int_{I_{y,\beta,n}} \phi(t, X(t,y)) dt d\beta dy, \end{aligned}$$

which is the Smirnov decomposition with

$$\eta = \int \int \sum_n \int \delta_{X(t,y)} \mathbf{1}_{I_{y,\beta,n}}(t) dt d\beta dy.$$

It is fairly standard to see that η is a Borel measure (exercise) and that

$$\|\eta\| \leq \|\rho_0\|_1 + \|\mu\|.$$

If ρ is just in $L^1(\mathbb{R}^{d+1})$, then one repeats the construction in a countable smooth partition of unity $\phi_n : \mathbb{R}^{d+1} \to [0,1]$ with compact support such that

$$\sum_n \int |\nabla_{1,x} \phi_n| (1 + |\mathbf{b}|) \rho \mathscr{L}^{d+1} < +\infty.$$

We leave the details as an exercise. □

To show that there are smooth approximations to $\rho(1, \mathbf{b})$, we take the convolution of the PDE (5.1) with a strictly positive space-time convolution kernel g^ϵ, e.g. the Gaussian:

$$\rho^\epsilon + \text{div}_x(\mathbf{b}^\epsilon \rho^\epsilon) = \mu^\epsilon, \quad \rho^\epsilon = g^\epsilon * \rho, \ \mu^\epsilon = g^\epsilon * \mu, \ \mathbf{b}^\epsilon = \frac{(\rho \mathbf{b})^\epsilon}{\rho^\epsilon}. \tag{5.3}$$

We now give an estimate on tightness. We need the following characterization of uniform integrability.

Definition 5.7. If ν_α is a family of probabilities and $f_\alpha \in L^1(\nu_\alpha)$ with integral equal to 1, we say that f_α is *uniformly integrable* if

$$\forall \epsilon > 0 \ \exists \delta > 0 \ \bigg(\rho_\alpha(A) < \delta \ \Rightarrow \ \int_A |f| \rho_\alpha < \epsilon \bigg).$$

Lemma 5.8. *The family f_α is uniformly integrable iff there exists a convex superlinear function $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$\int \omega(|f_\alpha|) \nu_\alpha < +\infty.$$

Proof. Indeed the uniform integrability there are increasing constant $y_n \to +\infty$ such that

$$\int_{|f_\alpha|>y_n} |f_\alpha| \nu_\alpha < 2^{-n}.$$

Define then the convex monotone function

$$\omega(y) = \sum_n 2^{\frac{n}{2}} [y - y_n]^+$$

and compute

$$\int \omega(|f_\alpha|) \nu_\alpha = \sum_n 2^{\frac{n}{2}} \int_{|f_\alpha|>y_n} |f_\alpha| \nu_\alpha < \sum 2^{-\frac{\mathbf{n}}{2}} < +\infty.$$

The converse is elementary. □

PROPOSITION 5.9 (Dunford-Pettis). *Assume $f_n \geq 0$ and that $\nu_n, f_n\nu_n \rightharpoonup \nu, f\nu$ in $\mathcal{P}(\mathbb{R}^d)$. Then f_n is uniformly integrable and viceversa (up to subsequences).*

PROOF. Consider the measure on R^{d+1} defined by

$$\varpi_n = (\mathrm{id}, f_n)_\sharp \nu_n.$$

If f_n are uniformly integrable, then it follows that ϖ_n is tight, so that up to subsequences $\varpi_n \to \varpi$. The projection of ϖ_n is ν_n, which narrowly converges (up to subsequences) to ν: being the projection a linear operation, it follows that the projection of ϖ in $\mathbb{R}^d$ is ν. Using the disintegration theorem

$$\varpi = \int \varpi_x \nu(dx), \quad f(x) = \int y \nu_x(dy),$$

it is easy to see that

$$f_n \nu_n \rightharpoonup f\nu.$$

If f_n are not uniformly integrable, then up to susbequences ϖ_n is not converging narrowly, which means that there is a subsequence weakly converging to ϖ with $\varpi(\mathbb{R}^{d+1}) < 1$. Hence $f_n\nu_n$ cannot converge to a probability a.c. to ν, because otherwise ϖ would have measure 1. □

We will not assume that the measures are smooth, because by a bootstrap argument we will have that the proposition holds also in the general case.

PROPOSITION 5.10. *Assume that*

(1) $\rho_\alpha \geq 0, \mu_\alpha$ are uniformly bounded measure, tight in $\mathbb{R}^{d+1}$,
(2) $\mathbf{b}_\alpha$ are uniformly integrable,
(3) the PDE (5.1) holds for all α.

Then the Smirnov representation η_α are tight in $\tilde{C}$.

PROOF. We will construct compact sets which leave outside η-arbitrarily small sets: we will do it for a single measure, and see that it depends only on the tightness assumptions of ρ_α, μ_α.

By the bound on the L^1-norm of $\mathbf{b}$ we have

$$\int \big(\mathscr{L}^1(I_\gamma) + \mathrm{Tot.Var.}(\gamma)\big)\eta_\alpha(d\gamma) \leq \|1 + |\mathbf{b}_\alpha|\|_{L^1(\rho_\alpha)},$$

so that up to a set of η-measure small we can assume that the trajectories have

$$\mathscr{L}^1(I_\gamma) + \int_{I_\gamma} |\dot{\gamma}(t)| dt \leq N. \tag{5.4}$$

Next, the measure of trajectories starting outside a ball $B_R(0)$ have measure

$$\eta_\alpha\big(\big\{\gamma : (t, \gamma(t)) \notin B_R(0) \ \forall t \in I_\gamma\big\}\big) \leq \mu_\alpha^+(\mathbb{R}^{d+1} \setminus B_R(0)),$$

so that we can assume that the trajectories intersect $B_R(0)$ up to an arbitrarily η-small set. Then by (5.4) we deduce that the trajectories are contained in $B_{R+N}(0)$.

Using again the uniform integrability of $\mathbf{b}_\alpha$ to get that there is a superlinear convex function $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\int \omega(|\mathbf{b}_\alpha|)\rho_\alpha = \int \bigg[\int_{I_\gamma} \omega(|\dot{\gamma}(t)|) dt\bigg] \eta_\alpha(d\gamma) < +\infty.$$

By removing an arbitrarily small set, we can thus assume that for some $M < +\infty$ it holds

$$\int_{I_\gamma} \omega(|\dot{\gamma}(t)|) dt \leq M.$$

Being ω convex, the above functional is l.s.c., so that we conclude that the compact set is given by

$$K_{N,R,H} = \big\{\gamma \in \tilde{C} : \mathrm{Graph}(\gamma) \subset B_{N+R}(0), \|\dot{\gamma}\|_1 \leq N, \|\omega(|\dot{\gamma}|)\|_{L^1} \leq M\big\}.$$

This gives the family of compact sets where the measure is concentrated. □

We can now pass to the limit.

THEOREM 5.11 (Smirnov/Lagrangian/Ambrosio representation). *The measure ρ solving (5.1) can be represented as a superposition solution.*

PROOF. We have constructed a family η^ϵ which is tight and represent ρ^ϵ: the only missing point for the tightness is the uniform integrability. Note that

$$(|\mathbf{b}|\rho)^\epsilon = (|\mathbf{b}|\rho\llcorner_{|\mathbf{b}|\leq N} + |b|\rho\llcorner_{|\mathbf{b}|>N})^\epsilon = (|\mathbf{b}|\rho\llcorner_{|\mathbf{b}|\leq N})^\epsilon + (|\mathbf{b}|\rho\llcorner_{|\mathbf{b}|>N})^\epsilon \leq N + (|\mathbf{b}|\rho\llcorner_{|\mathbf{b}|>N})^\epsilon.$$

Hence

$$\int_{\{|b^\epsilon|>2N\}} |\mathbf{b}^\epsilon|\rho^\epsilon \leq \int_{\{(|\mathbf{b}|\rho\llcorner_{|b|>N})^\epsilon > N\rho^\epsilon\}} (|\mathbf{b}|\rho\llcorner_{|\mathbf{b}|>N})^\epsilon \leq \int_{|\mathbf{b}|>N} |\mathbf{b}|\rho,$$

which converges to 0.

We pass to the limit of all quantities for suitable subsequence ϵ_n.

(1) Each side of

$$\int \phi\rho^{\epsilon_n} = \int \left[\int \phi(t,\gamma)dt\right]\eta^{\epsilon_n}$$

converges to the corresponding quantity, so that $\rho(t,dx) = \gamma(t)_\sharp\eta(d\gamma)$.

(2) The same for the initial and final data

$$\int \phi\mu^{\epsilon_n,\pm} = \int \phi(t_\gamma^-, \gamma(t_\gamma^-))\eta^{\epsilon_n}(d\gamma),$$

which passes to the limit.

(3) For the vector field the map

$$\gamma \mapsto \dot\gamma$$

is weakly continuous, so that

$$\int \rho g^{\epsilon_n} * (\mathbf{b}\rho) = \int \phi\mathbf{b}^{\epsilon_n}\rho^{\epsilon_n} = \int \left[\int \phi(t)\dot\gamma(t)dt\right]\eta^{\epsilon_n}$$

is converging. This proves that the projection of $\dot\gamma(t)$ is $\mathbf{b}(t,x)$. On the other hand by l.s.c.

$$\int \text{Tot.Var.}(\gamma, I_\gamma)\eta(d\gamma) \leq \limsup_\epsilon \int \text{Tot.Var.}(\gamma, I_\gamma)\eta_\epsilon(d\gamma) = \limsup \int |\mathbf{b}^\epsilon|\rho^\epsilon = \int |\mathbf{b}|\rho,$$

because of the convergence of the convolution. Hence we have that

$$\int \left[\int_{I_\gamma} |\dot\gamma(t)|dt\right]\eta(d\gamma) = \int |\mathbf{b}|\rho,$$

which implies that for $\mathscr{L}^1$-a.e. t

$$\int_{\gamma(t)=x} \dot\gamma(t)\eta_x(d\gamma) = \mathbf{b}(t,\gamma(t)), \quad \int_{\gamma(t)=x} |\dot\gamma(t)|\eta_x(d\gamma) = |\mathbf{b}(t,x)|.$$

By the strict convexity of $|\cdot|$ we deduce that $\dot\gamma(t) = \mathbf{b}(t,x)$ $\eta\times\mathscr{L}^1$-a.e..

Hence we obtain a representation. □

Some remarks are in order.

(1) First of all there is no uniqueness in general of the representation, the simple case being a vector field allowing two trajectories to cross and putting a Dirac delta on them: one can exchange the trajectories at the crossing time.

(2) The uniqueness of the representation does not imply that the trajectories do not intersect: however it implies some sort of future or past uniqueness.

(3) A similar decomposition can be done for the more general PDE

$$\operatorname{div}\mathbf{b} = \mu,$$

in this case as curves or solenoidal parts (Smirnov).

(4) In the case of initial data, we can partition the trajectories according to the initial data, obtaining that

$$\eta = \int \eta_y\rho_0(dy).$$

In the case of a flow, $\eta_y = \delta_{X(\cdot,y)}$, but in general it corresponds to a probability measure not supported on a single point.