6. Uniqueness

The condition for uniqueness of the Lagrangian representation is related to the uniqueness of the solution to the PDE

$$(\rho u)_t + \mathrm{div}_x(\mathbf{b}\rho u) = 0, \quad 0 \leq u \leq 1. \tag{6.1}$$

Recall ρ is considered as a background measure, while u is solving in the duality sense

$$u_t + \mathbf{b} \cdot \nabla u = 0, \quad u(t=0) = u_0. \tag{6.2}$$

Remark 6.1. We are assuming that (6.1) defines a measure solution when restricted to any interval $[0,T]$, i.e.

$$(\rho u)_t + \mathrm{div}_x(\mathbf{b}\rho u) = u_0\rho_0 \times \delta_{t=0} - u(T)\rho(T) \times \delta_T,$$

and ρ, ρ_0, ρ_T are bounded measure, $u \in L^\infty(\rho), u_0 \in L^\infty(\rho_0), u(T) \in L^\infty(\rho(T))$. We will not write the end interval T (for example by considering as test functions $\psi \in C^1_c([0,t) \times \mathbb{R}^d)$).

Definition 6.2. We say that $\rho(1,\mathbf{b})$ has the *uniqueness property* if for every $u_0 \in L^\infty(\rho)$ there is a unique ρ-solution in $L^\infty(\rho)$ to (6.2).

We say that ρ has the *renormalization property* if the following implication holds: for all $\beta \in C^1(\mathbb{R})$

$$u \text{ is a solution to (6.2) with } u(t=0) = u_0 \quad \Rightarrow \quad \beta(u) \text{ is a solution to (6.2) with } \beta(u)(t=0) = \beta(u_0).$$

Lemma 6.3. *For every $u_0 \in L^\infty$ there exists a solution to (5.1).*

Proof. Use the disintegration theorem to write

$$\eta = \int \eta_y \rho_0(dy),$$

and just check that

$$\eta' = \int \eta_y u_0(y)\rho_0(dy)$$

is a solution in L^∞. □

Next we prove the following implications:

Lemma 6.4. *If ρ has the uniqueness property, then it is has the renormalization property.*

Proof. If u_1, u_2 are two solutions to (6.2) with the same initial data, by linearity there is a solution with initial data 0. Clearly by taking $\beta(u) = u^2$, it follows that

$$0 = \int_0^T \int \mathrm{div}(u^2\rho(1,\mathbf{b})) = \int u^2(T)\rho(T) - \int u_0^2\rho_0 = \int u^2(T)\rho(T),$$

which means it is renormalized only if $u = 0$, obtaining a contradiction. □

Lemma 6.5. *If ρ has the renormalization property, then there is a unique Lagrangian representation which is a flow.*

Proof. Assume that there is a Lagrangian representation which is not a flow: then there is a set of positive ρ_0-measure such that the conditional probabilities η_y are not Dirac deltas. In particular, by a partition argument, one of the set

$$A_{r,\bar{t},x_1,x_2,\delta} = \Big\{ y : \eta_y\big(\{\gamma(\bar{t}) \in B_r(\bar{x}_i)\}\big) > \delta, i = 1,2, |x_1 - x_2| > 2r \Big\}$$

has positive ρ_0 measure: otherwise it follows that that for all $r, \bar{t}, x_1, x_2$ rational the set has measure 0, which gives that it is a flow.

The two solutions are

$$\rho u_i = \int_{A_{r,\bar{t},x_1,x_2,\delta}} \frac{\eta_y \llcorner \{\gamma(\bar{t}) \in B_r(\bar{x}_i)\}}{\eta_y(\{\gamma(\bar{t}) \in B_r(\bar{x}_i)\})} u_0(y)\rho_0(dy),$$

which are disjoint at time $\bar{t}$ being in $A_{r,\bar{t},x_1,x_2}$.

Hence every Lagrangian representation is a flow. Being the set of Lagrangian representation convex, it follows that that the same flow is used from any other η up to a negligible set of trajectories. □

Lemma 6.6. *If ρ has a unique Lagrangian representation which is a flow, then there is uniqueness.*

PROOF. Since every η has to use the same flow X, then unique solution is given by Lemma 6.3. $\square$

We now relate the uniqueness to a monotonicity principle.

COROLLARY 6.7. *If $u_0 \geq 0$ implies that every solution to (6.2) is positive, then uniqueness holds. and viceversa.*

PROOF. By Lemma 6.3 the implication (uniqueness $\Rightarrow$ positivity) is trivially true.
If monotonicity holds, it follows by the same reasoning as in Lemma 6.4 that

$$0 = \int_0^T \int \operatorname{div}(u\rho(1, \mathbf{b})) = \int u(T)\rho(T) - \int u_0\rho_0 = \int u(T)\rho(T),$$

which means $u = 0$. $\square$

REMARK 6.8. A more refined argument gives that η is concentrated on a set of trajectories γ such that the set

$$\big\{(t, \gamma(t)), t \in I_\gamma\big\}$$

are disjoint, i.e. the curves can only intersect in the final points, thus forming a disjoint partition of the space $\mathbb{R}^{d+1}$.