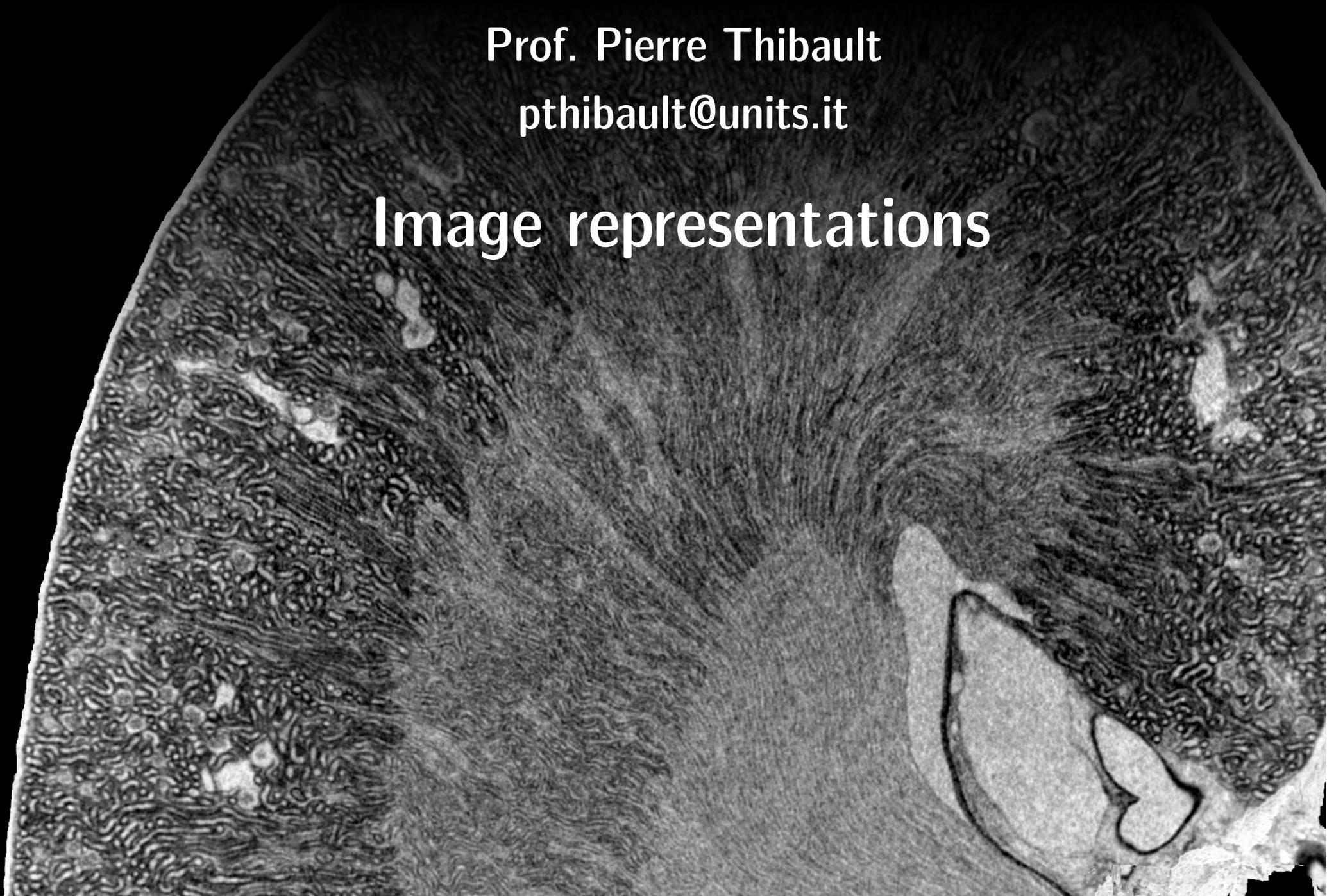


Image Processing for Physicists

Prof. Pierre Thibault

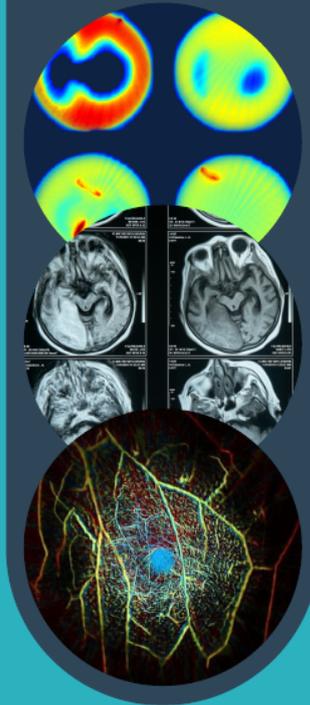
pthibault@units.it

Image representations



Inverse Problems and their Applications

A brief introduction to the topic and
the possibility to explore it further in
details



TOPICS AND DISCUSSION

Prof. Otmar Scherzer,
Universität Wien
*Regularization of Nonlinear
Inverse Problems: From
Functional Analysis to
Machine Learning*

Prof. Ronny Ramlau,
JKU Linz
*On the Use of Sobolev
Embedding Operators in
Solving Inverse Problems:
Theory and Applications
in Tomography*

Dr. Sonia Foschiatti,
Universität Wien
*Postdoc in Vienna: a
research experience*

Buffolini Andrea
Venier Giacomo
*Erasmus+ project: a
students' experience*

TIME AND LOCATION

Thursday, OCT 23
Time: 11:00-14:00
Building: D
Room: T_A

Tomorrow's lecture
replaced with this
interesting seminar
series



universität
wien

JKU

JOHANNES KEPLER
UNIVERSITÄT LINZ



UNIVERSITÀ
DEGLI STUDI
DI TRIESTE

Overview

- The Discrete Fourier Transform as a change of basis
- Discrete Cosine Transform
- Windowed Fourier Transform
- Wavelet Transform
- (many others omitted!)

Image representations

$$f(x, y) = \sum_n C_n B_n(x, y)$$

DFT:

$$f(m, n) = \sum_{k, l} F_{kl} \underbrace{e^{2\pi i \left(\frac{mk}{M} + \frac{nl}{N} \right)}}_{B_{kl}(m, n)} \quad \text{(shape of image is } (M, N))$$

DFT basis set

orthogonal basis $\langle kl | mn \rangle$
 " $\langle \vec{q} | \vec{x} \rangle = e^{i \vec{q} \cdot \vec{x}}$ "

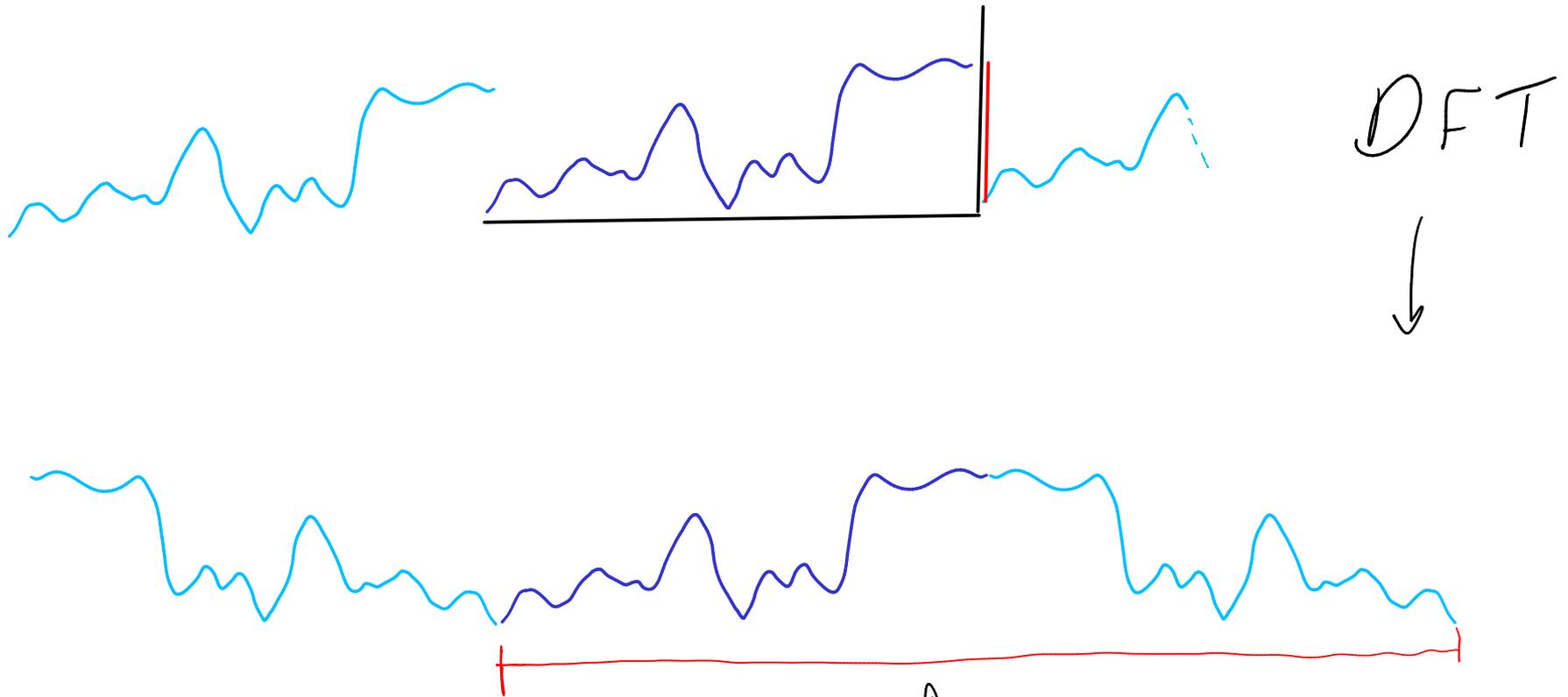
DFT (1D): $B_k(n) = e^{2\pi i \left(\frac{kn}{N} \right)}$

$$\begin{bmatrix} f \end{bmatrix} = \begin{bmatrix} | & | & | & \dots \\ | & z & z^2 & z^3 & \dots \\ | & z^2 & z^4 & & \\ | & z^3 & & & \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ | & & & & \end{bmatrix} \begin{bmatrix} F \end{bmatrix}$$

$$f_n = \sum_k F_k z^{nk} \quad \left. \begin{array}{l} \\ z = e^{2\pi i / N} \end{array} \right\} z\text{-transform}$$

Discrete Cosine Transform

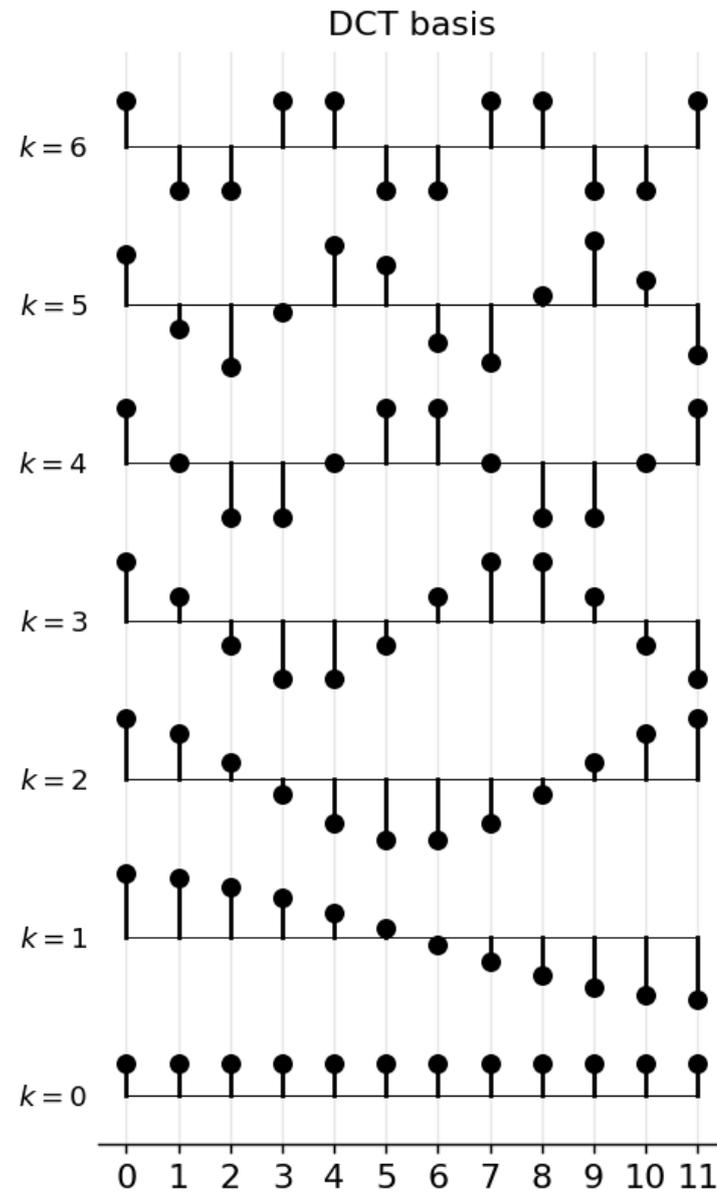
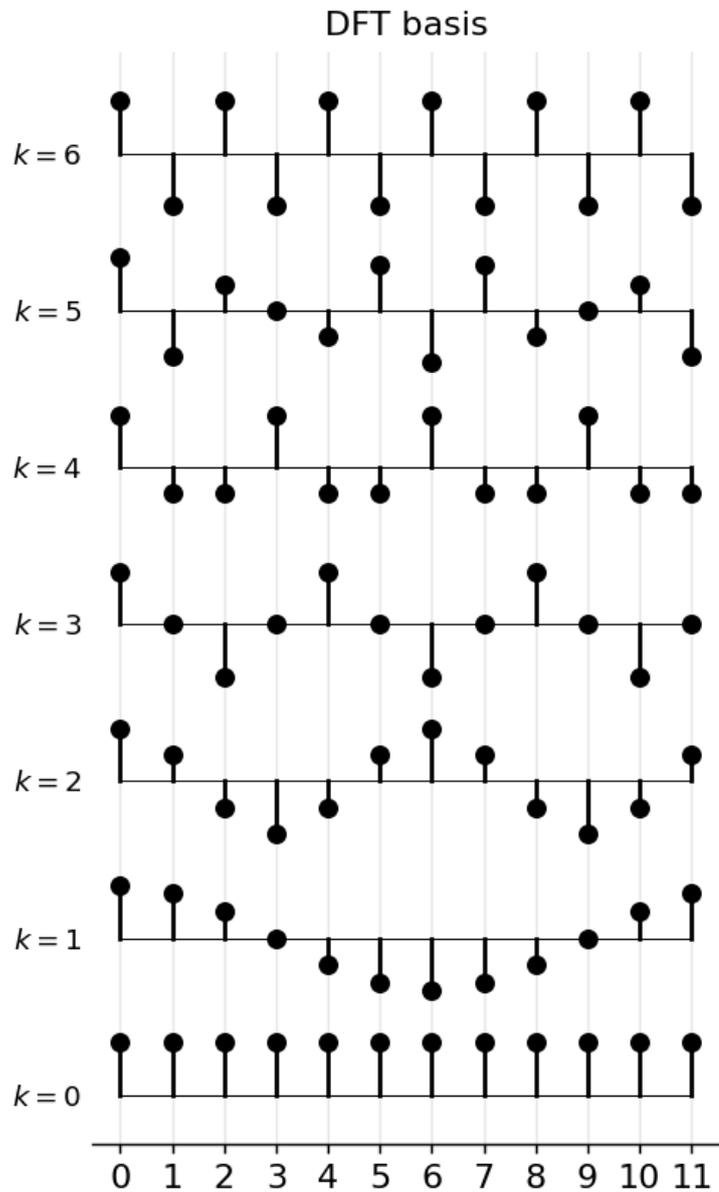
A variation on the theme of DFT



DCT: $F_k = \sum_n f_n \cos\left(\pi\left(n+\frac{1}{2}\right)\frac{k}{N}\right)$ can be decomposed as a sum of cosines

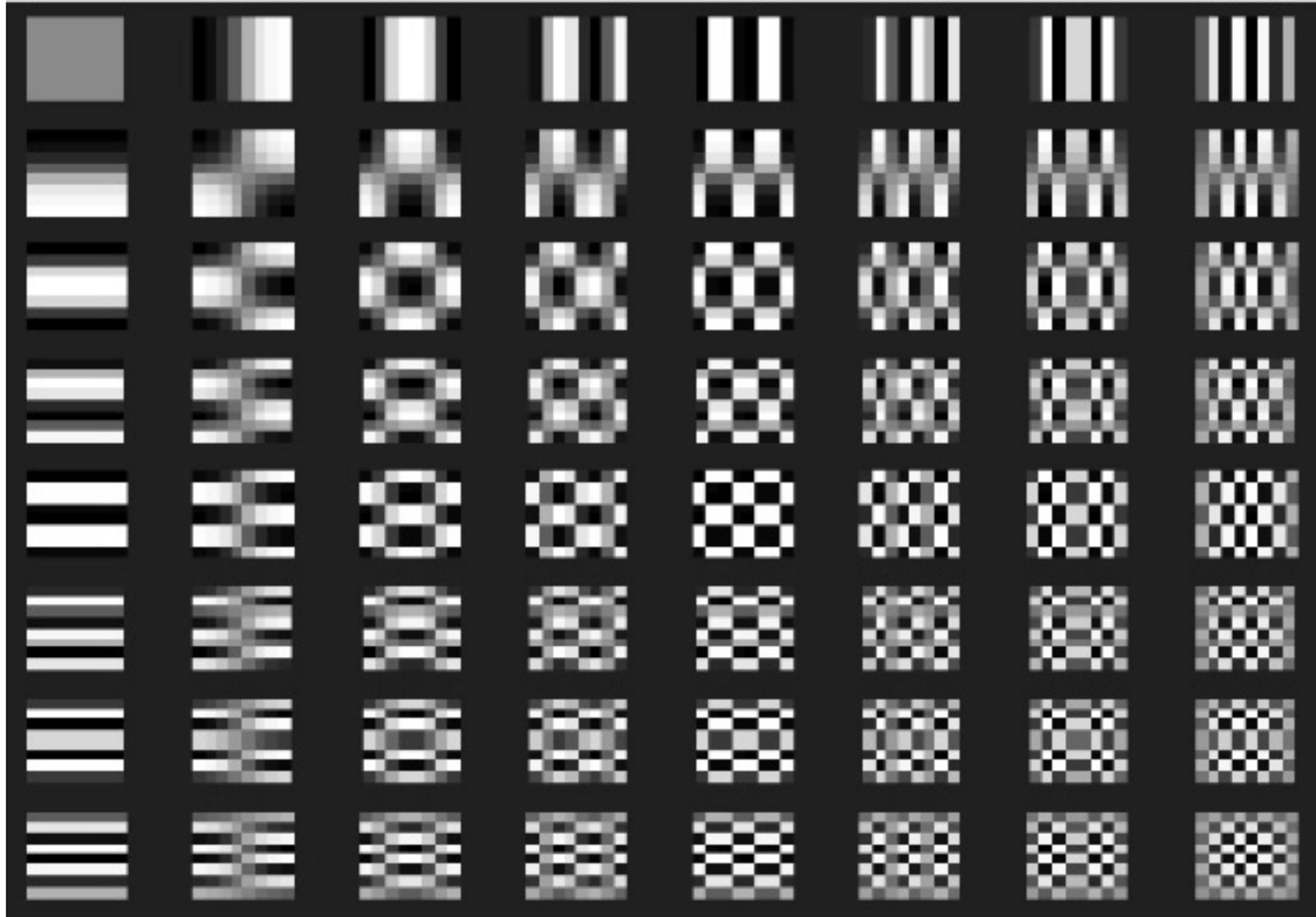
$f_0 = \sum_n \frac{1}{\sqrt{2}} f_n$

Discrete Cosine Transform



Discrete Cosine Transform

64 DCT basis vectors for 8x8 image



Discrete Cosine Transform

Image compression



1:1 bit rate



8:1 bit rate



32:1 bit rate



128:1 bit rate

JPEG
compression

keep $\frac{1}{4}$ of the
coefficients

Historical overview

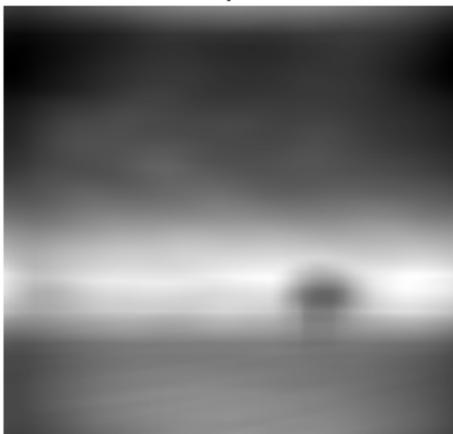
- 1822 Fourier: Fourier transform
- 1946 Gabor: “Gabor transform”, Short-time Fourier transform (STFT)
- 1974 Ahmed, Natarajan & Rao: Discrete Cosine Transform
- 1980s Morlet, Mallat, Daubechies, ... : Wavelets

Bandpass filtering

original



low pass

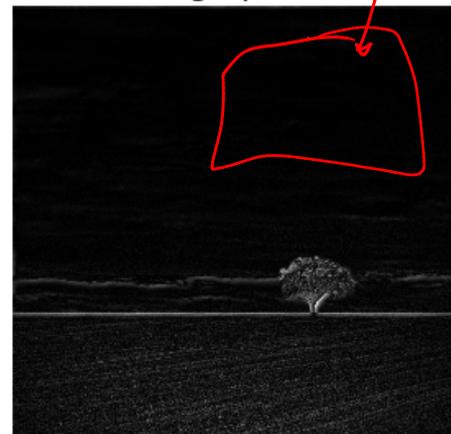


Don't need high spatial resolution

mid pass



high pass

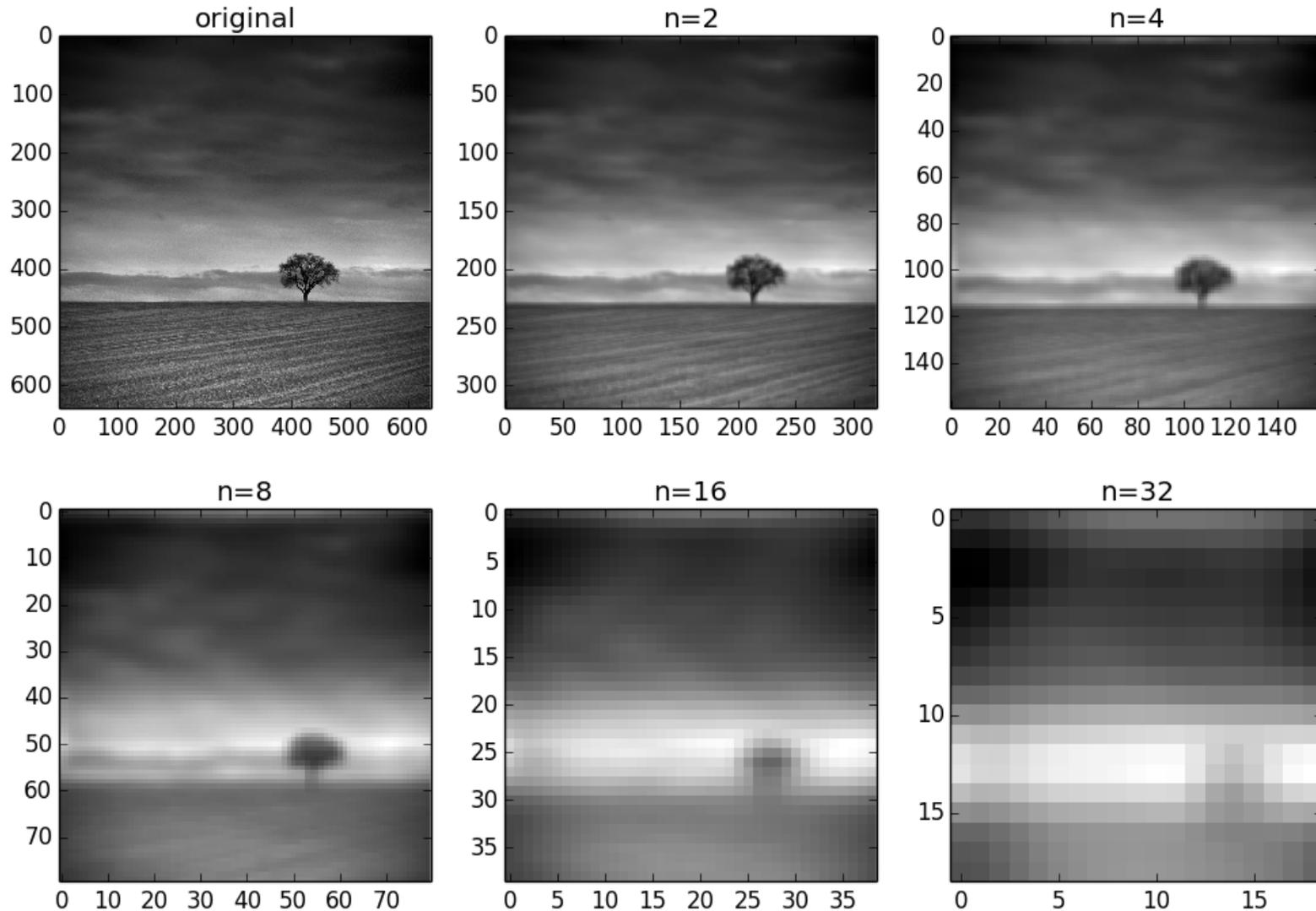


no signal

Need high spatial resolution

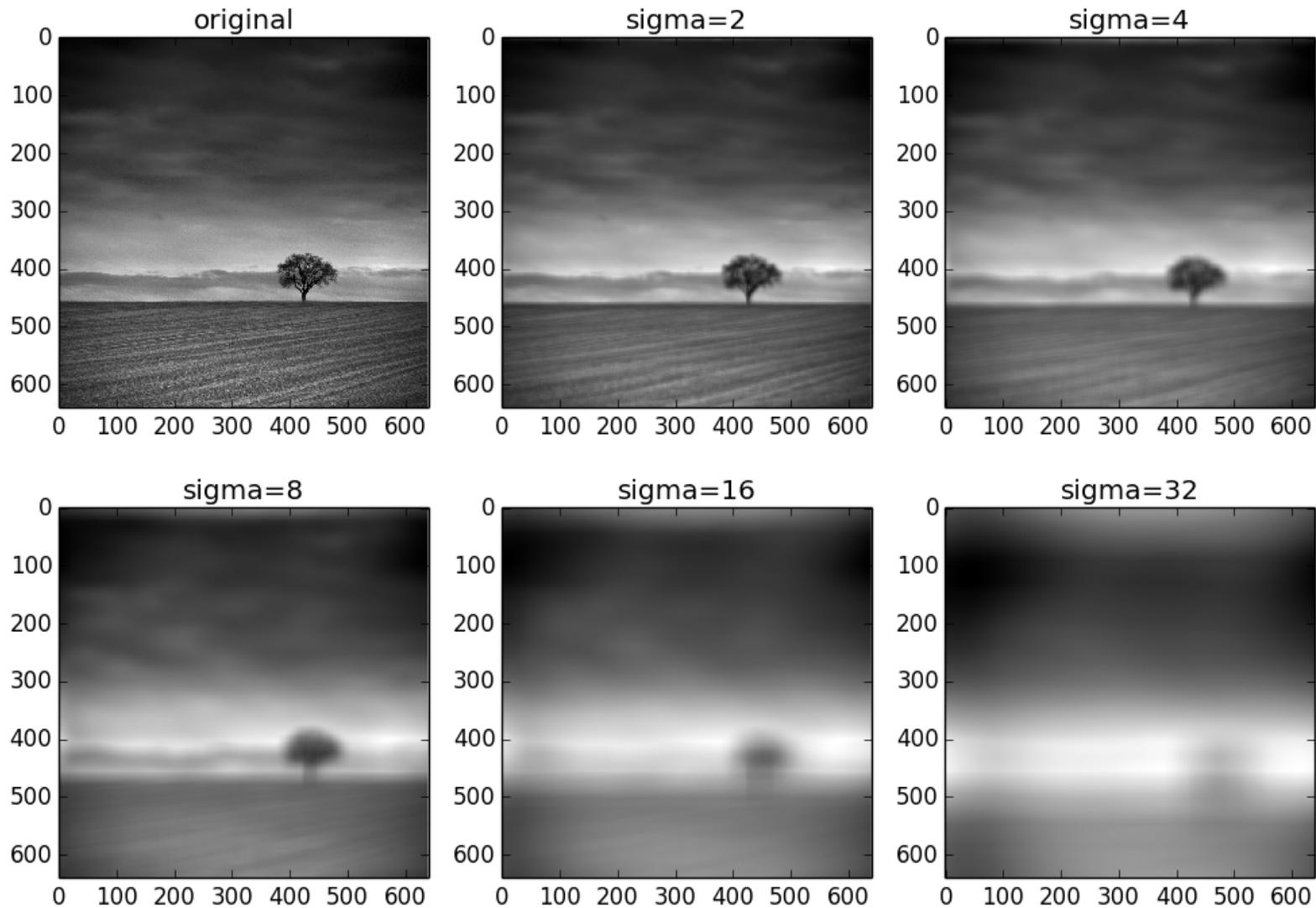
Multiresolution analysis

Subsampling (taking every n^{th} pixel) successively reduces high frequency content



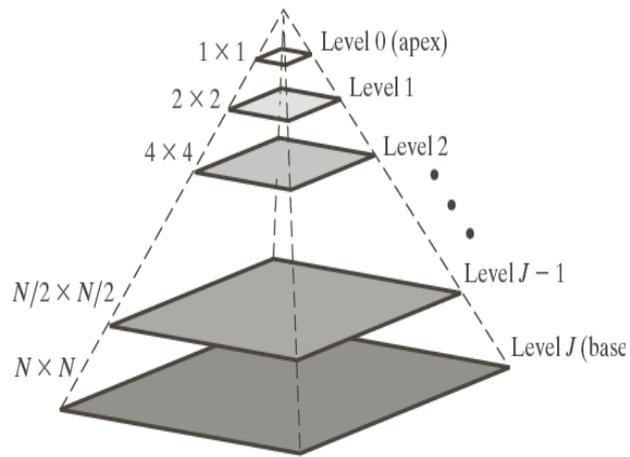
Multiresolution analysis

Multiple filtering with Gaussian filters, sigma determines resolution



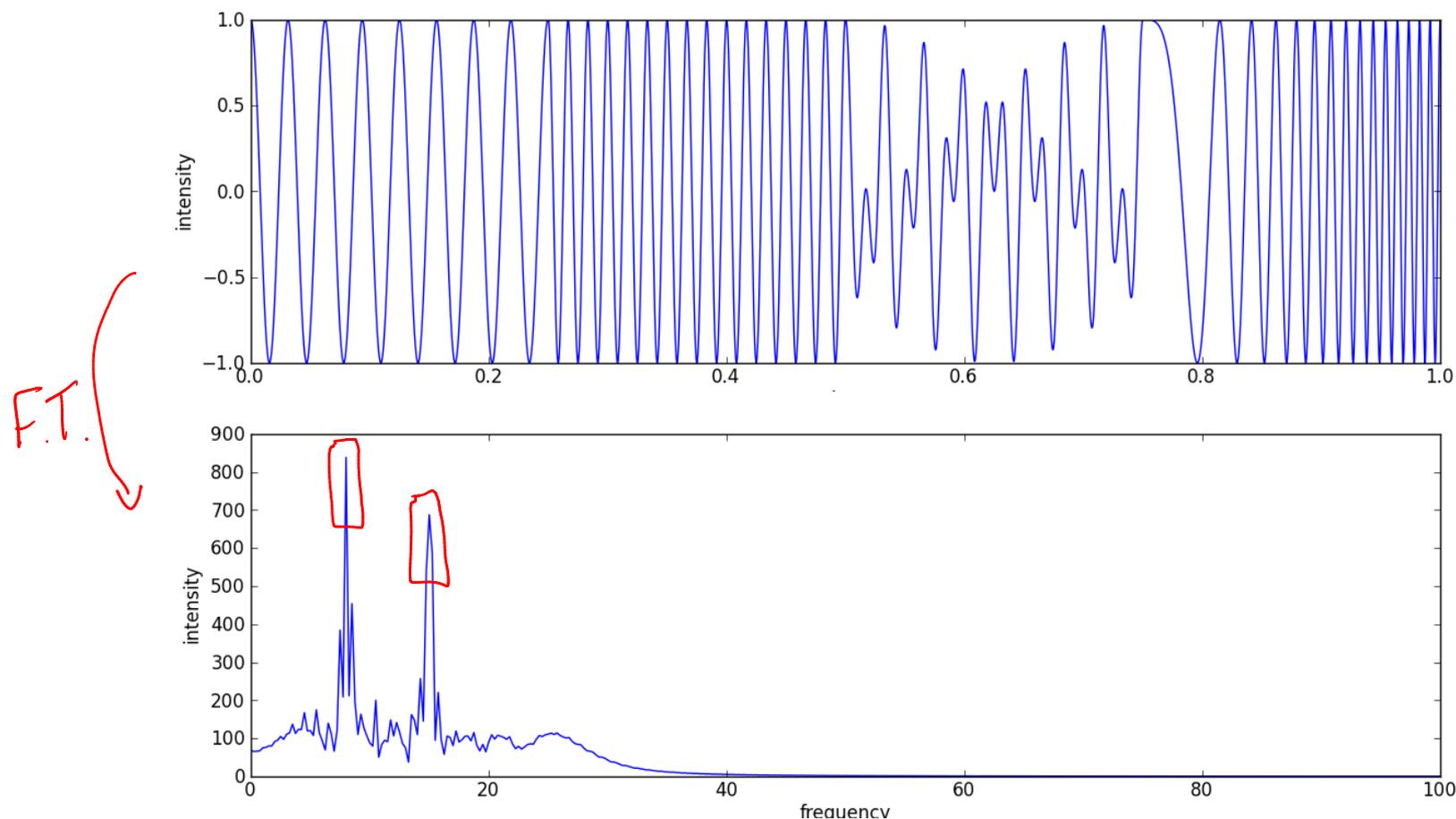
Pyramid representation

Scale-space representation, pyramidal representation



Stationary vs. non-stationary signals

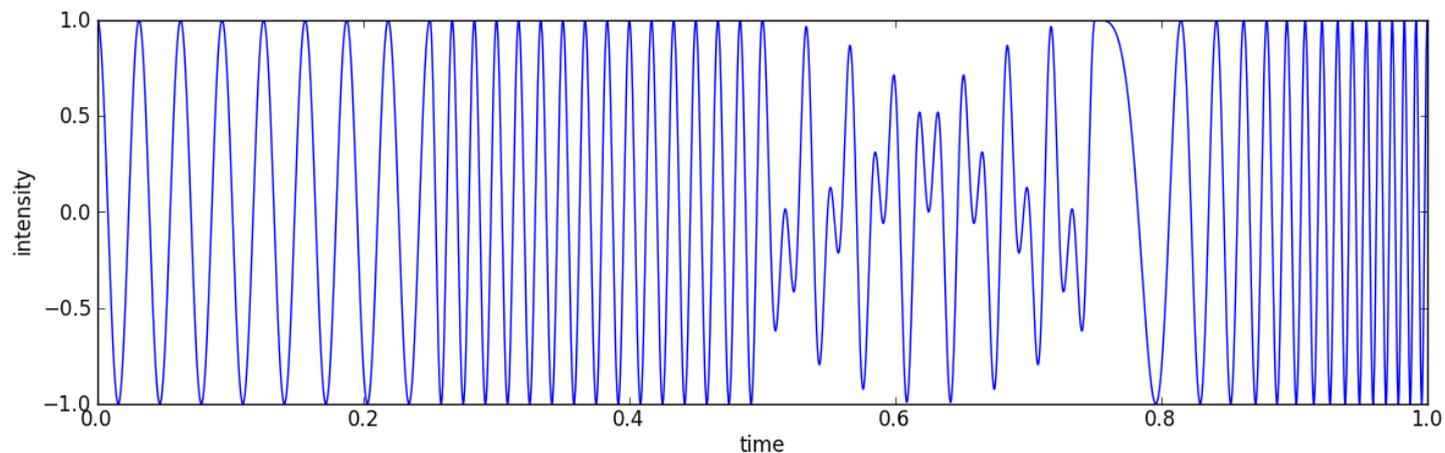
- Stationary signals: frequency doesn't change over time (spatially over the image)
- Non-stationary signals: frequency changes over time (spatially over the image)
- Examples of non-stationary signals: speech, most images



FT insufficient to localize the frequencies in our signal (image)

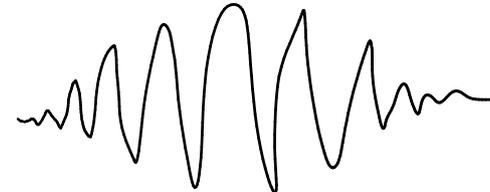
Windowed Fourier transform

- Windowed Fourier transform is part of the field of “time-frequency analysis”
- Also known as Short-time Fourier Transform (STFT)
- Time-frequency representations are used in many different contexts (Audio, image processing/optics, quantum mechanics)
- Idea: slice up signal into small parts, analyze each separately
 - Multiply with window function w (of width d) at position x_0
 - Take Fourier transform of result
 - Slide window to new position
 - repeat

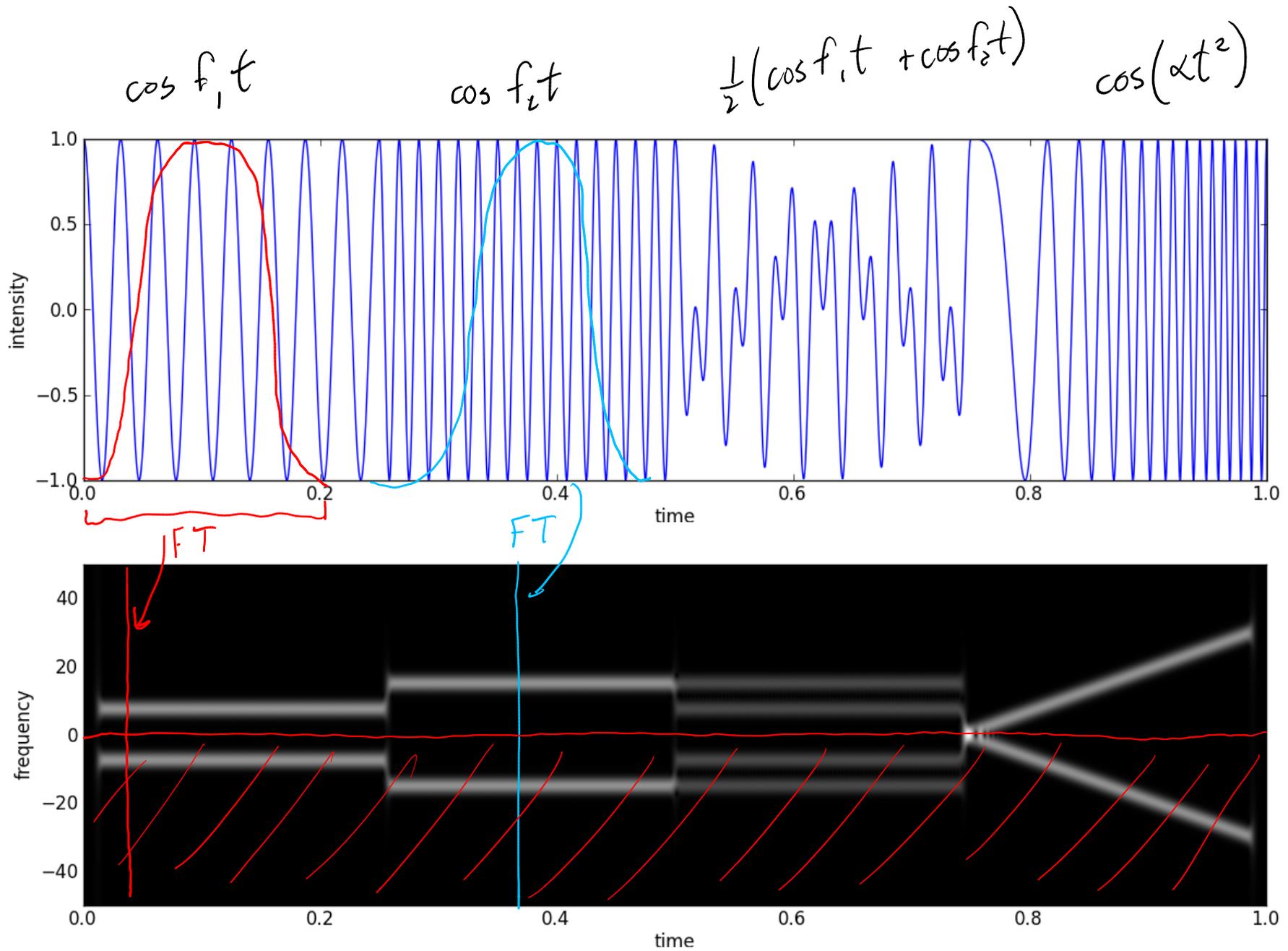


Analogy to audio signals



$$S(a, x_0) = \int_{-\infty}^{\infty} f(x) \underbrace{w(x-x_0)}_{\text{waveform}} e^{-2\pi i a x} dx$$


Spectrogram



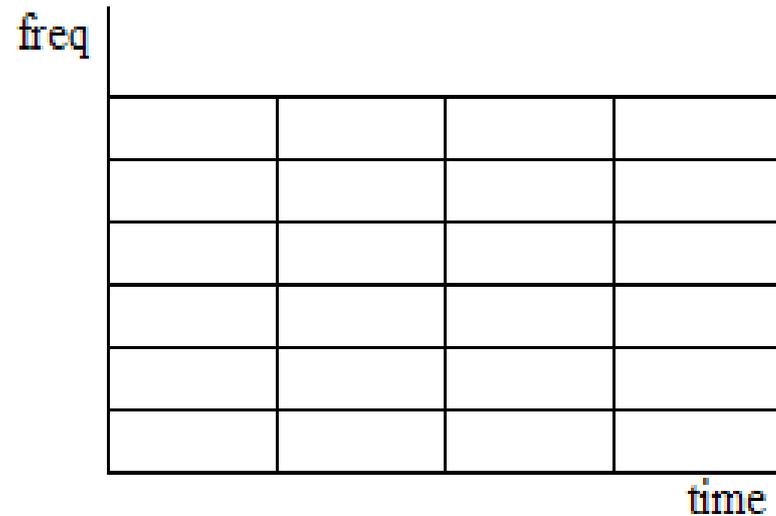
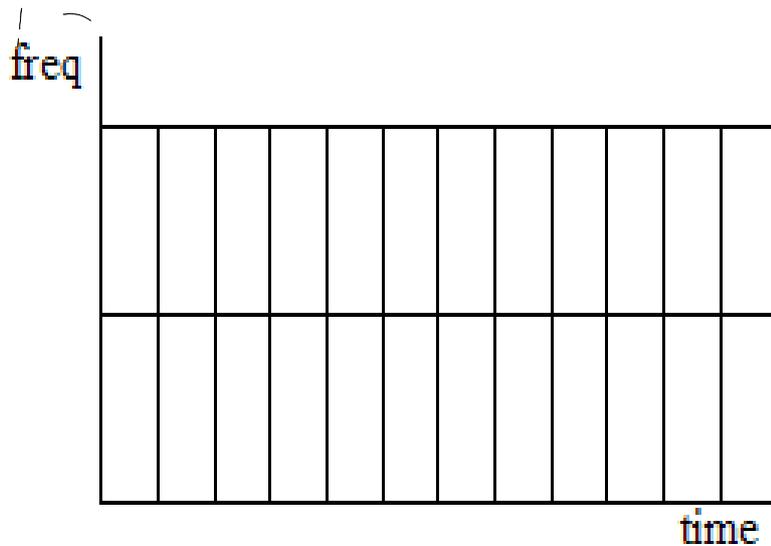
Uncertainty relation

$$\sigma_x \sigma_f \geq \frac{1}{4\pi}$$

$$\sigma_x \sigma_u$$

area in phase space

- Finite area in the time-frequency plane



- This is limitation of WFT and hence development of **wavelets**

Continuous wavelet transform (WT)

mother wavelet

- Parameters: translation and scaling

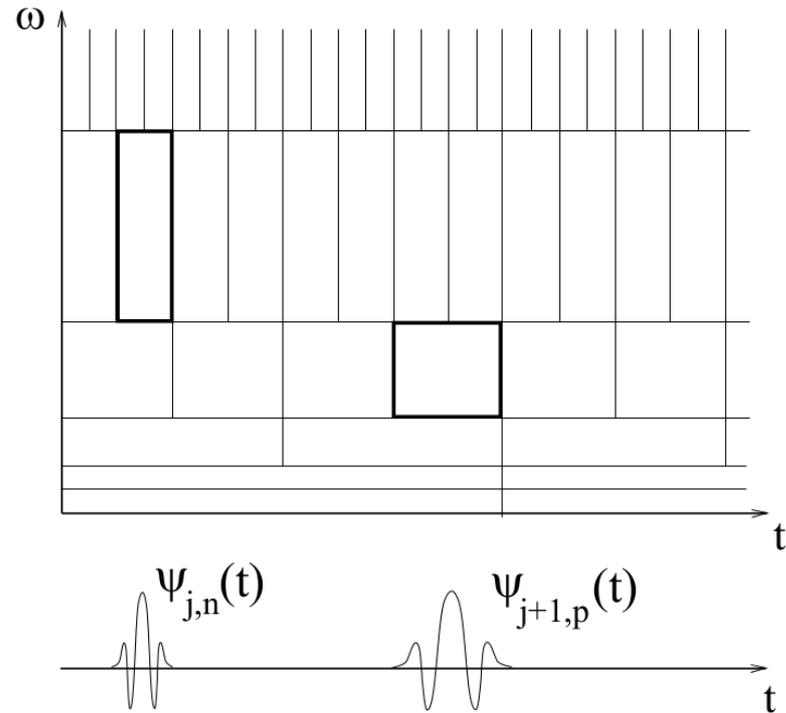
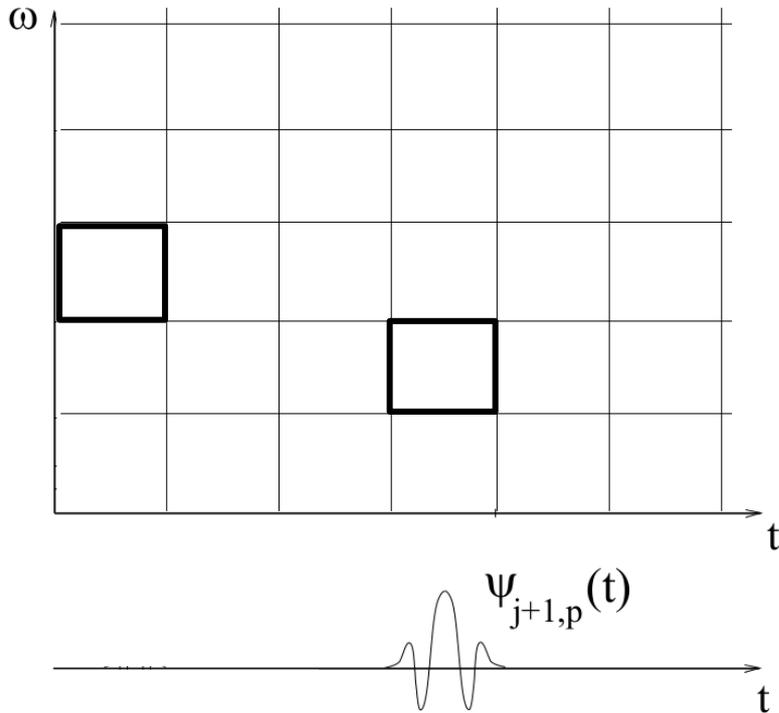
$$WT_{s,x_0}\{f\} = \int_{-\infty}^{\infty} f(x) \underbrace{\psi_{s,x_0}(x)}_{\text{wavelet}} dx$$

\uparrow scale \uparrow position

$$\psi_{s,x_0} = \frac{1}{\sqrt{s}} \psi\left(\frac{x-x_0}{s}\right)$$

\downarrow mother wavelet

- Analyze signal at different scales instead of different frequencies

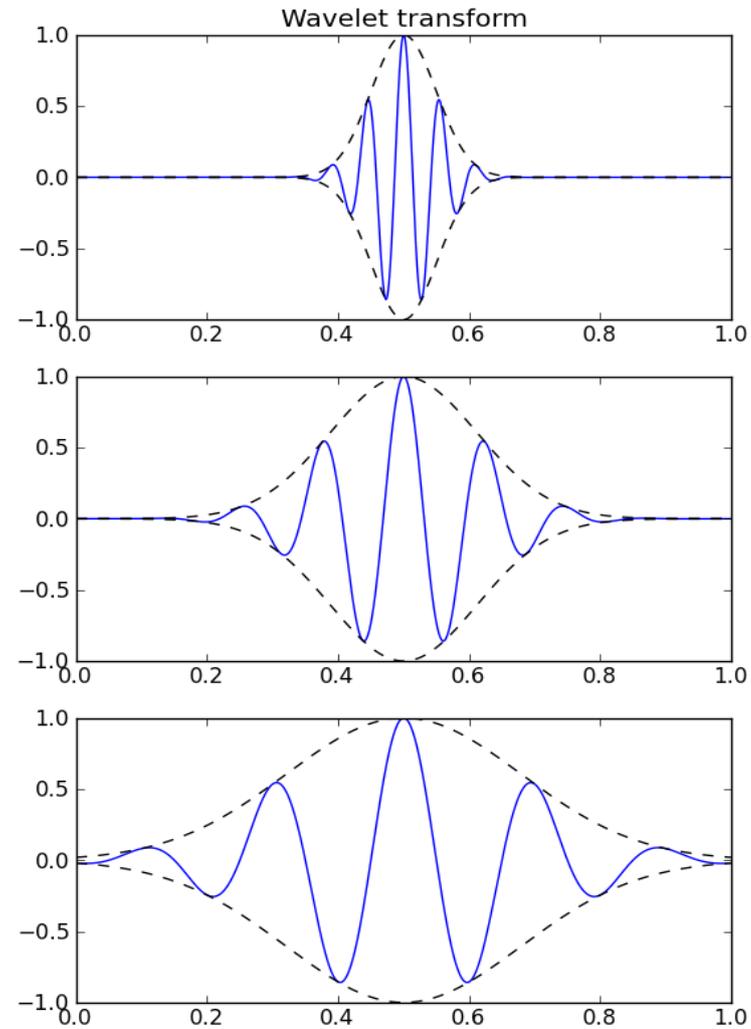
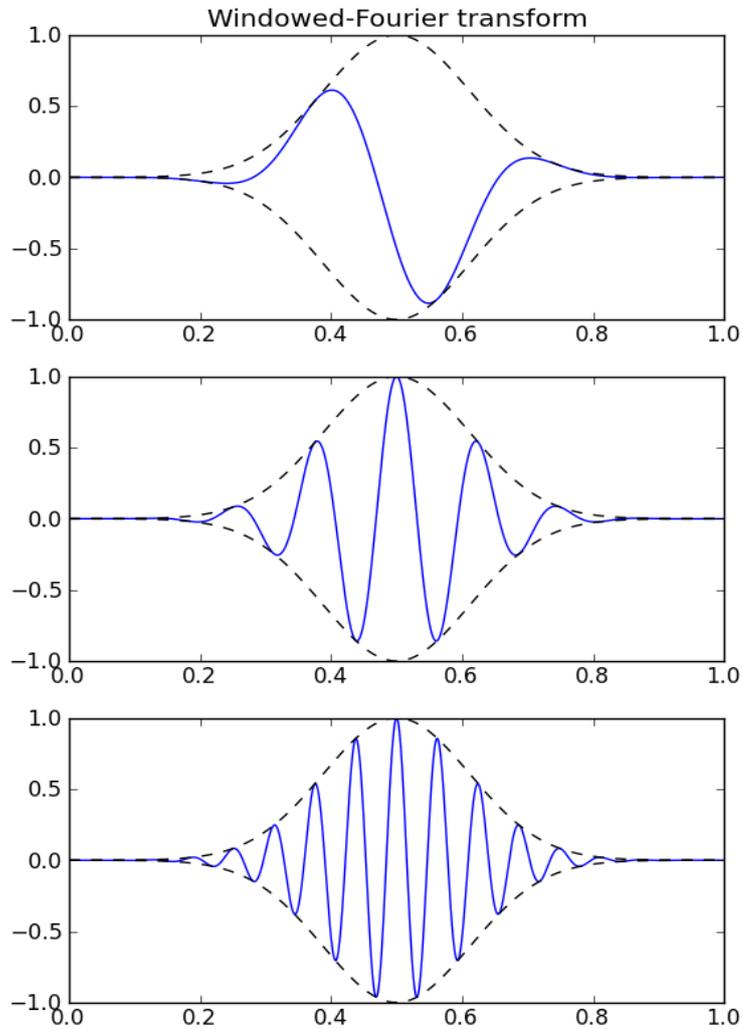


Source: Mallat, "A wavelet tour of signal processing"

WFT vs WT

WFT - keep window width constant
- change modulation

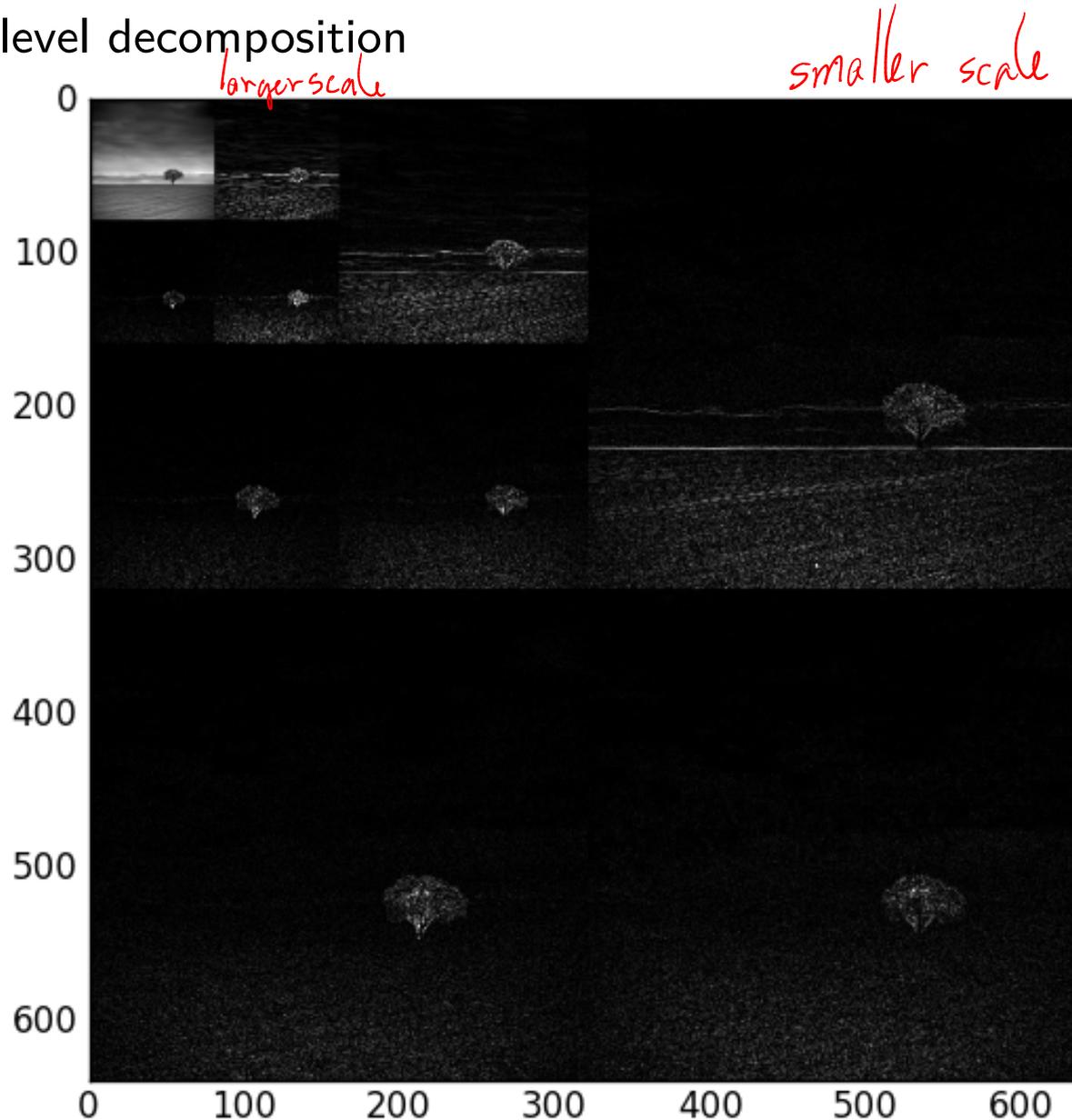
Wavelet - keep shape constant
- change scale



Discrete Wavelet decomposition of image

- Perform each DWT, collect and tile all coefficients

- Here: 3 level decomposition



Summary

- Images can be represented by different basis functions.
- Fourier basis: localized in frequency, delocalized in real space.
- Windowed Fourier Transform: localized – to some extent – in both spaces
- Wavelet analysis decomposes a signal in position and scale (instead of position and frequency as for WFT).
- Sparse representations are representations in which the image content is represented by a few relevant coefficients, while the other ~~pixels~~ are close to zero
- Sparse representations have advantages for compression, denoising, ...

values

regularization