October 30

<u>Df</u> X TVS $M \subseteq X$

$$M^{\perp} = \{ f \in X' : \langle f, x \rangle_{X' \times X} = 0 \quad \forall x \in M \}$$

$$N \subseteq X'$$

$$N^{\perp} = \{ x \in X : \langle f, x \rangle_{X' \times X} = 0 \quad \forall f \in N \}$$

<u>Remark</u> Notice that $M^{\perp} \subseteq X'$ and $N^{\perp} \subseteq X$ are vector spaces

$N^{\perp} = \bigcap_{f \in N} \ker f$ $\quad$ $N^{\perp}$ is closed vector space in X.

If X is normed, X' is Banach, X''

$M \subseteq X$

$$M^{\perp} = \{ f \in X' : \langle f, x \rangle_{X' \times X} = \langle f, Jx \rangle_{X' \times X''} = 0 \quad \forall x \in M \}$$

$M^{\perp} = \bigcap_{x \in M} \ker Jx$ is closed vector space in X'

<u>Lemma</u> $\underbrace{M \subseteq X}_{\text{vector space}}$ X normed $\Rightarrow (M^{\perp})^{\perp} = \overline{M}$ (a)

$N \subseteq X' \Rightarrow (N^{\perp})^{\perp} \supseteq \overline{N}$ (b)

Proof. First we show that $(M^{\perp})^{\perp} \supseteq \overline{M}$

$$* \quad M^{\perp} = \{ f \in X' : \langle f, x \rangle_{X' \times X} = 0 \quad \forall x \in M \}$$

$$(M^{\perp})^{\perp} = \{ y \in X : \langle f, y \rangle_{X' \times X} = 0 \quad \forall f \in M^{\perp} \}$$

It is clear from $*$ that $x \in M$ then

$\langle f, x \rangle_{X' \times X} = 0 \quad \forall f \in M^{\perp} \Rightarrow x \in (M^{\perp})^{\perp}$

$\Rightarrow \underbrace{(M^{\perp})^{\perp}}_{\text{closed}} \supseteq M \Rightarrow (M^{\perp})^{\perp} \supseteq \overline{M}$

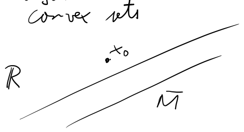
We want to show $(M^{\perp})^{\perp} = \overline{M}$.

Let us suppose $\exists$ $x_0 \in (M^{\perp})^{\perp} \setminus \overline{M}$

$\{x_0\}$, $\overline{M}$. Since M is a vector space

also $\overline{M}$ is a vector space in X.

$\underset{\text{compact}}{\{x_0\}}$ and $\underset{\text{closed}}{\overline{M}}$ disjoint convex sets



Then $\exists f \in X'$ an $\alpha \in \mathbb{R}$

s.t.

$f(x) < \alpha < f(x_0) \quad \forall x \in \overline{M}$

$f(x) = 0$

$f(x) \neq 0$ $\quad \sup \{ f(tx) : t \in \mathbb{R} \} = +\infty$ $\quad tf(x)$

$f(x) = 0 < f(x_0) \quad \forall x \in \overline{M}$

$\forall x \in M$

$\Rightarrow f \in M^{\perp}$ $\quad$ and $x_0 \in (M^{\perp})^{\perp}$

$\Rightarrow g(x_0) = 0 \quad \forall g \in M^{\perp}$

but we found an $f \in M^{\perp}$ s.t. $f(x_0) > 0$

$$(M^{\perp})^{\perp} = \overline{M}$$

Lemma vector space X normed

$M \subseteq X \Rightarrow (M^\perp)^\perp = \overline{M}$ (a)

$N \subseteq X' \Rightarrow (N^\perp)^\perp \supseteq \overline{N}$ (b)

Lemma Let $T : X \to Y$ continuous between B-spaces and $T^* : Y' \to X'$ be the adjointe.

$R(T) := TX$ is the range of T

$R(T^*) := T^* Y'$

Then

$$\ker T = R(T^*)^\perp$$

$$\ker T^* = R(T)^\perp$$

$$(\ker T)^\perp \supseteq \overline{R(T^*)}$$

$$(\ker T^*)^\perp = \overline{R(T)}$$

$$Tx = y$$

$$y \notin (\ker T^*)^\perp$$

$$\Downarrow$$

$$y \notin \overline{R(T)}$$

Banach Steinhaus

Def (Baire Space) A top space X is Baire if one of the following two equivalent statements holds

1) For any sequence $\{A_n\}_{n\in\mathbb{N}}$ of open dense subspaces of X, $\bigcap_{n\in\mathbb{N}} A_n$ is dense in X.

2) For any sequence $\{C_n\}_{n\in\mathbb{N}}$ st. $\mathring{C}_n=\phi$ of closed subspace ~~of X st. then~~ $\bigcup_{n\in\mathbb{N}} C_n$ ~~have non empty~~ has empty interior ~~interior, then $\mathring{C}_n=\phi$~~

Theorem The following spaces are Baire spaces

1) X a locally compact Hausdorff space

2) X a topological space whose topology is induced by a complete metric on X.

Example $\Omega\subseteq\mathbb{R}^d$ open $\mathcal{D}(\Omega)=C_c^\infty(\Omega,\mathbb{R})$ with an appropriate topology, sketched in the notes. ~~$\mathcal{D}(\Omega)$~~ The topology of $\mathcal{D}(\Omega)$ does not come from a metric.

Exercise Prove that every closed convex and absorbing subset of a B space X is a nbhd of 0.

Def let X and Y be B-spaces

Then a family $\{T_j\}_{j \in J}$ of elements of

$\mathcal{L}(X, Y)$ is equicontinuous if $\exists$ ~~+~~ $M > 0$

s.t. $\|T_j\|_{\mathcal{L}(X,Y)} \leq M \quad \forall j \in J$.

$(X, d_1) \xrightarrow{f_j} (Y, d_2) \qquad \{f_j\}_{j \in J}$

is equicontinuous if

$\forall x_0 \in X$ and $\forall \varepsilon > 0 \quad \exists \delta_{x_0 \varepsilon} > 0$

s.t. $d_1(x, x_0) < \delta_{x_0 \varepsilon} \Rightarrow d_2(f_j(x), f_j(x_0)) < \varepsilon \quad \forall j \in J$

Theorem let X and $\cancel{Y}$ be B-spaces

and consider a family $\{T_j\}_{j \in J}$ in $\mathcal{L}(X, Y)$.

Suppose that $\sup\limits_{j \in J} \|T_j x\|_Y < +\infty \quad \forall x \in X$.

Then $\{T_j\}_{j \in J}$ is equicontinuous, that is $\exists$

$M \in \mathbb{R}_+$ s.t. $\sup\limits_{j \in J} \|T_j\|_{\mathcal{L}(X,Y)} \leq M$.

If ~~it is not true that~~ $\sup\limits_{j \in J} \|T_j\|_{\mathcal{L}(X,Y)} = +\infty$

then $\{x : \sup\limits_{j \in J} \|T_j x\|_Y = +\infty\}$ ~~contains~~

~~a dense~~ is dense in X.

Theorem Let X and Y be B-spaces and consider a family $\{T_j\}_{j\in J}$ in $\mathcal{L}(X,Y)$.

Suppose that $\sup_{j\in J}\|T_j x\|_Y < +\infty \quad \forall x\in X$.

Then $\{T_j\}_{j\in J}$ is equicontinuous, that is $\exists$ $M\in\mathbb{R}_+$ s.t. $\sup_{j\in J}\|T_j\|_{\mathcal{L}(X,Y)} \le M$.

$$\uparrow$$

$$\sup_{\substack{j\in J\\ \|x\|_X\le 1}} \|T_j x\|_Y \le M$$

Pf By hypotesis $\forall x\in X$

$$\sup_{j\in J}\|T_j x\|_Y < +\infty$$

so $\forall x\in X \quad \exists\ n\in\mathbb{N}$ s.t.

$$\|T_j x\|_Y \le n \qquad \forall j\in J$$

$\forall n\in\mathbb{N}$ let

$$E_n = \{x\in X : \|T_j x\|_Y \le n \quad \forall j\in J\}$$

We know that $\forall x\in X \ \exists\, n\in\mathbb{N}$ s.t. $x\in E_n$

$$\bigcup_{n\in\mathbb{N}} E_n = X.$$

$$E_n = \bigcap_{j\in J} T_j^{-1}\,\overline{D_Y(0,n)}$$

$x\in E_n \Leftrightarrow \|T_j x\|_Y \le n \ \forall j \Leftrightarrow T_j x\in \overline{D_Y(0,n)} \quad \forall j$

$\Leftrightarrow x\in T_j^{-1}\overline{D_Y(0,n)} \quad \forall j$

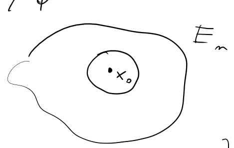
$$\overline{D_Y(0,n)} = \{y\in Y : \|y\|_Y \le n\}$$

E_n is closed for $\forall n$.

$\bigcup_{n\in\mathbb{N}} E_n = X$. X is Baire

$\exists\, n\in\mathbb{N}$ s.t. $\mathring{E}_n \ne \emptyset$

$\exists\ x_0\in E_n$ and $r>0$ s.t. $D_X(x_0,r) \subseteq E_n$



$x_0 + rD_X(0,1) = \{x_0 + rz : z\in D_X(0,1)\}$

$\|T_j(x_0+rz)\|_Y \le n \qquad \forall\ j\in J \quad \forall$ z with $\|z\|_X < 1$.

$n \ge \|T_j x_0 + rT_j z\|_Y \ge \|rT_j z\|_Y - \|T_j x_0\|_Y$

$(\ \|a+b\| \ge \|a\| - \|b\|\ )$

$r\|T_j z\|_Y \le n + \|T_j x_0\|_Y \qquad \forall j\in J \quad \forall z\in D_X(0,1)$

$r\|T_j z\|_Y \le 2n$

$\|T_j z\|_Y \le \frac{2n}{r} \qquad \forall j$

$\sup_{\substack{z\in X\\ \|z\|_X<1}}$

$\|T_j\|_{\mathcal{L}(X,Y)} \le \frac{2n}{r} =: M \quad \forall j$

Suppose $\sup_{j \in J} \|T_j\|_{\mathcal{L}(X,Y)} = +\infty$

By the previous part of the proof

for each n we have $\mathring{E}_n = \phi$

E_n are closed $\Rightarrow$ $X \setminus E_n$ is open and dense

Its is dense because for any $A \subseteq X$ open inst

$A \cap (X \setminus E_n) \neq \phi$. Otherwise

ih $A \cap (X \setminus E_n) = \phi \Rightarrow A \subseteq E_n$ but $\mathring{E}_n = \phi$

So each $X \setminus E_n$ is open and dense

$\bigcap_{n \in \mathbb{N}} (X \setminus E_n)$ is dense subspace of X because X is Baire

$= X \setminus \left(\bigcup_{n \in \mathbb{N}} E_n \right)$ is dense

$x \in X \setminus \left(\bigcup_{n \in \mathbb{N}} E_n \right) \Rightarrow x \notin E_n \ \forall n$

$\sup_{j \in J} \|T_j x\|_Y > n \Rightarrow \sup_{j \in J} \|T_j x\|_Y = +\infty$

We have shown that the set $\{ x \in X : \sup_{j \in J} \|T_j x\|_Y = +\infty \}$ is dense in X.

Trigonometric series $[-\pi,\pi]$

$f(x) = P(\cos x, \sin x)$ $\quad$ $P(\cancel{x}, \cancel{y})$ a polinomial

is a trigonometric polynomial.

Thanks to a repeated use of the prostoferese formulas

$$f(x) = \frac{a_0}{2} + \sum_{\ell=1}^{n} \left(a_\ell \cos(\ell x) + b_\ell \sin(\ell x) \right)$$

Lemma

$$a_\ell = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(\ell x)\, dx$$

$$b_\ell = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(\ell x)\, dx$$

Def Given $f \in L^1(-\pi,\pi)$ then Fourier series of f is the series

$$\frac{a_0}{2} + \sum_{\ell=1}^{\infty} \left(a_\ell \cos(\ell x) + b_\ell \sin(\ell x) \right)$$

$$e^{i\ell x} = \cos(\ell x) + i \sin(\ell x), \qquad \cos(\ell x) = \frac{e^{i\ell x} + e^{-i\ell x}}{2}, \quad \sin(\ell x) = \frac{e^{i\ell x} - e^{-i\ell x}}{2i}$$

$$\sum_{\ell \in \mathbb{Z}} \hat{f}(\ell) e^{i\ell x}$$

$$\hat{f}(\ell) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-i\ell x}\, dx \qquad \ell \in \mathbb{Z}$$

$$|\hat{f}(\ell)| \le \frac{1}{2} \int_{-\pi}^{\pi} |f(x)|\, dx$$

$$f \in L^1(-\pi,\pi) \longrightarrow \hat{f} : \mathbb{Z} \to \mathbb{C}$$

$$f \qquad\qquad \hat{f} \in \ell^\infty(\mathbb{Z}, \mathbb{C})$$

Lemma (Riemann Lebesgue theorem)

$$\lim_{\ell \to \infty} \hat{f}(\ell) = 0$$