Lezione 20 - 31/10/2025

limiti fondamentali

$$\lim_{x\to 0} \frac{e^x-1}{x} = 1, \qquad \lim_{x\to 0} \frac{\log(1+x)}{x} = 1$$

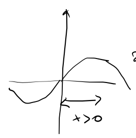
oss. se $\lim_{x\to x_0} f(x) = 0$ (e $f(x)\neq 0$ se $x \neq x_0$)

allora $\displaystyle \lim_{x\to x_0} \frac{e^{f(x)}-1}{f(x)} = 1$, $\boxed{\displaystyle \lim_{x\to x_0} \frac{\log(1+f(x))}{f(x)} = 1}$

infatti $x \longmapsto f(x) = y \longmapsto \dfrac{e^y-1}{y}$ ← e uso il limite della f. composta

$x \longmapsto f(x) = y \longmapsto \dfrac{\log(1+y)}{y}$

es. $\displaystyle \lim_{x\to 0^+} \frac{e^{\sqrt{x}}-1}{\sqrt{\operatorname{sen} x}}$ $\quad \frac{0}{0}$



idea $\dfrac{e^{\sqrt{x}}-1}{\sqrt{x}} \longrightarrow 1$

$$\lim_{x\to 0^+} \underbrace{\frac{e^{\sqrt{x}}-1}{\sqrt{x}}}_{\downarrow\, 1} \cdot \underbrace{\frac{\sqrt{x}}{\sqrt{\operatorname{sen} x}}}_{\downarrow\, 1} = 1$$

resta da calcolare

$$\lim_{x\to 0^+} \frac{\sqrt{x}}{\sqrt{\operatorname{sen} x}} = \lim_{x\to 0^+} \sqrt{\underbrace{\frac{x}{\operatorname{sen} x}}_{\downarrow\, 1}} = 1$$

$$\lim_{x\to 0} \frac{e^{\alpha x} - \sqrt{1-x}}{\operatorname{sen} x} \qquad \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{e^{\alpha x} - 1 + 1 - \sqrt{1-x}}{\operatorname{sen} x} \qquad A - B = \frac{A^2-B^2}{A+B}$$

$$= \lim_{x\to 0} \frac{e^{\alpha x}-1}{\operatorname{sen} x} + \lim_{x\to 0} \frac{\overbrace{1-\sqrt{1-x}}^{A-B}}{\operatorname{sen} x}$$

$$= \lim_{x\to 0} \frac{e^{\alpha x}-1}{\alpha x} \cdot \frac{\alpha x}{\operatorname{sen} x} + \lim_{x\to 0} \frac{(1-\sqrt{1-x})}{\operatorname{sen} x} \cdot \frac{(1+\sqrt{1-x})}{(1+\sqrt{1-x})}$$

$$= \lim_{x\to 0} \underbrace{\frac{e^{\alpha x}-1}{\alpha x}}_{\downarrow\, 1} \cdot \underbrace{\alpha}_{\downarrow\, \alpha} \cdot \underbrace{\frac{x}{\operatorname{sen} x}}_{\downarrow\, 1} + \lim_{x\to 0} \underbrace{\frac{\overbrace{\cancel{1}-(\cancel{1}-x)}^{x}}{\operatorname{sen} x}}_{\downarrow\, 1} \cdot \underbrace{\frac{1}{1+\sqrt{1-x}}}_{\downarrow\, \frac{1}{2}}$$

$$= \alpha + \frac{1}{2}$$

$$\lim_{x\to 0} \frac{\log(\cos x)}{x^2} \qquad \frac{0}{0}$$

idea: far saltar fuori $\displaystyle \lim_{x\to x_0} \frac{\log(1+f(x))}{f(x)}$

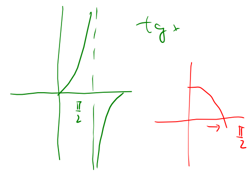
$$= \lim_{x\to 0} \underbrace{\frac{\log(1+(\cos x - 1))}{\cos x - 1}}_{\to\, 1} \cdot \frac{\cos x - 1}{x^2}$$

$$= \lim_{x\to 0} \underbrace{\frac{\log(1+(\cos x-1))}{\cos x-1}}_{\to 1} \cdot \frac{\cos x-1}{x^2}$$

resta da calcolare $\lim_{x\to 0} \frac{\cos x-1}{x^2} = -\frac{1}{2}$

$\left(\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2} \right.$ limite fondamentale $\left.\right)$

$$\lim_{x\to\frac{\pi}{2}^-} \underbrace{\operatorname{tg} x}_{\to +\infty} \cdot \underbrace{(e^{\cos x}-1)}_{\to 0} \qquad +\infty \cdot 0$$



$$= \lim_{x\to\frac{\pi}{2}^-} \frac{\sin x}{\cos x} \cdot (e^{\cos x}-1) \qquad \operatorname{tg} x = \frac{\sin x}{\cos x}$$

$$= \lim_{x\to\frac{\pi}{2}^-} \underbrace{\sin x}_{\to 1} \cdot \underbrace{\frac{e^{\cos x}-1}{\cos x}}_{\to 1} = 1$$

$$\lim_{x\to\frac{\pi}{2}^-} \operatorname{tg} x \underbrace{\frac{e^{\cos x}-1}{\cos x}}_{\to 1} \cos x \qquad \lim_{x\to\frac{\pi}{2}^-} \operatorname{tg} x \cdot \cos x$$

$$\lim_{x\to+\infty} \frac{\log_2(1+e^x)}{x+\sin x} \qquad \frac{+\infty}{+\infty}$$

$-1 \le \sin x \le 1$

$\underbrace{x-1}_{\to +\infty} \le x+\sin x \le x+1$

$$\log_2(y) = \frac{\log y}{\log 2} \qquad \log_b x = \frac{\log_a x}{\log_a b}$$

$$= \lim_{x\to+\infty} \frac{\log(1+e^x)}{\log 2} \cdot \frac{1}{x+\sin x}$$

$$\log(a\cdot b) = \log a + \log b$$

$$= \lim_{x\to+\infty} \frac{\log\left(e^x\left(1+\frac{1}{e^x}\right)\right)}{\log 2} \cdot \frac{1}{x+\sin x}$$

$$= \frac{1}{\log 2} \lim_{x\to+\infty} \left(\log(e^x) + \log(1+e^{-x})\right) \cdot \frac{1}{x+\sin x}$$

$$\log(e^x) = x$$

$$= \frac{1}{\log 2} \cdot \lim_{x\to+\infty} \frac{x+\log(1+e^{-x})}{x+\sin x}$$

$$= \frac{1}{\log 2} \lim_{x\to+\infty} \frac{\cancel{x}\left(1+\underbrace{\frac{\log(1+e^{-x})}{x}}_{\to 0}\right)}{\cancel{x}\left(1+\underbrace{\frac{\sin x}{x}}_{\to 0}\right)} = \frac{1}{\log 2}$$

om. ricordo che $\lim_n \frac{a^n}{n} = +\infty$, se $a>1$,

$$\frac{(1+\varepsilon)^n}{n} \geq \frac{1 + \cancel{n}\varepsilon + \frac{\cancel{n}(n-1)}{2}\varepsilon^2}{n}$$

$$\frac{(1+\varepsilon)^n}{n} \geq \underbrace{\frac{1}{n}}_{\to 0} + \underbrace{\varepsilon}_{\to \varepsilon} + \underbrace{\frac{n-1}{2}\varepsilon^2}_{\xrightarrow{n\to+\infty} +\infty}$$

voglio calcolare

$a>1 \qquad \lim_{x\to+\infty} \frac{a^x}{x}$

$n \leq x < n+1$

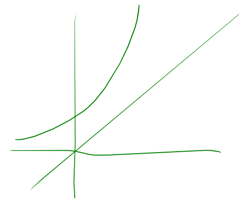
$n = [x]$ è la parte intera di x

$$[x] \leq x < [x]+1$$

$$a^{[x]} \leq a^x < a^{[x]+1}$$

$$\frac{a^{[x]}}{[x]+1} < \frac{a^x}{x} < \frac{a^{[x]+1}}{[x]}$$

$$\underbrace{\frac{a^n}{n+1}}_{\to+\infty} < \frac{a^x}{x} < \underbrace{\frac{a^{n+1}}{n}}_{\to+\infty}$$



$$\boxed{\lim_{x\to+\infty} \frac{a^x}{x} = +\infty}$$

$\lim_{x\to+\infty} \frac{a^x}{x^k} \qquad k \in \mathbb{Z}$

$= \lim_{x\to+\infty} \left(\frac{(a^{\frac{1}{k}})^x}{x}\right)^k = +\infty \qquad \left((a^{\frac{1}{k}})^x\right)^k = a^x$

$\frac{(a^{\frac{1}{k}})^x}{x} \to +\infty \qquad \lim_{x\to+\infty} \frac{b^x}{x} = +\infty \quad$ dove $b = a^{\frac{1}{k}} > 1$

conclusione $\lim_{x\to+\infty} \frac{a^x}{x^k} = +\infty$ | l'esponenziale cresce più di ogni potenza

$$\boxed{\lim_{x\to+\infty} \frac{\lg x}{x^\varepsilon}} \qquad (\varepsilon>0)$$

$\lg x = y \qquad x = e^y$

$$\lim_{y\to+\infty} \frac{y}{(e^y)^\varepsilon} = \lim_{y\to+\infty} \frac{y}{\underbrace{(e^\varepsilon)}_{>1}{}^y} = 0$$