October 31

$$L^1([-\pi,\pi]^d) \sim L^1(\mathbb{T}^d) \qquad \mathbb{T}^d = \mathbb{R}^d / 2\pi\mathbb{Z}^d$$

$$\hat{f}(n) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x)\, e^{-in\cdot x}\, dx$$

$n \in \mathbb{Z}^d$

$$L^1(\mathbb{T}^d, \mathbb{C}) \longrightarrow \ell^\infty(\mathbb{Z}^d, \mathbb{C})$$

$$|\hat{f}(n)| \leq \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} |f(x)|\, dx$$

$$\hat{f} \in c_0(\mathbb{Z}^d, \mathbb{C}) = \{ g \in \ell^\infty(\mathbb{Z}^d, \mathbb{C}) : \lim_{n\to\infty} g(n) = 0 \}$$

Reimann-Lebesgue lemma.

$d = 1$

<u>Def</u> $\forall\, n \geq 1$ $D_n(x)$ Dirichlet kernel

$$D_n(x) = \frac{1}{2} + \sum_{\ell=1}^{n} \cos(\ell x) = \frac{\sin((n+\frac{1}{2})x)}{2 \sin \frac{x}{2}}$$

where the 2° equality follows from

$$\sin((n+\tfrac{1}{2})x) = \sin(\tfrac{x}{2}) + \sum_{\ell=1}^{n} \left(\sin((\ell+\tfrac{1}{2})x) - \sin((\ell-\tfrac{1}{2})x) \right)$$

$$= \sin(\tfrac{x}{2}) + \sum_{\ell=1}^{n} 2 \sin(\tfrac{x}{2}) \cos(\ell x)$$

<u>Lemma</u> $f \in L^1(\mathbb{T})$ $\qquad \mathbb{T}$ — $(-\pi,\pi)$, $\mathbb{R}/2\pi\mathbb{Z}$

$$S_n f(x) = \frac{a_0}{2} + \sum_{\ell=1}^{n} \left(a_\ell \cos(\ell x) + b_\ell \sin(\ell x) \right)$$

$$a_\ell = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(\ell x)\, dx$$

$b_\ell =$ - - - $\sin(\ell x)$

$$S_n f(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\, D_n(x-t)\, dt$$

<u>Pf</u>

$$S_n f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, dt + \sum_{\ell=1}^{n} \left(\cos(\ell x) \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(\ell t)\, dt + \sin(\ell x) \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(\ell t)\, dt \right)$$

$\cos(\ell(x-t))$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, dt + \sum_{\ell=1}^{n} \frac{1}{\pi} \int_{\pi}^{\pi} f(t) \cos(\ell(x-t))\, dt$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \left(\frac{1}{2} + \sum_{\ell=1}^{n} \cos(\ell(x-t)) \right) dt$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\, D_n(x-t)\, dt$$

Rudin

$$D_n \in L^1(\mathbb{T})$$

$$\lim_{n\to+\infty} \|D_n\|_{L^1(\mathbb{T})} = +\infty \qquad 0 < x < \pi$$

$$0 < \sin\frac{x}{2} < \frac{x}{2}$$

$$\|D_n\|_{L^1(\mathbb{T})} = 2\int_0^\pi \frac{|\sin((n+\frac{1}{2})x)|}{2|\sin\frac{x}{2}|}\,dx \geq$$

$$\geq 2\int_0^\pi |\sin(\underbrace{(n+\tfrac{1}{2})x}_{y})| \frac{dx}{x} =$$

$$= 2\int_0^{(n+\frac{1}{2})\pi} |\sin(y)| \frac{dy}{y}$$

$$\geq 2\int_0^{n\pi} |\sin(y)| \frac{dy}{y} =$$

$$= 2\sum_{j=1}^{n} \int_{(j-1)\pi}^{j\pi} |\sin(y)| \frac{dy}{y}$$

$$> \frac{2}{\pi}\sum_{j=1}^{n} \frac{1}{j} \underbrace{\int_{(j-1)\pi}^{j\pi} |\sin(y)|\,dy}_{\underbrace{\int_0^\pi \sin(y)\,dy}_{2}}$$

$$= \frac{4}{\pi}\sum_{j=1}^{n}\frac{1}{j} \xrightarrow{n\to+\infty} +\infty$$

Theorem $\forall x \in \mathbb{T}$ $\exists$ $f \in C^0(\mathbb{T})$ s.t.

$\lim_{n\to+\infty} S_n f(0)$ does not converge.

Pf $x = 0$

$g(t) = \operatorname{sign}(D_n(t))$ where

$$\operatorname{sign} x = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$

$g \in L^1(\mathbb{T})$

$$S_n g(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)\, D_n(0-t)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)\, D_n(t)\,dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_n(t)|\,dt = \frac{1}{2\pi}\|D_n\|_{L^1(\mathbb{T})}$$

Lusin theorem

$\forall j$ $\exists$ $f_j \in C^0(\mathbb{T})$ s.t.

$$\|f_j\|_{L^\infty(\mathbb{T})} \le \|g\|_{L^\infty(\mathbb{T})}$$

and such that

$$|\{x : f_j(x) \neq g(x)\}| < \frac{1}{j}.$$

Then it follows that $f_j \to g$ in $L^1(\mathbb{T})$

$$\int_{\mathbb{T}} |f_j(x) - g(x)|\,dx = \int_{\{x : f_j(x) \neq g(x)\}} \underbrace{|f_j(x) - g(x)|}_{\le \|f_j\|_{L^\infty} + \|g\|_{L^\infty}}\,dx$$

$$\le \frac{1}{j}\left(\|f_j\|_{L^\infty} + \|g\|_{L^\infty}\right) \le \frac{2}{j} \xrightarrow[j\to\infty]{} 0$$

$$\begin{tikzcd} C^0(\mathbb{T}) \arrow[d] \arrow[dr] & \\ L^1(\mathbb{T}) \arrow[r] & \mathbb{C} \end{tikzcd}$$

$\varphi \longrightarrow S_n \varphi(0)$

$f_j \xrightarrow{j\to+\infty} g$ in $L^1(\mathbb{T})$

$S_n f_j(0) \xrightarrow[j\to\infty]{} S_n g(0) = \frac{1}{2\pi}\|D_n\|_{L^1(\mathbb{T})}$ $\quad \|f_j\|_{L^\infty(\mathbb{T})} \le 1$

If by contradiction we assume that

$\{S_n f(0)\}_{n\in\mathbb{N}}$ is a bounded sequence for any

$f \in C^0(\mathbb{T})$

$\{T_n f\}_{n\in\mathbb{N}}$ $\quad T_n f = S_n f(0)$, $T_n \in (C^0(\mathbb{T}))'$

have $\sup_{n\in\mathbb{N}} \{|T_n f|\} < +\infty$ $\quad \forall f \in C^0(\mathbb{T})$

then by Banach-Steinhaus $\exists$ $M \in \mathbb{R}_+$

s.t. $\|T_n\| \le M$ $\quad \forall n \in \mathbb{N}$

But recall that we found for any n

a sequence f_j in $C^0(\mathbb{T})$ with $\|f_j\|_{L^\infty(\mathbb{T})} \le 1$

$$\underbrace{|S_n f_j(0)|}_{|T_n f_j| \le \|T_n\| \ \forall j} \xrightarrow{j\to+\infty} \frac{1}{2\pi}\|D_n\|_{L^1(\mathbb{T})}$$

$$\Rightarrow \quad \frac{1}{2\pi}\underbrace{\|D_n\|_{L^1(\mathbb{T})}}_{\downarrow\, n\to+\infty \atop +\infty} \le \|T_n\| \le M$$

and this gives a contradiction.

Therefore $\exists$ $f \in C^0(\mathbb{T})$ s.t.

$\{S_n f(0)\}$ is an unbounded sequence

Such f's form a dense subspace of $C^0(\mathbb{T})$

Theorem (Open Mapping)

Let E and F be B-spaces and $T: E \to F$ continuous surjective. Then there exists a $c>0$ s.t.

$$T(D_E(0,1)) \supseteq D_F(0,c).$$

(Corollary: if T is bijective and continuous, also T^{-1} is continuous)

(X, τ)

(X, d_1), (X, d_2)

d_1 and d_2 are equivalent if $\exists\, C>1$

s.t. $\frac{1}{C} d_1(x,y) \leq d_2(x,y) \leq C d_1(x,y) \quad \forall x,y \in X$

If d_1 and d_2 are equivalent, they induce on X the same topology, but there are easy cases of pairs of metrics which induce the same topology but are not equivalent

X

$$d_1(x,y) = \|x-y\|$$

$$d_2(x,y) = \frac{\|x-y\|}{1+\|x-y\|}$$

X $\quad \|\ \|_1 \quad \|\cdot\|_2$ are equivalent if $\exists\, C>1$

s.t. $\frac{1}{C}\|x\|_1 \leq \|x\|_2 \leq C\|x\|_1 \quad \forall x \in X$

$(X, \|\cdot\|_1) \xrightarrow{id} (X, \|\ \|_2) \Rightarrow$ if id is continuous the two norms are equivalente.

Theorem (Open Mapping)

Let E and F be B-spaces and $T: E \to F$ continuous surjective. Then there exists a $c>0$ s.t.

$$T(D_E(0,1)) \supseteq D_F(0,c).$$

Pf Initially we will prove $\exists\, c>0$ st

$$\overline{T(D_E(0,1))} \supseteq D_F(0,2c)$$

$$X_n = n\overline{T(D_E(0,1))} = \overline{nT(D_E(0,1))} =$$

$$= \overline{T(D_E(0,n))}$$

$$F = TE = T\bigcup_{n=1}^{\infty} D_E(0,n) = \bigcup_{n=1}^{\infty} TD_E(0,n)$$

$$\subseteq \bigcup_{n=1}^{\infty} \underbrace{\overline{TD_E(0,n)}}_{X_n}$$

$$F = \bigcup_{n=1}^{\infty} X_n$$

The X_n are closed sets. F is Baire space

$\exists\, n$ s.t. $\overset{\circ}{X}_n \neq \emptyset$ $\qquad X_n = nX_1$

$\overset{\circ}{X}_1 \neq \emptyset \qquad X_1 = \overline{TD_E(0,1)}$

$\exists\; y_0 \in X_1$ s.t. $D_F(y_0, 4c) \subseteq \overline{TD_E(0,1)}$

$-y_0 \in -X_1 \qquad \{-y_0\} \in \overline{TD_E(0,1)}$

$$D_F(0,4c) \subseteq \overline{TD_E(0,1)} + \overline{TD_E(0,1)} \subseteq$$

$$\subseteq 2\overline{TD_E(0,1)}$$

$$\frac{1}{2}\overline{TD_E(0,1)} + \frac{1}{2}\overline{TD_E(0,1)} \subseteq \overline{TD_E(0,1)}$$

$$D_F(0,2c) \subseteq \overline{TD_E(0,1)}$$

Initially we will prove $\exists\ c>0$ st

$$\overline{T(D_E(0,1))} \supseteq D_F(0,2c) \qquad (1)$$

Now we show $T(D_E(0,1)) \supseteq D_F(0,c)$

Let $y \in D_F(0,c)$. Want ~~to~~ show $\exists\ x \in D_E(0,1)$ st.

$y = Tx$

We claim that

$\exists\ z_1 \in D_E(0,\frac{1}{2})$ st. $\|y - Tz_1\|_F < c\,\frac{1}{2}$ Claim 1

$$(1) \iff \overline{T(D_E(0,\tfrac{1}{2}))} \supseteq D_F(0,c) \qquad (2)$$

so either our y is in $T(D_E(0,\frac{1}{2}))$ and so

$y = Tz_1$ for some $z_1 \in D_E(0,\frac{1}{2})$

or y is an accumulation point of $T(D_E(0,\frac{1}{2}))$

and so $\forall \varepsilon>0\ \exists\ z_1 \in D_E(0,\frac{1}{2})$ st

$\|y - Tz_1\|_F < \varepsilon \qquad \varepsilon = \frac{c}{2}$

Suppose by induction that we have found

$z_j \in D_E(0,2^{-j})$ for $j=1,\dots,n$ such that

$$\|y - \sum_{j=1}^{n} Tz_j\|_F < c\,2^{-n}$$

$$\Updownarrow$$

$$y - \sum_{j=1}^{n} Tz_j \in D_F(0,c\,2^{-n})$$

We claim $\exists\ z_{n+1} \in D_E(0,2^{-n-1})$ st

$$\|y - \sum_{j=1}^{n+1} Tz_j\|_F < c\,2^{-n-1} \qquad (3)$$

$$\overline{TD_E(0,1)} \supseteq D_E(0,2c) \qquad 2^{-n-1}$$

$$\overline{TD_E(0,2^{-n-1})} \supseteq D_E(0,2^{-n}c)$$

$$y - \sum_{j=1}^{n} Tz_j \in D_E(0,2^{-n}c)$$

If $y - \sum_{j=1}^{n} Tz_j \in TD_E(0,2^{-n-1})$

then $\exists\ z_{n+1} \in D_E(0,2^{-n-1})$ st.

$$y - \sum_{j=1}^{n} Tz_j = Tz_{n+1}$$

$$\iff y - \sum_{j=1}^{n+1} Tz_j = 0$$

If $y - \sum_{j=1}^{n} Tz_j \notin TD_E(0,2^{-n-1})$ then

it is an accumulation point for $TD_E(0,2^{-n-1})$

Hence $\forall \varepsilon>0\ \exists\ z_{n+1} \in D_E(0,2^{-n-1})$

st $\|y - \sum_{j=1}^{n+1} Tz_j\|_F < \varepsilon \qquad \varepsilon = c\,2^{-n-1}$

$$D_F(0,c) \ni y = \sum_{j=1}^{\infty} Tz_j = \lim_{n\to+\infty} T\sum_{j=1}^{n} z_j = Tx$$

$\sum_{j=1}^{\infty} z_j$ is a series in E

and if I show that it converges

to $x \in D_E(0,1)$

$$\sum_{j=1}^{\infty} \|z_j\|_E < \sum_{j=1}^{\infty} 2^{-j} = \qquad \|z_j\|_E < 2^{-j}$$

$$= 1$$

$x = \sum_{j=1}^{\infty} z_j$

$$\sum_{j=0}^{\infty} 2^{-j} = 1 + \sum_{j=1}^{\infty} 2^{-j} = \frac{1}{1-\frac{1}{2}} = 2$$

$\|x\|_E < 1$