November 6

Today at 16:15

BCAM

Last time

Corollary X, Y B-spaces

$T \in \mathcal{L}(X,Y)$, T bijective, then T is an isomorphism

Corollary $L^1(\mathbb{T}) = L^1([-\pi,\pi]) \ni f \longrightarrow \underset{\underset{c_0(\mathbb{Z})}{\pi}}{\hat{f}} (: \mathbb{Z} \to \mathbb{C})$

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\,dx$$

The map $f \in L^1(\mathbb{T}) \longrightarrow \hat{f} \in c_0(\mathbb{Z})$ is not onto.

Pf First of all, it easy to see that

$$f \xrightarrow{\mathcal{F}} \hat{f} \quad \text{is injective.}$$

So, if by contradiction this map were surjective, since it is continuous linear map, it would be an isomorphism.

$$D_n(x) = \frac{1}{2} + \sum_{\ell=1}^{n} \cos(\ell x) =$$

$$= \frac{1}{2}\left(1 + \sum_{\ell=1}^{n}\left(e^{i\ell x} + e^{-i\ell x}\right)\right)$$

$$= \frac{1}{2} \sum_{|\ell|\le n} e^{i\ell x}$$

$$\hat{D}_n(k) = \begin{cases} \frac{1}{2} & \text{if } |k| \le n \\ 0 & |k| > n \end{cases}$$

$$\hat{D}_n(k) = \frac{1}{2}\chi_{[-n,n]}(k) \quad \text{in } \mathbb{Z}$$

$$f \longleftrightarrow \hat{f}$$
$$\longleftarrow$$

If $\mathcal{F}$ was an isomorphism there would be a $c > 0$ s.t. $\|\mathcal{F}^{-1}\|_{\mathcal{L}(c^0(\mathbb{Z}), L^1(\mathbb{T}))} \le c$

In particular

$$\|D_n\|_{L^1(\mathbb{T})} = \|\mathcal{F}^{-1}\tfrac{1}{2}\chi_{[-n,n]}\|_{L^1(\mathbb{T})}$$

$$\le \underbrace{\|\mathcal{F}^{-1}\|_{\mathcal{L}(c_0(\mathbb{Z}), L^1(\mathbb{T}))}}_{\le c} \underbrace{\|\tfrac{1}{2}\chi_{[-n,n]}\|_{\ell^\infty(\mathbb{Z})}}_{\frac{1}{2}}$$

$$+\infty \xleftarrow{n\to+\infty} \|D_n\|_{L^1(\mathbb{T})} \le \frac{c}{2} \qquad \forall n$$

Def If E and F are B-spaces then $E \times F$ is a B-space for the norm

$\|(x,y)\| = \|x\|_E + \|y\|_F$ $\quad$ $\sqrt[p]{\|x\|_E^p + \|y\|_F^p}$ $\quad \forall \quad p \geq 1$ $\quad \frac{d}{dx} \quad L^2(\mathbb{R}) \to L^2(\mathbb{R})$

Theorem Let E and F be B-spaces and let $T: E \to F$ be a linear map. Then T is continuous if and only if the graph of T

$$G(T) = \{(x, Tx) \in E \times F : x \in E\}$$

is closed in $E \times F$.

Pf It is a general fact that if T is continuous then $G(T)$ is closed.
This follows from the fact that the map

$$(x,y) \xrightarrow{\Phi} Tx - y$$

$$E \times F \xrightarrow{\Phi} F \quad \Rightarrow$$

$G(T) = \Phi^{-1}(0)$ is closed

We need to show that if $G(T)$ is closed and $T: E \to F$ is linear and E, F are B-spaces then $T \in \mathcal{L}(E,F)$.

Since T is linear, it follows that

$$G(T) = \{(x, Tx) \in E \times F : x \in E\}$$

is a vector space and since it is a closed subspace of the B-space $E \times F$
$G(T)$ is a B-space

$$\begin{array}{ccc} G(T) & \subseteq & E \times F \\ & \searrow & \downarrow \pi \\ & & E \end{array}$$

$(x, Tx) \xrightarrow{\pi} x \in E$ is continuous map
$G(T) \xrightarrow{\pi} E$

$$\|\pi(x,Tx)\|_E = \|x\|_E \leq \|x\|_E + \|Tx\|_F$$

$$\|\pi\|_{\mathcal{L}(G(T),E)} \leq 1$$

and π is surjective
But this map is also injective

$$(x, Tx) \xrightarrow{\pi} x = x' \xleftarrow{\pi} (x', Tx')$$

π is a continuous and bijective linear map between $G(T)$ and E.
It is necessarily an isomorphism.

$\pi^{-1}: x \longrightarrow (x, Tx)$ is bounded i.e.

$\exists \quad C > 0$ s.t.

$$\|(x, Tx)\| = \|x\|_E + \|Tx\|_F \leq C\|x\|_E$$

$$\|Tx\|_F \leq (C-1)\|x\|_E \quad \forall x \in E$$

$\Rightarrow$ T is a continuous operator from $E \to F$.

Projections and complemented subspaces

Def A vector subspace F of a TVS E is "complemented" if F is closed and if $\exists$ a closed vector subspace of E, let me call it G, such that $E = F \oplus G$

($\forall x \in E$ there exists a unique pair $(x_1, x_2) \in F \times G$ s.t. $x = x_1 + x_2$)

Theorem If F is subspace of E (B-space) with $\dim F = n < +\infty$ then F is complemented

Pf First of all F is isomorphic to K^n. The latter is complete so F is complete.
$\Rightarrow$ F is a closed subspace of E.

Let $F = \mathrm{Span}\{f_1, \dots, f_n\}$ ← a basis

$\Rightarrow$ $x = \sum_{j=1}^{m} x_j f_j$ for some $x_j \in K$.

$\phi_j(x) = x_j$ $\qquad \phi_j : F \longrightarrow K$ linear

and they are all $\phi_j \in F'$.

By Hahn-Banach I can extend

$\phi_j : E \longrightarrow K$. $\qquad \|\phi_j\|_{E'} = \|\phi_j|_F\|_{F'}$

$G := \bigcap_{j=1}^{m} \ker \phi_j$. G is closed and is a vector subspace of E

We want to show that $E = F \oplus G$.

First of all we show $E = F + G$

Given $z \in E$ $\qquad x_j := \phi_j z$ and

$x := \sum_{j=1}^{m} x_j f_j \in F$

$z = x + (z - x) \in F + G$

$\phi_j(z - x) = \phi_j z - \phi_j x = x_j - x_j = 0 \Rightarrow z - x \in G$

Next we need to show $F \cap G = 0$

$x \in F \cap G$

$x \in G \Rightarrow \phi_j x = 0$

$x \in F \Rightarrow x = \sum_{j=1}^{m} \underset{\substack{\| \\ 0}}{\phi_j x}\, f_j = 0$

Theorem F closed subspace of a B-space E. Then if $\operatorname{codim} F < +\infty$ then F is complemented.

$\ell^\infty(\mathbb{N})$

$c_0(\mathbb{N})$ is not complemented in $\ell^\infty(\mathbb{N})$

Def Given a TVS E a $P \in \mathcal{L}(E)$ is a projection if $P^2 = P$.

Exercise P is projection $\Rightarrow$ $1-P$ is a projection

$$(1-P)(1-P) = 1-2P+P^2 = 1-2P+P =$$
$$= 1-P$$

Exercise If $E = X \oplus Y$ with X and Y closed E B-space, then $P(x+y) = x$ and $Q(x+y) = y$ are projections.

Pf I focus on P. Let us show $P^2 = P$

$z \in E$, $z = x+y$ in unique way

$Pz = x$ $\qquad$ $P^2 z = P\,Pz = Px = P(x+0) = x$

Now we show that $P \in \mathcal{L}(E)$

$$\begin{array}{ccc} (x,y) & \xrightarrow{T} & x+y \in E \\ X \times Y & & \end{array}$$

$\|\cdot\|_{X\times Y}$

$$\|x+y\|_E \le \|x\|_E + \|y\|_E =: \|(x,y)\|_{X \times Y}$$

So T is a continuous map $\|T\|_{\mathcal{L}(X\times Y, E)} \le 1$

But T is bijective. So T is an isomorphism.

$$T^{-1} z = T^{-1}(x+y) = (x,y)$$

$\exists\, C > 0$ st

$$\|(x,y)\|_{X\times Y} = \|x\|_E + \|y\|_E \le C\,\|z\|_E \qquad z = x+y$$

$$Pz = P(x+y) = x$$

$$\|x\|_E = \|Pz\|_E \le C\,\|z\|_E \qquad \forall z \in E$$

$\Rightarrow P$ is continuous.

E B space

$P \in \mathcal{L}(E)$ projection

$$E = X \oplus Y$$

$$Y = \ker P$$

$$X = \ker(1-P)$$

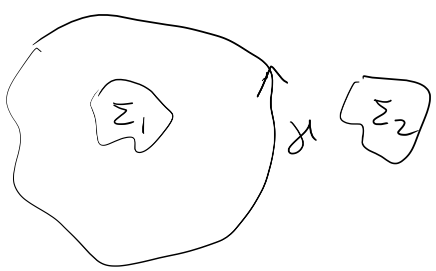
$$z = Pz + (1-P)z$$

$$(1-P)Pz = Pz - P^2 z = Pz - Pz = 0$$

X B-space $A \in \mathcal{L}(X)$

$\sigma(A) = \Sigma_1 \cup \Sigma_2$

Σ_1 and Σ_2 two closed disjoint subsets of $\mathbb{C}$.



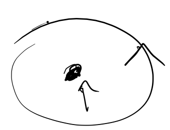
γ a closed path in $\rho(A)$

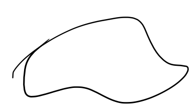
$$X \to X$$

$$P = -\frac{1}{2\pi i} \int_\gamma R_A(z)\, dz =$$

$$= -\frac{1}{2\pi i} \int_\gamma (A-z)^{-1}\, dz$$

P is a projection





$R(P) = \ker(1-P)$

$X = R(P) \oplus \ker P$

A preserves the direct sum

$$A = \left(\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array}\right) \qquad \begin{array}{l} A_1 = A|_{R(P)} \\ A_2 = A|_{\ker P} \end{array}$$

$\sigma(A_1) = \Sigma_1$
$\sigma(A_2) = \Sigma_2$