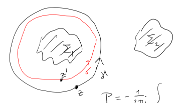
Nov 7

X B-space, $A \in \mathcal{L}(X)$

$\sigma(A) = \Sigma_1 \cup \Sigma_2$



$P = -\frac{1}{2\pi i} \int_\gamma R_A(z)\, dz$ it is a projection.

$\gamma : I \to \mathbb{C}$

$:= -\frac{1}{2\pi i} \int_I R_A(\gamma(t))\, \gamma'(t)\, dt$

$P^2 = P$

$P \; P = -\frac{1}{2\pi i} \int_\sigma R_A(z')\, dz' \left(-\frac{1}{2\pi i}\right) \int_\gamma R_A(z)\, dz$

$= \left(\frac{1}{2\pi i}\right)^2 \int_\sigma \int_\gamma R_A(z') R_A(z)\, dz\, dz'$

$R_A(z') R_A(z) = \frac{1}{A - z'} \frac{1}{A - z} = \frac{1}{z' - z}\left(\frac{1}{A - z'} - \frac{1}{A - z}\right)$

$= \frac{1}{z' - z} \frac{\cancel{A} - z - (\cancel{A} - z')}{(A - z')(A - z)}$

$= \left((A - z')(A - z)\right)^{-1} = R_A(z) R_A(z')$

$= \left(\frac{1}{2\pi i}\right)^2 \int_\sigma \int_\gamma \frac{1}{z' - z}\left(R_A(z') - R_A(z)\right) dz\, dz' =$

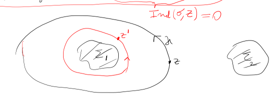
$R_A(z') R_A(z) = (z' - z)^{-1}\left(R_A(z') - R_A(z)\right)$

$= \left(\frac{1}{2\pi i}\right)^2 \int_\sigma \int_\gamma \frac{1}{z' - z} R_A(z')\, dz\, dz'$

$- \left(\frac{1}{2\pi i}\right)^2 \int_\sigma \int_\gamma \frac{1}{z' - z} R_A(z)\, dz\, dz'$

$= \frac{1}{2\pi i} \int_\sigma dz' R_A(z') \frac{1}{2\pi i} \int_\gamma \frac{1}{z' - z}\, dz$

$- \frac{1}{2\pi i} \int_\gamma dz\, R_A(z) \underbrace{\frac{1}{2\pi i} \int_\sigma \frac{1}{z' - z}\, dz'}_{\mathrm{Ind}(\sigma, z) = 0} = 0$



$= -\frac{1}{2\pi i} \int_\sigma dz' R_A(z') \underbrace{\frac{1}{2\pi i} \int_\gamma \frac{1}{z - z'}\, dz}_{\mathrm{Ind}(\gamma, z') = 1}$

$= -\frac{1}{2\pi i} \int_\sigma dz' R_A(z') = P$

$X = RP \oplus \ker P$ $\qquad RP = \ker(1 - P)$

$A = PA + (1 - P)A$ $\qquad [P, A] = PA - AP = 0$

$R_A(z)\, A = A\, R_A(z)$ $\qquad [A, R_A(z)] \equiv 0 \quad \forall z \in \rho(A)$

$[A, P] = \left[A, \frac{-1}{2\pi i} \int_\gamma R_A(z)\, dz\right] =$

$= -\frac{1}{2\pi i} \int_\gamma \underbrace{[A, R_A(z)]}_{0}\, dz \equiv 0$

$(A - z)^{-1} A = A (A - z)^{-1}$

$\cancel{(A - z)}\cancel{(A - z)^{-1}} A (A - z) = (A - z) A \cancel{(A - z)^{-1}}\cancel{(A - z)}$

$A(A - z) = (A - z) A$

$A^2 - zA = A^2 - zA$

If $\quad x \in RP \Rightarrow Ax \in RP$

$x = Px \qquad Ax = APx = PAx$

$A \quad RP \oplus \ker P$ $\qquad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ $\quad 0$

$A = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}$

$\sigma(A_{11}) = \Sigma_1$

$P = -\frac{1}{2\pi i} \int_\gamma R_A(z)\, dz$

$A : RP \circlearrowleft$ $\qquad \lambda \in \Sigma_2$

$A - \lambda : RP \circlearrowleft$

$T = -\frac{1}{2\pi i} \int_\gamma (z - \lambda)^{-1} R_A(z)\, dz$

$A - \lambda = -\frac{1}{2\pi i} \int_\gamma (z - \lambda) R_A(z)\, dz$ $\;\cancel{\ast}$

RP

$(A - \lambda)\, T =$

$= \frac{1}{2\pi i} \int_\sigma (z' - \lambda) R_A(z')\, dz' \; \frac{1}{2\pi i} \int_\gamma (z - \lambda)^{-1} R_A(z)\, dz$

$= \left(\frac{1}{2\pi i}\right)^2 \int_\sigma \int_\gamma (z' - \lambda)(z - \lambda)^{-1} \underbrace{R_A(z') R_A(z)}_{(z' - z)^{-1}(R_A(z') - R_A(z))}\, dz\, dz'$

$= \left(\frac{1}{2\pi i}\right)^2 \int_\sigma \int_\gamma (z' - \lambda)(z - \lambda)^{-1}(z' - z)^{-1} R_A(z')\, dz\, dz'$

$- \left(\frac{1}{2\pi i}\right)^2 \int_\sigma \int_\gamma (z' - \lambda)(z - \lambda)^{-1}(z' - z)^{-1} R_A(z)\, dz\, dz'$

$- \frac{1}{z' - \lambda}\,\mathrm{Ind}(\gamma, z')$

$= \frac{1}{2\pi i} \int_\sigma dz' R_A(z') \frac{z' - \lambda}{2\pi i} \int_\gamma \frac{1}{z - \lambda} \frac{1}{z' - z}\, dz = 1$

$- \frac{1}{2\pi i} \int_\gamma dz\, R_A(z) \frac{1}{2\pi i} \frac{1}{z - \lambda} \int_\sigma \frac{z' - \lambda}{z' - z}\, dz'$

$(z - \lambda)\, \mathrm{Ind}(\sigma, z) = 0$

$= -\frac{1}{2\pi i} \int_\sigma dz' R_A(z') = P = \mathrm{id}_{RP}$

$$\mathbb{T}^d = \mathbb{R}^d / 2\pi\mathbb{Z}^d \qquad [0,2\pi]^d$$

$$f \in L^2(\mathbb{T}^d, \mathbb{R}^d) \qquad f : \mathbb{T}^d \to \mathbb{R}^d$$

$$\int_{\mathbb{T}^d} |f(x)|^2 \, dx < +\infty \qquad L^2(\mathbb{T}^d) \longleftrightarrow L^2([0,2\pi]^d)$$

$$H := \{ f \in L^2(\mathbb{T}^d, \mathbb{R}^d) : \operatorname{div} f = 0 \}$$

$$\operatorname{div} f = \sum_{j=1}^{n} \partial_j f_j \iff n \cdot \hat{f}(n) = 0 \quad \forall n \in \mathbb{Z}^d$$

$$\hat{f} : \mathbb{Z}^d \to \mathbb{R}^d$$

$$\hat{f}(n) = \frac{1}{(2\pi)^d} \int_{[0,2\pi]^d} f(x) \, e^{-in\cdot x} \, dx$$

$$\widehat{\partial_j f}(n) = i \, n_j \, \hat{f}(n)$$

$$f \to \partial_j f$$

$$\hat{f} \to i n_j \hat{f}(n)$$

$$L^2(\mathbb{T}^d, \mathbb{R}^d) = H \oplus \nabla \dot{H}^1$$

$\mathbb{P}$ Leray

Def Let E be a TVS and consider E'.

Then the $\sigma(E, E')$ weak topology in E is the topology which has as a subbasis of seminorms $\{|f|\}_{f \in E'}$

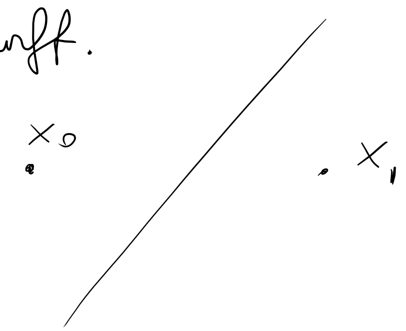
Exercise Show that given any $x_0 \in E'$ there is a basis of nbhds of x_0 of the form

$$V_{x_0}(f_1, \dots, f_n, \varepsilon) = \{x \in E : |f_j(x - x_0)| < \varepsilon \text{ for } j = 1, \dots, n\}$$

for any finite family $f_1, \dots, f_n \in E'$ and any $\varepsilon > 0$.

Lemma If E is locally convex, then $(E, \sigma(E, E'))$ is Hausdorff.

Pf



By Hahn Banach $\exists$ $f \in E'$ and $\alpha \in \mathbb{R}$

s.t. $f(x_0) < \alpha < f(x_1)$

Here $f^{-1}(-\infty, \alpha)$ and $f^{-1}(\alpha, +\infty)$ are nbhds of x_0 and x_1 for this topology and are disjoint.

<u>Notation</u> We will write $x_n \xrightarrow{n\to+\infty} x$ if

$x = \lim_{n\to+\infty} x_n$ in the $\sigma(E,E')$ topology

$\left(x_n \xrightarrow{n\to+\infty} x \right)$

$\sigma(E,E')$

<u>Lemma</u> E B-space and $\{x_n\}$ a sequence

1) $x_n \rightharpoonup x \iff f(x_n) \xrightarrow{n\to+\infty} f(x) \quad \forall f \in E'$

2) $x_n \xrightarrow{n\to+\infty} x$ strongly $\Rightarrow x_n \xrightarrow{n\to+\infty} x$

3) If $x_n \rightharpoonup x \Rightarrow \sup_n \|x_n\|_E < +\infty$

4) If $x_n \rightharpoonup x$ and $f_n \longrightarrow f$ in norm in E'

$\Rightarrow f_n(x_n) \longrightarrow f(x)$

<u>Pf</u> 3) follows by Banach-Steinhaus

$J : E \hookrightarrow E''$

Then $Jx_n, Jx \in E''$

$\forall f \in E' \quad \langle Jx_n, f \rangle_{E'' \times E'} = f(x_n) \to f(x) \quad \forall f \in E'$

$\sup_{n\in\mathbb{N}} |\langle Jx_n, f \rangle| < +\infty \quad \forall f \in E'$

$Jx_n \in \mathcal{L}(E', K)$

$\Rightarrow \exists M > 0$ s.t. $\underbrace{\|Jx_n\|_{E''}}_{= \|x_n\|_E} \le M$

Theorem Let E be B-space and $C \subseteq E$ be convex.

Then C is closed for the strong topology if and only if C is closed for $\sigma(E, E')$ topology

$\Longleftrightarrow$

Pf $\Leftarrow$ Since the strong topology is finer than the $\sigma(E, E')$ topology the closed set of the less refined topology are also ~~open~~ closed sets for the more refined topology.

$\Rightarrow$

Suppose C is convex and closed for the strong topology. To show that it closed for the $\sigma(E, E')$ topology is enough to show that $E \setminus C$ is open for $\sigma(E, E')$ topology.

Using Hahn Banach $\exists$ $f \in E'$ and $\alpha \in \mathbb{R}$

s.t. $f(x) < \alpha < f(x_0)$ $\forall x \in C$

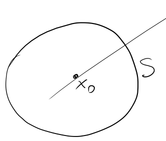
$\Rightarrow$ $f^{-1}(\alpha, +\infty)$ is an open nbhd of x_0 for $\sigma(E, E')$ contained in $E \setminus C$

Lemma Let E be a Banach space $\dim E = +\infty$ and let $S = \partial D_E(0,1) = \{x \in E : \|x\|_E = 1\}$

$\overline{S}|_{\sigma(E,E')} = \overline{D_E(0,1)} = \{x \in E : \|x\|_E \le 1\}$

Pf $\overline{D_E(0,1)} \supseteq S$

closed for the weak topology



Let $\|x_0\|_E \le 1$

$\forall$ $V_{x_0}(f_1, \dots, f_n, \varepsilon) =$

$= \{x \in E : |f_j(x - x_0)| < \varepsilon \ \ j = 1, \dots, n\}$

$V_{x_0}(f_1, \dots, f_n, \varepsilon) \cap S \neq \phi$

Pf: $y_0 \in \ker f_1 \cap \dots \cap \ker f_n \neq 0$

$\{x : x = x_0 + t y_0 \quad t \in \mathbb{R}\}$ is a line

$g(t) = \|x_0 + t y_0\|_E$ $\quad \exists$ t st $g(t) = 1$

$x_0 + t y_0 \in V_{x_0}(f_1, \dots, f_n, \varepsilon)$

$f_j(\cancel{x_0} + t y_0 - \cancel{x_0}) = t f_j(y_0) = 0$