10 novembre

$f : I \to \mathbb{R}$ $\quad x_0 \in I$ $\quad f^{(k)}(x_0)$ $\quad k = 0, \dots, n$

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

Lemma Supponiamo di essere nelle ipotesi indicate sopra. Allora se $q(x)$ è un polinomio con $\deg q \leq n$ e se

$$f(x) = q(x) + o((x-x_0)^n) \quad (1)$$

si ha che $q(x) = P_n(x)$.

Dim Utilizziamo il Teorema di Peano per il quale

$$\begin{cases} f(x) = P_n(x) + o((x-x_0)^n) & (2) \\ f(x) = q(x) + o((x-x_0)^n) & (1) \end{cases}$$

se noi consideriamo la differenza

$$0 = (P_n(x) - q(x)) + o((x-x_0)^n) \quad (3)$$

$$\lim_{x \to x_0} \frac{o((x-x_0)^n) - o((x-x_0)^n)}{(x-x_0)^n} = \lim_{x \to x_0} \left[\underbrace{\frac{o((x-x_0)^n)}{(x-x_0)^n}}_{\downarrow\, 0} - \underbrace{\frac{o((x-x_0)^n)}{(x-x_0)^n}}_{\downarrow\, 0} \right]$$

$$= 0 - 0 = 0$$

$$q(x) - P_n(x) = o((x-x_0)^n)$$

$\deg q \leq n$ per ipotesi, $\deg P_n \leq n$

$\deg(q - P_n) \leq n$. Per il lemma precedente

$$q(x) - P_n(x) \equiv 0 \qquad \boxed{P_n = q}$$

Scrivere tutti i polinomi di McLaurin di $f(x) = x^2 \sin(x^3)$

$$P_n f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$$

Si parte da $\left(f(x) = x^2 \sin(x^3) \right)$

$$\sin(y) = \sum_{k=0}^{n} (-1)^k \frac{\cancel{x} y^{2k+1}}{(2k+1)!} + o(\cancel{x} y^{2n+1})$$

$$\sin(x^3) = \sum_{k=0}^{n} (-1)^k \frac{x^{6k+3}}{(2k+1)!} + o(x^{6n+3})$$

$$f(x) = x^2 \sin(x^3) = \sum_{k=0}^{n} (-1)^k \frac{x^{6k+5}}{(2k+1)!} + x^2 o(x^{6n+3})$$

Qui $x^2 o(x^{6n+3}) = o(x^{6n+5})$

$= o(x^2 x^{6n+3}) =$?

$$\lim_{x \to 0} \frac{x^2 o(x^{6n+3})}{x^{6n+5}} = 0 \quad = \lim_{x \to 0} \frac{o(x^{6n+3})}{x^{6n+3}} = 0$$

ma allora $x^2 o(x^{6n+3}) = o(x^{6n+5})$

$$f(x) = x^2 \sin(x^3) = \sum_{k=0}^{n} (-1)^k \frac{x^{6k+5}}{(2k+1)!} + x^2 o(x^{6n+3})$$

$$f(x) = x^2 \sin(x^3) = \underbrace{\sum_{k=0}^{n} (-1)^k \frac{x^{6k+5}}{(2k+1)!}}_{deg = 6n+5} + o(x^{6n+5})$$

è il polinomio di McLaurin $P_{6n+5}(x)$

P_5, P_{11}, P_{17}, ...

$$P_5(x) = \sum_{k=0}^{0} (-1)^k \frac{x^{6k+5}}{(2k+1)!} = x^5$$

$P_0 = ?$ $P_1 = ?$, $P_2 = ?$, $P_3 = ?$

$P_4 = ?$.

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$$

$$P_{n+1}(x) = \sum_{k=0}^{n+1} \frac{f^{(k)}(0)}{k!} x^k = \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k}_{P_n(x)} + \frac{f^{(n+1)}(0)}{(n+1)!} x^{n+1}$$

$$P_{n+1}(x) = P_n(x) + \frac{f^{(n+1)}(0)}{(n+1)!} x^{n+1}$$

$n = 4$

$P_5 = x^5 = 0 + x^5$

$P_4 = 0 = P_3 = P_2 = P_1 = P_0$

P_5 , P_{11} , P_{17} , ...

$P_5(x) = \sum_{k=0}^{0} (-1)^k \frac{x^{6k+5}}{(2k+1)!} = x^5$

$P_0 = ?$ $P_1 = ?$, $P_2 = ?$, $P_3 = ?$

$P_4 = ?$.

$P_0 = P_1 = P_2 = P_3 = P_4 = 0$

$P_5(x) = x^5$ $P_{11}(x) = \sum_{k=0}^{1} (-1)^k \frac{x^{6k+5}}{(2k+1)!} =$

$P_{11}(x) = x^5 - \frac{x^{11}}{6}$

$P_6 = ?$ $P_7 = ?$ $P_8 = ?$ $P_9 = ?$ $P_{10} = ?$

$P_6 = P_7 = P_8 = P_9 = P_{10} = x^5$

P_m $6n+5 \leq m < 6(n+1)+5$

$\Rightarrow P_m = P_{6n+5}$

$$P_{6n+5}(x) = \sum_{k=0}^{n} (-1)^k \frac{x^{6k+5}}{(2k+1)!}$$

$$P_{6(n+1)+5}(x) = P_{6n+5}(x) + (-1)^{n+1} \frac{x^{6n+11}}{(2(n+1)+1)!}$$

Se $6n+5 \leq m < 6n+1 \Rightarrow P_m = P_{6n+5}$

$$P_{6n+5}(x) = \sum_{k=0}^{n} (-1)^k \frac{x^{6k+5}}{(2k+1)!}$$

Qual'è il valore $f(x) = x^2 \sin(x^3)$

$f^{(m)}(0) = ?$ $\forall m$

$$P_{6n+5}(x) = \sum_{k=0}^{n} (-1)^k \frac{x^{6k+5}}{(2k+1)!}$$

$$= \sum_{m=0}^{6n+5} \frac{f^{(m)}(0)}{m!} x^m$$

Se quando divido m per 6 ottengo un resto

$0 \leq r \leq 5$ con $r \neq 5$

$m = q6 + r$ allora $\Rightarrow f^{(m)}(0) = 0$

Se invece $m = k6 + 5$ allora

$$\frac{f^{(6k+5)}(0)}{(6k+5)!} = \frac{(-1)^k}{(2k+1)!} \Rightarrow f^{(6k+5)}(0) = \frac{(-1)^k (6k+5)!}{(2k+1)!}$$

Esempi Sia $x \neq 1$

$$\sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x} =$$

$$= \frac{1}{1-x} - \frac{x^{n+1}}{1-x}$$

$$\frac{1}{1-x} = \underbrace{\sum_{k=0}^{n} x^k}_{P_n(x)} + \underbrace{\frac{x^{n+1}}{1-x}}_{o(x^n)}$$

$$\lim_{x\to 0} \frac{\frac{x^{n+1}}{1-x}}{x^n} = \lim_{x\to 0} \frac{x^{\cancel{n}+1}}{\cancel{x^n}(1-x)} =$$

$$= \lim_{x\to 0} \frac{x}{1-x} = \frac{0}{1} = 0$$

pertanto $P_n(x)$ è il polinomio di Mc Laurin di ordine n di $f(x) = \frac{1}{1-x}$

$$f(x) = (1+x)^a$$

$$f(x) = \sum_{k=0}^{n} \binom{a}{k} x^k + o(x^n)$$

$$\binom{\frac{1}{2}}{2} = \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!} = \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2} = -\frac{\frac{1}{2}\cdot\frac{1}{2}}{2} = -\frac{1}{8}$$

$$\binom{a}{k} = \frac{\prod_{j=1}^{k}\left(a-(j-1)\right)}{k!}$$

$$(1+x)^{-1} = \sum_{k=0}^{n} \binom{-1}{k} x^k + o(x^n)$$

$$\frac{1}{1-x} = \sum_{k=0}^{n} x^k + o(x^n)$$

sostituendo x con $-x$

$$\frac{1}{1+x} = \sum_{k=0}^{n} (-x)^k + o((-x)^n)$$

$$= \sum_{k=0}^{n} (-1)^k x^k + \underbrace{o((-1)^n x^n)}_{o(x^n)}$$

$c \neq 0$

$$\lim_{x\to 0} \frac{o(cx^n)}{cx^n}\, c = 0$$

c

$o(cx^n) = o(x^n)$

$$\frac{1}{1+x} = \underbrace{\sum_{k=0}^{n} (-1)^k x^k}_{} + o(x^n)$$

questi sono i polinomi di McLaurin di ordine n di $\frac{1}{1+x}$

$$\frac{1}{1+x^2} = \sum_{k=0}^{n} (-1)^k x^{2k} + o(x^{2n})$$

$(x^2)^k$

$$\frac{1}{1-x^2} = \sum_{k=0}^{n} (-1)^k (-x^2)^k + o(x^{2n})$$

$$= \sum_{k=0}^{n} \cancel{(-1)^k} \cancel{(-1)^k} x^{2k} + o(x^{2n})$$

$$= \sum_{k=0}^{n} x^{2k} + o(x^{2n})$$

$$(1+x)^{\frac{1}{2}} = \sqrt{1+x} = \sum_{k=0}^{n} \binom{1/2}{k} x^k + o(x^n)$$

$$P_0 = 1 = \binom{1/2}{0}$$

$$P_1 = 1 + \binom{\frac{1}{2}}{1} x = 1 + \frac{1}{2} x$$

$$P_2 = P_1 + \binom{\frac{1}{2}}{2} x^2 = 1 + \frac{1}{2} x - \frac{1}{8} x^2$$

$$\sqrt{1+x^2} = \sum_{k=0}^{n} \binom{\frac{1}{2}}{k} x^{2k} + o(x^{2n})$$

$$\sqrt{1-x^2} = \sum_{k=0}^{n} \binom{\frac{1}{2}}{k} (-1)^k x^{2k} + o(x^{2n})$$