

DYNKIN DIAGRAMS

La matrice di Cartan può essere encodata nel corrispondente diagramma di Dynkin.

- $\alpha_i \leftrightarrow$ nodo (\forall simple root)
- $A_{ij} A_{ji} \in \{0, 1, 2, 3\} \leftrightarrow$ # linee che connettono i nodi i e j
- se $A_{ij} A_{ji} > 1$ lunghezza di α_i e α_j sono diverse
 $\text{length } \alpha_i > \text{length } \alpha_j \leftrightarrow$ freccia da α_i a α_j

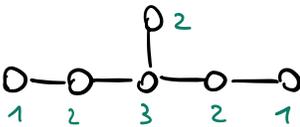
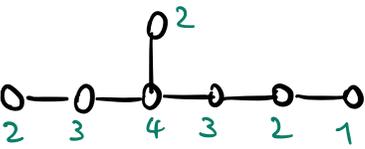
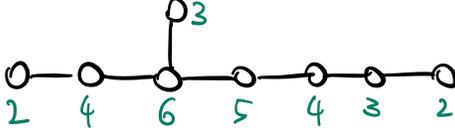
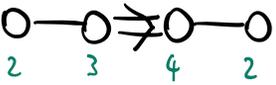
Si può CLASSIFICARE le alg. di Lie semplici, trovando quali sono i possibili diagrammi di Dynkin compatibili con le proprietà della matrice di Cartan.

$$A_r \quad \begin{array}{c} \text{O} - \text{O} - \dots - \text{O} \\ \underset{1}{\text{O}} \quad \underset{1}{\text{O}} \quad \dots \quad \underset{1}{\text{O}} \end{array} \quad r \geq 1 \quad \rightsquigarrow \quad \mathfrak{su}(r+1) \quad d = r(r+2)$$

$$B_r \quad \begin{array}{c} \text{O} - \text{O} - \dots - \text{O} \Rightarrow \text{O} \\ \underset{1}{\text{O}} \quad \underset{2}{\text{O}} \quad \dots \quad \underset{2}{\text{O}} \quad \underset{2}{\text{O}} \end{array} \quad r \geq 2 \quad \rightsquigarrow \quad \mathfrak{so}(2r+1) \quad d = r(2r+1)$$

$$C_r \quad \begin{array}{c} \text{O} - \text{O} - \dots - \text{O} \Leftarrow \text{O} \\ \underset{2}{\text{O}} \quad \underset{2}{\text{O}} \quad \dots \quad \underset{2}{\text{O}} \quad \underset{1}{\text{O}} \end{array} \quad r \geq 1 \quad \rightsquigarrow \quad \mathfrak{sp}(2r)$$

$$D_r \quad \begin{array}{c} \text{O} - \text{O} - \dots - \text{O} \\ \underset{1}{\text{O}} \quad \underset{2}{\text{O}} \quad \dots \quad \underset{2}{\text{O}} \quad \begin{array}{l} \text{O} \quad \underset{1}{\text{O}} \\ \text{O} \quad \underset{1}{\text{O}} \end{array} \end{array} \quad r \geq 2 \quad \rightsquigarrow \quad \mathfrak{so}(2r) \quad d = r(2r-1)$$

E_6		$d = 78$
E_7		$d = 133$
E_8		$d = 248$
F_4		$d = 52$
G_2		$d = 14$

Numeri sui nodi sono i Coxeter labels a_i .

Si possono notare isomorfismi tra le algebre:

$$A_1 \cong B_1 \quad (\mathfrak{su}(2) \cong \mathfrak{so}(3)) \quad , \text{ ma la normalizzazione \u00e8 diversa.}$$

$$A_1 \cong C_1 \quad (\mathfrak{su}(2) \cong \mathfrak{sp}(1)) \quad , \text{ con stessa normal.}$$

$$D_2 \cong A_1 \times A_1 \quad (\mathfrak{so}(4) \cong \mathfrak{su}(2) \times \mathfrak{su}(2))$$

$$D_3 \cong A_3 \quad (\mathfrak{so}(6) \cong \mathfrak{so}(4))$$

Dato Dynkin diagram (\Rightarrow Cartan matrix) one gets info about a basis of simple roots and which combinations of them are roots $\rightarrow \Delta$ reconstructed $\rightarrow \mathfrak{g}$ with its comm. rel.

ES. A_1 $\begin{matrix} \alpha_1 \\ 0 \end{matrix}$ $n\alpha_1$ non è root \Rightarrow roots sono

$$\rightarrow d_{A_1} = \underset{\substack{\uparrow \\ \Delta_+}}{1} + \underset{\substack{\uparrow \\ \Delta_-}}{1} + \underset{\substack{\uparrow \\ \mathcal{H}}}{1} = 3 \quad (= \dim SU(2))$$

$$\alpha_1 \in -\alpha_1$$

Generatori: $E^{\alpha_1}, E^{-\alpha_1}, [E^{-\alpha_1}, E^{\alpha_1}]$.

ES. A_2 $\begin{matrix} \alpha_1 & \alpha_2 \\ 0 & -0 \end{matrix}$ $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

$$-1 = 2 \frac{(\alpha_1, \alpha_2)}{|\alpha_1|^2} = -\rho_1 + \rho_2$$

$$-1 = 2 \frac{(\alpha_2, \alpha_1)}{|\alpha_2|^2} = -\rho_2 + \rho_1$$

Partiamo da α_1 , $\rho_2 = 0$ perché $\alpha_1 - \alpha_2$ non è root (α_1, α_2 sono sempl.)

$\Rightarrow \rho_1 = 1 \Rightarrow \alpha_1 + \alpha_2$ è root, ma $\alpha_1 + 2\alpha_2 \notin \Delta$

Stesse partendo da α_2

\rightarrow roots sono $\alpha_1, \alpha_2, \alpha_1 + \alpha_2, -\alpha_1, -\alpha_2, -\alpha_1 - \alpha_2$

$$\Rightarrow d_{A_2} = \underset{\substack{\uparrow \\ \Delta_+}}{3} + \underset{\substack{\uparrow \\ \Delta_-}}{3} + \underset{\substack{\uparrow \\ \mathcal{H}}}{2} = 8 \quad (= \dim SU(3))$$

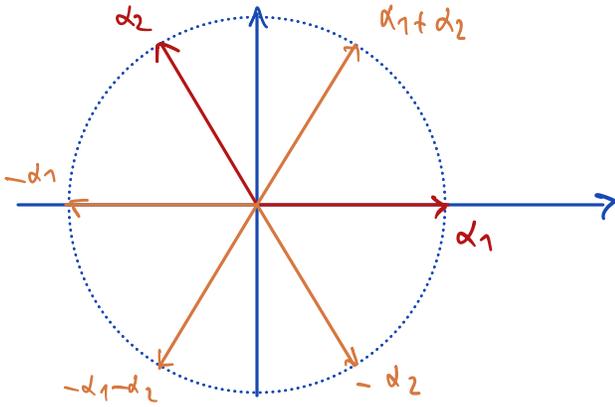
Generatori: $E^{\alpha_1}, E^{\alpha_2}, [E^{\alpha_1}, E^{\alpha_2}], E^{-\alpha_1}, E^{-\alpha_2}, [E^{-\alpha_1}, E^{-\alpha_2}], [E^{\alpha_1}, E^{\alpha_1}], [E^{\alpha_2}, E^{\alpha_2}]$

Root system of A_2 .

$$\cos \theta_{12} = -\frac{\sqrt{A_{12}A_{21}}}{2} = -\frac{1}{2} \rightarrow \theta_{12} = \frac{2\pi}{3}$$

$$\alpha_1 = \sqrt{2}(1, 0)$$

$$\alpha_2 = \sqrt{2}\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$



ES. G_2

$$\alpha_1 \neq 0$$

$$\alpha_2 \neq 0$$

$$A = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$$

$$\frac{|\alpha_2|^2}{|\alpha_1|^2} = \frac{A_{21}}{A_{12}} = \frac{1}{3}$$

$$-3 = A_{12} = 2 \frac{(\alpha_1, \alpha_2)}{|\alpha_2|^2} = -p_2 + q_2$$

$$-1 = A_{21} = 2 \frac{(\alpha_2, \alpha_1)}{|\alpha_1|^2} = -p_1 + q_1$$

$$A_{ij} = \frac{2(\alpha_i, \alpha_j)}{|\alpha_j|^2}$$

$$2 \frac{(\alpha_i, \alpha_j)}{|\alpha_j|^2} = -(p_i - q_i) \in \mathbb{Z}$$

(α_1, α_2 simple)

Parto da α_1 ; $p_2 - q_2 = 3$, ma se $\alpha_1 - \alpha_2$ non è una root $\Rightarrow q_2 = 0$

$\Rightarrow p_2 = 3 \Rightarrow \underline{\alpha_1 + \alpha_2}, \underline{\alpha_1 + 2\alpha_2}, \underline{\alpha_1 + 3\alpha_2}$ are roots

Parto da α_2 ; $p_1 - q_1 = 1$, $q_1 = 0$, ottengo ancora $\alpha_1 + \alpha_2$.

Parto da $\alpha_1 + \alpha_2$; $2 \frac{(\alpha_1 + \alpha_2, \alpha_1)}{|\alpha_1|^2} = 2 + (-1) = 1$, so che $q_1 = 1$
 $\Rightarrow p_1 = 0$

$\Rightarrow 2\alpha_1 + \alpha_2$ non è una root

Parto da $\alpha_1 + 2\alpha_2$; $2 \frac{(\alpha_1 + 2\alpha_2, \alpha_1)}{|\alpha_1|^2} = 2 + 2(-1) = 0$, so che $q_1 = 0 \Rightarrow p_1 = 0$

$$2 \frac{(\alpha_1 + 3\alpha_2, \alpha_1)}{|\alpha_1|^2} = 2 + 3(-1) = -1 = q_1 - p_1$$

$$\hookrightarrow q_1 = 0 \Rightarrow p_1 = 1$$

$\rightarrow \underline{2\alpha_1 + 3\alpha_2}$ è una root

Ho usato che se α è root, allora $n\alpha$ non è una root.

→ le roots sono $\alpha_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, \alpha_1 + 3\alpha_2, 2\alpha_1 + 3\alpha_2$ e le comp. $-\alpha$

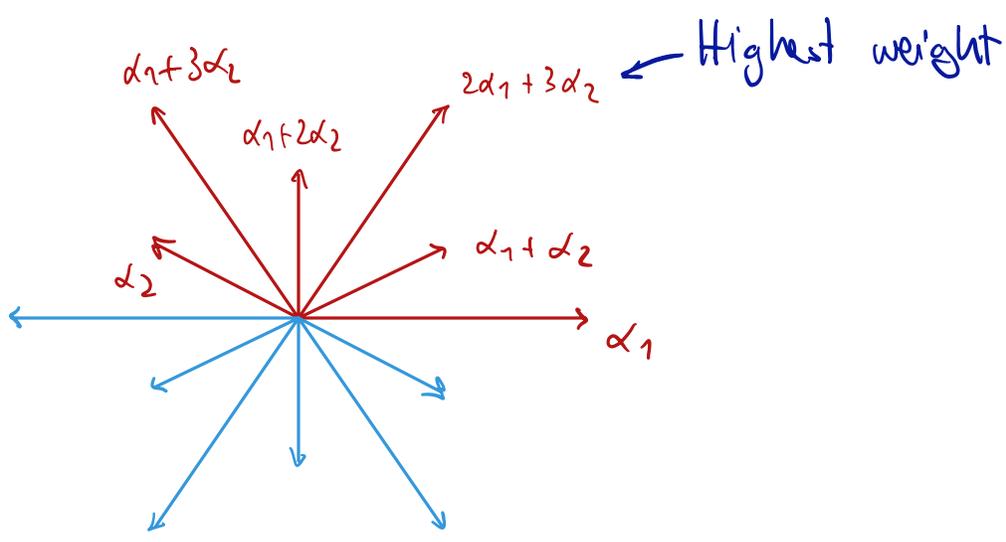
$\Rightarrow d_{G_2} = 6 + 6 + 2 = 14$

 \uparrow \uparrow \uparrow
 $\# \Delta_+$ $\# \Delta_-$ r

General.: $E^{\alpha_1} E^{\alpha_2} [E^{\alpha_1}, E^{\alpha_2}] [E^{\alpha_2}, [E^{\alpha_1}, E^{\alpha_2}]]$
 $[E^{\alpha_2}, [E^{\alpha_2}, [E^{\alpha_1}, E^{\alpha_2}]]] [E^{\alpha_1}, [E^{\alpha_1}, [E^{\alpha_2}, [E^{\alpha_1}, E^{\alpha_2}]]]]$
 e stesso con $E^{-\alpha_1}, E^{-\alpha_2}, \dots$
 $[E^{-\alpha_1}, E^{\alpha_1}]$ e $[E^{-\alpha_2}, E^{\alpha_2}] \in \mathcal{H}$

Root system of G_2 :

$$\begin{cases} \frac{|\alpha_1|^2}{|\alpha_2|^2} = \frac{A_{12}}{A_{21}} = 3 \\ \cos \theta_{12} = -\frac{\sqrt{A_{12} A_{21}}}{2} = -\frac{\sqrt{3}}{2} \end{cases}$$



Roots: $\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, \alpha_1 + 3\alpha_2, 2\alpha_1 + 3\alpha_2$
 $-\alpha_1, -\alpha_1 - \alpha_2, -\alpha_1 - 2\alpha_2, -\alpha_1 - 3\alpha_2, -2\alpha_1 - 3\alpha_2$

CLASSIFICAZIONE SIMPLE LIE ALGEBRAS

- Prodotto di Killing permette di dare a $\mathcal{H}_{\mathbb{R}}^*$ la struttura di uno spazio Euclideo.

• Def. Sia E uno sp. euclideo di $\dim. = r$.

Un sottos. $\{\alpha_1, \dots, \alpha_r\} \subset E$ è chiamato SCHEMA se

a) $\forall \alpha_i, \alpha_j \in S \quad \alpha_i \neq \alpha_j \rightarrow A_{ij} = \frac{2(\alpha_j, \alpha_i)}{(\alpha_i, \alpha_i)} \in \{0, -1, -2, \dots\}$

b) $\det A \neq 0 \quad (\Rightarrow \alpha_1, \dots, \alpha_r \text{ are lin. indep.})$

r è chiamato il RANGO dello schema.

- Un sistema di SIMPLE ROOTS è uno schema con $n_{ij} = A_{ij}A_{ji} \in \{0, 1, 2, 3\} \quad i \neq j$.

- Uno schema è rappresentato da un Dynkin diagram.

- Se $r \cdot \kappa_S = r \Rightarrow$ il numero di links p di S è minore di r .

Dim. $\xi \equiv \sum_{i=1}^r \frac{\alpha_i}{(\alpha_i, \alpha_i)^{1/2}} \neq 0 \Rightarrow 0 < (\xi, \xi) = r + 2 \sum_{i < j} \cos \theta_{ij} \Rightarrow$

$\Rightarrow r > \sum_{i < j} -2 \cos \theta_{ij} \geq \sum_{i < j} 1 = p \quad \text{for each linked pair} \quad = p. //$

- Gli schemi non contengono circuiti chiusi!

Dim. $S' = \{\alpha_1, \dots, \alpha_m\}$ sia un circuito chiuso in $S \Rightarrow$

$\Rightarrow p' \geq m$; ma esso sarebbe un sottoschema e quindi $p' < m. //$

• Sia S' un sottoschema connesso di uno schema $S \neq S'$.

Allora ogni $\alpha \in S$, ma $\alpha \notin S'$, è connesso al massimo con un elemento di S' .

• Il numero di linee che escono da un nodo è al massimo 3.

Dim. $\alpha_i \in S$; siano $\{\beta_1, \dots, \beta_m\} \in S \setminus \{\alpha_i\}$ siano i nodi connessi ad $\alpha_i \Rightarrow (\beta_j, \alpha_i) \neq 0$.

Il numero di linee connessi ad α_i è $N = \sum_{j=1}^m n_{ij}$
 con $n_{ij} = \frac{4(\alpha_i, \beta_j)(\alpha_i, \beta_j)}{(\alpha_i, \alpha_i)(\beta_j, \beta_j)} = 4 \frac{(\gamma_j, \alpha_i)^2}{(\alpha_i, \alpha_i)}$ $\gamma_j \equiv \beta_j / (\alpha_i, \alpha_i)^{1/2}$

Inoltre $(\gamma_i, \gamma_j) = 0$ perché non ci sono circuiti chiusi.

$\{\alpha_i, \beta_1, \dots, \beta_m\}$ sono lin. indep. $\Rightarrow \exists$ unit vect. $\gamma_0 \in \langle \alpha_i, \beta_1, \dots, \beta_m \rangle$

che è $\perp \beta_j \forall j=1, \dots, m$, ma $(\alpha_i, \gamma_0) \neq 0$.

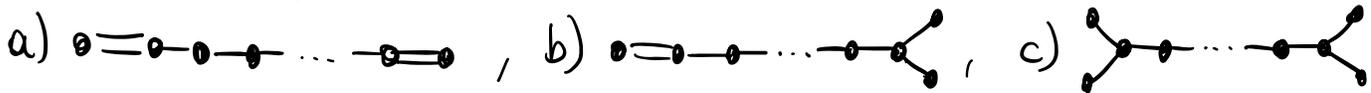
$\Rightarrow \{\gamma_0, \gamma_1, \dots, \gamma_m\}$ è un sist. o.n. \Rightarrow

$$\Rightarrow \begin{cases} \alpha_i = \sum_{j=0}^m (\alpha_i, \gamma_j) \gamma_j \\ \|\alpha_i\|^2 = \sum_{j=0}^m (\alpha_i, \gamma_j)^2 \end{cases} \Rightarrow N = 4 \underbrace{\sum_{j=0}^m \frac{(\alpha_i, \gamma_j)^2}{(\alpha_i, \alpha_i)}}_{=1} - 4 \frac{(\alpha_i, \gamma_0)^2}{(\alpha_i, \alpha_i)} < 4 //$$

• Se $r \geq 3$ allora il Diagramma non contiene tre linee fra due nodi.

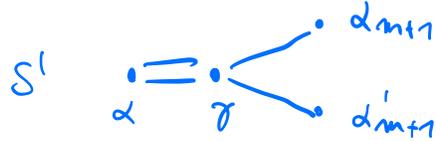
Dim. Se ho $\bullet \equiv \bullet$, non posso più attaccarci una linea a un nodo, altrimenti viene 4 linee da pl. nodi.

- Il diagramma di Dynkin di uno schema NON contiene un sottodiagramma del tipo

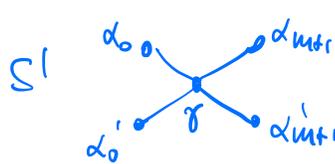
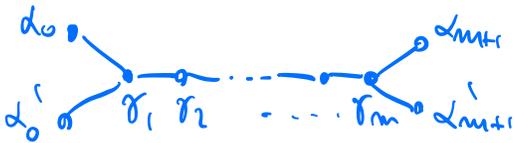


Dim $\alpha_0 = \alpha_1 = \dots = \alpha_m = \alpha_{m+1}$ S' $\alpha_0 = \alpha_1 = \alpha_{m+1}$ ma γ non può essere 4 linee se S' è schema //

$$\frac{2(\alpha_0, \gamma)}{(\alpha_0, \alpha_0)} \frac{2(\alpha_0, \gamma)}{(\gamma, \gamma)} = 4 \frac{(\alpha_0, \alpha_1)}{(\alpha_0, \alpha_0)} \frac{(\alpha_0, \alpha_1)}{(\alpha_1, \alpha_1)} \frac{(\alpha_1, \alpha_1)}{(\gamma, \gamma)} = 2$$

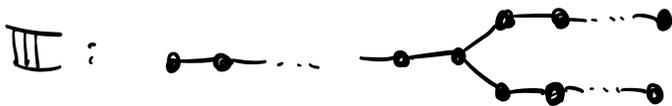


ancora non ammissibile.



ancora non ammissibile. //

- I possibili diagrammi, secondo quanto detto finora, sono



- La classe I non ha ulteriori vincoli. Esso definisce una Cartan Matrix, che produce un'Algebra di Lie che chiameremo A_r .

$\{\hat{\alpha}_1, \dots, \hat{\alpha}_m\}$ cat. ortog. con $(\hat{\alpha}_i, \hat{\alpha}_i) = a$, $\hat{\gamma} = \sum_{j=1}^m \hat{\alpha}_j \Rightarrow (\hat{\gamma}, \hat{\gamma}) = a \frac{m(m+1)}{2}$

Dim $(\hat{\gamma}, \hat{\gamma}) = \sum_{j=1}^m j \cdot j \cdot a = \sum_{j=1}^m j^2 a = \sum_{j=1}^m j(j+1)a = m^2 a - a \sum_{j=1}^{m-1} j = a(m^2 - \frac{m^2 - m}{2}) = a \frac{m(m+1)}{2} //$

• I diagrammi di classe II che corrispondono a schemi sono



Dim. $\alpha_1 \alpha_2 \dots \alpha_p \alpha_{p+1} \alpha_{p+2} \dots \alpha_{p+q}$ $\leftarrow \alpha_{p+1} = \beta_1, \alpha_{p+2} = \beta_{-1}, \dots, \alpha_{p+q} = \beta_1$

$\alpha_{p+1} = \beta_1, \alpha_{p+2} = \beta_{-1}, \dots, \alpha_{p+q} = \beta_1$

$0 \quad A_{p+1,p+1} = -2 \quad \text{e} \quad A_{p+1,p} = -1$

$n_{p,p+1} = A_{p,p+1} \quad A_{p+1,p} = 2 \Rightarrow$

oppure $A_{p+1,p+1} = -1 \quad \text{e} \quad A_{p+1,p} = -2$

$$\Rightarrow \frac{(\alpha_p, \alpha_p)}{(\alpha_{p+1}, \alpha_{p+1})} = \frac{A_{p+1,p}}{A_{p,p+1}} = \frac{1}{2} \quad \text{o} \quad 2$$

Prendiamo $1/2$ (senza perdere generalità visto simm. del diagramma)

cioè $(\beta_q, \beta_q) \equiv (\alpha_{p+1}, \alpha_{p+1}) = 2(\alpha_p, \alpha_p) \equiv 2a$

Definiamo $\alpha \equiv \sum_{i=1}^p i \alpha_i \quad \beta \equiv \sum_{j=1}^q j \beta_j$

$\rightarrow (\alpha, \alpha) = \frac{1}{2} a p(p+1) \quad (\beta, \beta) = a q(q+1)$

$(\alpha, \beta) = pq (\alpha_p, \beta_q) = pq \frac{2(\alpha_p, \alpha_{p+1})}{(\alpha_p, \alpha_p)} \frac{(\alpha_p, \alpha_p)}{2} = pq A_{p,p+1} \frac{a}{2}$

$= -p \cdot q a$

α e β sono lin. indep. $\Rightarrow (\alpha, \beta)^2 < (\alpha, \alpha)(\beta, \beta) \Rightarrow$

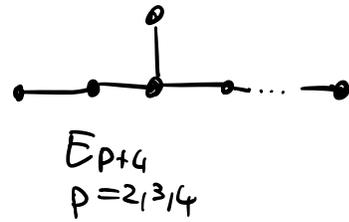
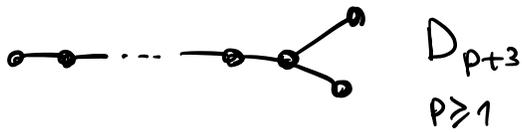
$\Rightarrow p^2 q^2 a^2 < \frac{1}{2} a^2 p(p+1) q(q+1) \Rightarrow 2pq < (p+1)(q+1) \Rightarrow$

$\Rightarrow 2pq - p - q - 1 < 2 \Rightarrow (p-1)(q-1) < 2 \Rightarrow$

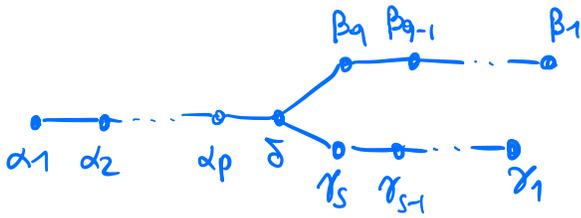
$\Rightarrow \begin{cases} p=1, q=1, 2, \dots & B_{q+1} \\ q=1, p=1, 2, \dots & C_{p+1} \\ p=2, q=2 & F_4 \end{cases}$

$B_2 = C_2$

• I diagrammi di classe III che corrispondono a schemi sono



Dim.
m.



$$\gamma \equiv \sum_{k=1}^s k \gamma_k$$

$$(\alpha_i, \beta_j) = (\beta_j, \gamma_k) = (\gamma_k, \alpha_i) = 0 \Rightarrow (\alpha, \beta) = (\beta, \gamma) = (\gamma, \alpha) = 0$$

$\{\alpha_i, \beta_j, \gamma_k, \delta\}$ sono lin. indep. $\Rightarrow \delta$ indep. da ogni comb. lin. di α, β, γ . (*)

Prendiamo $\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta}$ paralleli a $\alpha, \beta, \gamma, \delta$, ma di norma = 1.

$$\hat{\delta} = \hat{\delta}_{\parallel} + \hat{\delta}_{\perp} \neq 0 \quad \text{dove } \hat{\delta}_{\parallel} \in \langle \alpha, \beta, \gamma \rangle \quad \text{e } \hat{\delta}_{\perp} \in \langle \alpha, \beta, \gamma \rangle^{\perp}$$

$$\hookrightarrow \hat{\delta}_{\parallel} = \hat{\delta} \cdot \hat{\alpha} \cdot \hat{\alpha} + \hat{\delta} \cdot \hat{\beta} \cdot \hat{\beta} + \hat{\delta} \cdot \hat{\gamma} \cdot \hat{\gamma} \quad \text{perché } \hat{\alpha}, \hat{\beta}, \hat{\gamma} \text{ è base o.n. in } \langle \alpha, \beta, \gamma \rangle$$

$$1 = \|\hat{\delta}\|^2 = \|\hat{\delta}_{\parallel}\|^2 + \|\hat{\delta}_{\perp}\|^2 > \|\hat{\delta}_{\parallel}\|^2 = (\hat{\delta} \cdot \hat{\alpha})^2 + (\hat{\delta} \cdot \hat{\beta})^2 + (\hat{\delta} \cdot \hat{\gamma})^2$$

$$\text{dove } \hat{\delta} \cdot \hat{\alpha} = \cos \theta_{\alpha\delta} \quad \hat{\delta} \cdot \hat{\beta} = \cos \theta_{\beta\delta} \quad \hat{\delta} \cdot \hat{\gamma} = \cos \theta_{\gamma\delta}$$

$$\Rightarrow \cos^2 \theta_{\alpha\delta} + \cos^2 \theta_{\beta\delta} + \cos^2 \theta_{\gamma\delta} < 1 \quad (*)$$

$$\text{Abbiamo anche } (\alpha_i, \alpha_i) = (\beta_j, \beta_j) = (\gamma_k, \gamma_k) = (\delta, \delta) = a \quad \text{e}$$

$$(\alpha, \delta) = p(\alpha_p, \delta) = \frac{1}{2} p \frac{2(\alpha_p, \delta)}{(\delta, \delta)} (\delta, \delta) = -\frac{pa}{2}, \quad (\beta, \delta) = -\frac{qa}{2}, \quad (\gamma, \delta) = -\frac{sa}{2}$$

$$\Rightarrow \cos^2 \theta_{\alpha\delta} = \frac{(\alpha, \delta)^2}{(\alpha, \alpha)(\delta, \delta)} = \frac{\frac{p^2 a^2}{4}}{2 \frac{a^2}{p(p+1)}} \frac{1}{a} = \frac{1}{2} \frac{p}{p+1} = \frac{1}{2} \left(1 - \frac{1}{p+1}\right)$$

$$\cos^2 \theta_{\beta\delta} = \frac{1}{2} \left(1 - \frac{1}{q+1}\right) \quad \cos^2 \theta_{\gamma\delta} = \frac{1}{2} \left(1 - \frac{1}{s+1}\right) \quad (x)$$

$$(*), (x) \Rightarrow \frac{1}{p+1} + \frac{1}{q+1} + \frac{1}{s+1} > 1$$

Se prendiamo $p \geq q \geq s$ allora $\frac{1}{p+1} \leq \frac{1}{q+1} \leq \frac{1}{s+1}$

$$\text{Quindi } \frac{3}{s+1} \geq \frac{1}{p+1} + \frac{1}{q+1} + \frac{1}{s+1} > 1 \Rightarrow s=1$$

$$\Rightarrow \frac{1}{p+1} + \frac{1}{q+1} > \frac{1}{2} \text{ e quindi } \frac{2}{1+q} > \frac{1}{2} \Rightarrow q=1,2$$

$$\Rightarrow \frac{1}{p+1} > \begin{cases} 0 & \text{se } q=1 \rightarrow (p,q,s) = (p,1,1) \quad p \geq 1 \quad D_{p+3} \\ \frac{1}{6} & \text{se } q=2 \end{cases}$$

\downarrow
 $2=q \leq p < 5$

$$(p,q,s) = \begin{cases} (2,2,1) & E_6 \\ (3,2,1) & E_7 \\ (4,2,1) & E_8 \end{cases} //$$