11 Nov.

Esempio Polinomi di $\lg(1+x)$

$$\left(\lg(1+x)\right)' = \lg'(1+x) \quad (1+x)'$$

$$= \frac{1}{1+x}$$

$$\left(\lg(1+x)\right)^{(n)} = \left(\left(\lg(1+x)\right)'\right)^{(n-1)} =$$

$$\left(\frac{d}{dx}\right)^n f = \left(\frac{d}{dx}\right)^a \left(\frac{d}{dx}\right)^b f \qquad n = a+b$$

$$= \left((1+x)^{-1}\right)^{(n-1)} = \prod_{j=1}^{n-1} (-\cancel{1}-j+\cancel{1}) (1+x)^{-\cancel{1}-(n-\cancel{1})}$$

$$= \prod_{j=1}^{n-1} (-j) \quad (1+x)^{-n}$$

$$= (-1)^{n-1} \prod_{j=1}^{n-1} j \quad (1+x)^{-n}$$

$$= (-1)^{n-1} (n-1)! \; (1+x)^{-n} \Big|_{x=0} = (-1)^{n-1} (n-1)!$$

$$\left(\lg(1+x)\right)^{(n)} \Big|_{x=0} = (-1)^{n-1} (n-1)!$$

$$P_n(x) = \sum_{j=0}^{n} \frac{\left(\lg(1+x)\right)^{(j)}\Big|_{x=0}}{j!} \; x^j$$

$$= \sum_{j=1}^{n} \frac{(-1)^{j-1} \cancel{(j-1)!}}{j \; \cancel{(j-1)!}} \; x^j \qquad j! = (j-1)!\, j$$

$$= \sum_{j=1}^{n} (-1)^{j-1} \frac{x^j}{j} = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n-1}\frac{x^n}{n}$$

Formula di Lagrange

Teorema Sia $f : (a,b) \to \mathbb{R}$

$x_0 \in (a,b)$ e supponiamo che $f^{(0)}, \dots, f^{(n+1)}$ siano definite in (a,b). Allora si ha

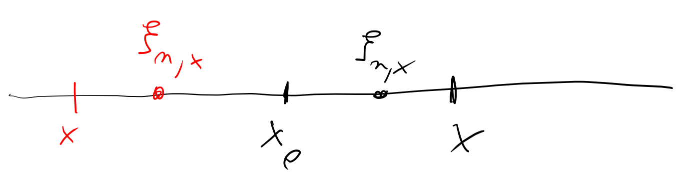
$$f(x) = P_n(x) + R_n(x) \quad \text{dove}$$

$$P_n(x) = \sum_{j=0}^{n} \frac{f^{(j)}(x_0)}{j!} (x-x_0)^j$$

dove $R_n(x_0) = 0$ e se $x \neq x_0$ vale la seguente formula:

$$R_n(x) = \frac{f^{(n+1)}(\xi_{n,x})}{(n+1)!} (x-x_0)^{n+1}$$

(x, ξ) dove $\xi_{n,x}$ è un punto nell'intervallo aperto di estremi x_0 e x



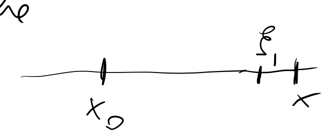
Osservazione Nel caso $n=0$

$$\boxed{f(x) = P_0(x) + f^{(1)}(\xi)(x-x_0)}$$

$$f(x) = f(x_0) + f'(\xi)(x-x_0) \qquad x \neq x_0$$

$$f'(\xi) = \frac{f(x) - f(x_0)}{x - x_0}$$

Dim $x_0 < x$ Vogliamo dimostrare che



$$R_n(x) = \frac{f^{(n+1)}(\xi_n)}{(n+1)!}(x-x_0)^{n+1}$$

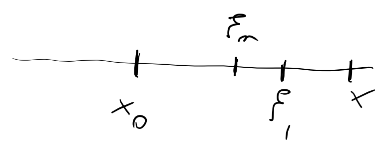
$$\Updownarrow$$

$$\frac{R_n(x)}{(x-x_0)^{n+1}} = \frac{f^{(n+1)}(\xi_n)}{(n+1)!}$$

$$\frac{R_n(x) - \overbrace{R_n(x_0)}^{0}}{(x-x_0)^{n+1} - \underbrace{(x_0-x_0)^{n+1}}_{0}}$$

$$\left(\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}\right)$$

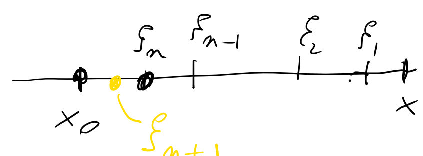
per Cauchy
$$= \frac{R_n'(\xi_1) - \overbrace{R_n'(x_0)}^{0}}{(n+1)(\xi_1-x_0)^n - (n+1)(x_0-x_0)^n}$$



$$R_n^{(k)}(x_0) = 0 \qquad k \le n$$

$$= \frac{R_n''(\xi_2)}{(n+1)n(\xi_2-x_0)^{n-1}} = \cdots = \frac{R_n^{(n)}(\xi_n)}{\underbrace{(n+1)\cdots 2}_{(n+1)!}(\xi_n-x_0)^{\cancel{n}+1-\cancel{n}}}$$

$$= \frac{R_n^{(n)}(\xi_n) - \overbrace{R_n^{(n)}(x_0)}^{0}}{(n+1)!\,(\xi_n-x_0)} =$$



$$= \frac{1}{(n+1)!}\,\frac{R_n^{(n)}(\xi_n) - R_n^{(n)}(x_0)}{\xi_n - x_0} = \frac{1}{(n+1)!}\,R_n^{(n+1)}(\xi_{n+1})$$

$$\|$$

$$\frac{R_n(x)}{(x-x_0)^{n+1}}$$

$$R_n(x) = \frac{R_n^{(n+1)}(\xi_{n+1})}{(n+1)!}(x-x_0)^{n+1}$$

Esercizio   Il numero e è irrazionale.

Approssimarlo con un numero razionale

facendo un errore $< \frac{1}{10^3}$.

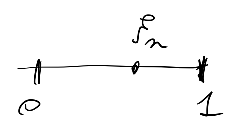
Consideriamo il polinomio di McLaurin di ordine n $P_n(x)$ di e^x

$$P_n(x) = \sum_{j=0}^{n} \frac{x^j}{j!}$$

$$P_n(1) = \sum_{j=0}^{n} \frac{1}{j!} \in \mathbb{Q}$$

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$$

$$e = P_n(1) + R_n(\xi_n)$$



$$0 < R_n(\xi_n) = \frac{e^{\xi_n}}{(n+1)!} < \frac{e}{(n+1)!} < \frac{3}{(n+1)!} < 10^{-3}$$

$$\Longleftrightarrow \quad (n+1)! > 3000$$

n	$n+1$	$(n+1)!$
1	2	$2! = 2$
2	3	$3! = 6$
3	4	24
4	5	120
5	6	720
6	7	$5040 > 3000$

$P_6(1)$

Esercizio e è irrazionale.

Supponiamo per assurdo che e sia razionale

$$e = \frac{a}{b} \qquad a, b \in \mathbb{N}$$

$$e = e^1 = P_n(1) + R_n(1) = P_n(1) + \frac{e^{\xi_n}}{(n+1)!}$$

dove $0 < \xi_n < 1$.

$$\frac{a}{b} = P_n(1) + \frac{e^{\xi_n}}{(n+1)!}$$

$$P_n(1) = \sum_{k=0}^{n} \frac{1}{k!}$$

è vero $\forall n \in \mathbb{N}$

$$n!\, P_n(1) = \sum_{k=0}^{n} \frac{n!}{k!} \in \mathbb{N} \qquad n! = k!\,(k+1)\dots n$$

$$n!\,\frac{a}{b} = n!\, P_n(1) + \frac{n!}{(n+1)!}\, e^{\xi_n}$$

$\mathbb{N}$? ove scelgo $n \geq b \Rightarrow n!\,\frac{a}{b} \in \mathbb{N}$

$$0 < \frac{e^{\xi_n}}{n+1} = \underbrace{n!\,\frac{a}{b}}_{\in\mathbb{N}} - \underbrace{n!\,P_n(1)}_{\in\mathbb{N}} \in \mathbb{N} = \{1, \dots\}$$

≥ 1 $\forall n$

$\frac{e^{\xi_n}}{n+1} \xrightarrow{n \to +\infty} 0$

$0 < \xi_n < 1$

$$\Rightarrow 0 < \frac{e^{\xi_n}}{n+1} < \frac{e}{n+1}$$

$\frac{e^{\xi_n}}{n+1} \to 0$, $\frac{e}{n+1} \to 0$

Ho due cose

$$\frac{e^{\xi_n}}{n+1} \geq 1 \quad \text{e} \quad \frac{e^{\xi_n}}{n+1} \xrightarrow{n \to +\infty} 0$$

$$\Rightarrow 0 \geq 1 \qquad \text{assurdo.}$$

Es 4 13/1/2025.

P_4 $\qquad f(x) = \lg(1+\sin x)$

$$P_4(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(IV)}(0)}{4!}x^4.$$

$$\lg(1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + o(y^4) =$$

$$\lg(1+\sin x) = \sin x - \frac{\sin^2 x}{2} + \frac{\sin^3 x}{3} - \frac{\sin^4 x}{4} + \underbrace{o(\sin^4 x)}_{o(x^4)}$$

Notare che $o(\sin^4(x)) = o(x^4)$

$$\lim_{x\to 0}\frac{o(\sin^4(x))}{x^4} = \lim_{x\to 0}\underbrace{\frac{o(\sin^4 x)}{\sin^4(x)}}_{\to 0}\underbrace{\left(\frac{\sin x}{x}\right)^4}_{\to 1} = 0$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + o(x^5) = x - \frac{x^3}{3!} + o(x^4)$$

$$= \left(x - \frac{x^3}{3!}\right) - \frac{\left(x - \frac{x^3}{3!}\right)^2}{2} + \frac{\left(x - \frac{x^3}{3!}\right)^3}{3} - \frac{\left(x - \frac{x^3}{3!}\right)^4}{4} + o(x^4)$$

$$= x - \frac{x^3}{6} - \frac{x^2 - \frac{1}{3}x^4}{2} + \frac{x^3}{3} - \frac{x^4}{4} + o(x^4)$$

$$= x - \frac{1}{2}x^2 + x^3\left(\frac{1}{3} - \frac{1}{6}\right) + \frac{x^4}{2}\left(\frac{1}{3} - \frac{1}{2}\right) + o(x^4)$$

$$= x - \frac{1}{2}x^2 + \frac{x^3}{6} + \frac{x^4}{2}\,\frac{2-3}{6} + o(x^4)$$

$$= x - \frac{1}{2}x^2 + \frac{x^3}{6} - \frac{x^4}{12} + o(x^4)$$