13 November

E Banach

$(E, \sigma(E,E'))$

Last time we showed that if

V is a neigh of $x_0 \in E$ for the weak top

$V \supseteq V(x_0, f_1, \dots, f_m, \varepsilon) = \{x \in E : |f_j(x-x_0)| < \varepsilon \ \forall j = 1 \dots m\}$, $\varepsilon > 0$

If $\dim E = +\infty \Rightarrow V$ contains a line

Indeed $f_1, \dots, f_n$

$\bigcap_{j=1}^{n} \ker f_j \neq 0$

$\vec{f} = (f_1, \dots, f_n) : E \longrightarrow K^n = \mathbb{R}^n$

and we need to have $\ker \vec{f} \neq 0$

because otherwise if $\ker \vec{f} = 0$

it would follow $\vec{f} : E \to R(\vec{f})$

would be bijective and would be an algebraic

isomorphism $+\infty = \dim E = \dim R(\vec{f}) \leq \dim \mathbb{R}^n = n$

$+\infty \leq n$ contradiction

$\Rightarrow \ker \vec{f} \neq 0 \quad \exists \ \underset{\in E}{x_1} \neq 0$ s.t.

$\vec{f}(x_0) = (\overset{= 0}{f_1(x_1)}, \dots, \overset{= 0}{f_n(x_1)}) = (0, \dots, 0) \in \mathbb{R}^n$

so we found $\bigcap_{j=1}^{n} \ker f_j \ni x_1 \neq 0 \Rightarrow$ contains the line $\mathbb{R} x_1$

$\Rightarrow \ \forall \ x = x_0 + t x_1 \quad \forall \ t \in \mathbb{R}$

$f_j(x - x_0) = f_j(t x_1) = 0 \quad \forall j$

$\Rightarrow V(x_0, f_1, \dots, f_n) = \{x \in E : |f_j(x - x_0)| < \varepsilon \ \forall j = 1 \dots m\}$

$x_0 + t x_1 \in V(x_0, f_1, \dots, f_n) \quad \forall t \in \mathbb{R}$

Corollary $(E, \sigma(E,E'))$ if $\dim E = +\infty$

is not metrizable.

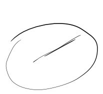
Pf Suppose by contradiction that there exists

a metric d inducing the above topology,

We can consider $U_n = \{x : d(x,0) < \frac{1}{n}\}$

These disks contain lines and in particular

$\forall n \ \exists \ x_n \in U_n$ s.t. $\|x_n\|_E = n$.



$d(x_n, 0) < \frac{1}{n} \xrightarrow{n \to +\infty} 0$

$\Rightarrow x_n \to 0$ for the $(E, \sigma(E,E'))$ topology

but here $\lim_{n \to +\infty} \|x_n\|_E = \lim_{n \to +\infty} n = +\infty$

but this contradicts a result discussed

last time. $(x_n \rightharpoonup x_0 \Rightarrow \sup \|x_n\|_E < +\infty)$

Last time we showed that

$\dim E = +\infty$ $\quad$ E Banach

then $\overline{\partial D_E(0,1)}\Big|_{\sigma(E,E')} = \overline{D_E(0,1)} =$

$= \{x : \|x\|_E \leq 1\}$

Let us see some examples

$1 \le p < \infty$

$$\ell^p(\mathbb{Z}) = \{ f : \mathbb{Z} \to \mathbb{R} \text{ s.t. } \sum_{n\in\mathbb{Z}} |f(n)|^p < +\infty$$

$1 < p < +\infty$

$f \in \ell^p(\mathbb{Z})$

$\tau_n f(k) = f(k-n)$

Let us show that $\|\tau_n f\|^p_{\ell^p(\mathbb{Z})} = \|f\|^p_{\ell^p(\mathbb{Z})}$

and that $\tau_n f \xrightarrow{n\to+\infty} 0$

$$\|\tau_n f\|^p_{\ell^p(\mathbb{Z})} = \sum_{k\in\mathbb{Z}} |(\tau_n f)(k)|^p = \sum_{k\in\mathbb{Z}} |f(k-n)|^p$$

$$= \sum_{k\in\mathbb{Z}} |f(k)|^p = \|f\|^p_{\ell^p(\mathbb{Z})}$$

$1 < p < +\infty$ $\qquad p' = \frac{p}{p-1} \in (1, +\infty)$

$\tau_n f \to 0 \qquad \sigma(\ell^p(\mathbb{Z}), (\ell^p(\mathbb{Z}))') =$

$f \in \ell^p(\mathbb{Z}),\ g \in \ell^{p'}(\mathbb{Z}) \quad \sigma(\ell^p(\mathbb{Z}), \ell^{p'}(\mathbb{Z}))$

$$\langle f, g\rangle_{\ell^p \times \ell^{p'}} = \sum_{k=-\infty}^{+\infty} f(k) g(k) =$$

$$= \lim_{N\to+\infty} \sum_{|k|\le N} f(k) g(k)$$

$|\langle f, g\rangle_{\ell^p\times\ell^{p'}}| \le \|f\|_{\ell^p} \|g\|_{\ell^{p'}}$ Hölder inequality.

$\tau_n f \to 0$ in $\ell^p(\mathbb{Z})$ for $\sigma(\ell^p(\mathbb{Z}), \ell^{p'}(\mathbb{Z}))$

$$\langle \tau_n f, g \rangle \xrightarrow{n\to\infty} 0$$

$\forall g \in \ell^{p'}(\mathbb{Z})$. ~~We proved~~

In general $\forall f \in \ell^p(\mathbb{Z})$ and $g \in \ell^{p'}(\mathbb{Z})$ $1 < p < +\infty$ we have

$$\lim_{n\to+\infty} \langle \tau_n f, g \rangle = 0$$

Case 1 Suppose supp f and supp g are compact $f, g : \mathbb{Z} \to \mathbb{R}$

$\exists$ $M > 0$ s.t. for $|k| \geq M \Rightarrow$

$f(k) = 0$ and $g(k) = 0$

$$\langle \tau_n f, g \rangle = \sum_{k=-\infty}^{+\infty} f(k-n)\, g(k)$$

Let $|n| > 2M$.

$f(k-n)\, g(k) = 0$ if $|k| \geq M$

If $|k| < M \Rightarrow |k-n| \geq |n| - |k| \geq$

$\geq |n| - M > 2M - M = M$

so $|k| < M \Rightarrow |k-n| > M \Rightarrow f(k-n) = 0$

Conclusion: $f(k-n)\, g(k) = 0 \quad \forall k$

So we have shown that if f and g have compact support then $\langle \tau_n f, g \rangle = 0$ for $n \gg 1 \Rightarrow \lim_{n\to+\infty} \langle \tau_n f, g \rangle = 0$

Now we will show that

$\lim_{n\to+\infty} \langle \tau_n f, g\rangle = 0$ $\quad \forall f \in \ell^p(\mathbb{Z})$

$\forall g \in \ell^{p'}(\mathbb{Z})$

$1 < p < +\infty$

$c_c^0(\mathbb{Z}) = \{ f : \mathbb{Z} \to \mathbb{R} : \text{supp} f \text{ is compact}\}$

$c_c^0(\mathbb{Z}) \subseteq \ell^p(\mathbb{Z})$

$c_c^0(\mathbb{Z}) \subseteq \ell^{p'}(\mathbb{Z})$ and in both cases

$c_c^0(\mathbb{Z})$ is ~~dense~~ dense in these spaces.

$f \in \ell^p(\mathbb{Z})$

$1_{[-N,N]} f \in c_c^0(\mathbb{Z})$

$\lim_{N\to+\infty} 1_{[-N,N]} f = f$ in $\ell^p(\mathbb{Z})$

$\| f - 1_{[-N,N]} f \|^p_{\ell^p(\mathbb{Z})} \xrightarrow{N\to+\infty} 0$

$= \| 1_{\mathbb{Z}\setminus[-N,N]} f \|^p_{\ell^p(\mathbb{Z})} =$

$= \| 1_{\mathbb{Z}\setminus[-N,N]} |f|^p \|_{\ell^1(\mathbb{Z})}$

$= \lim_{N\to+\infty} \int_{\mathbb{Z}} 1_{\mathbb{Z}\setminus[-N,N]}(k) |f(k)|^p \, d\mu(k) = 0$

$\le |f(k)|^p$

$\downarrow N \to +\infty$

0

dominated convergence.

let $\varepsilon > 0$

$\frac{\varepsilon}{3}$

and let $\tilde f$ and $\tilde g \in c_c^0(\mathbb{Z})$ st

$\|\tilde f - f\|_{\ell^p(\mathbb{Z})} < \frac{\varepsilon}{3}$ and $\|\tilde g - g\|_{\ell^{p'}(\mathbb{Z})} < \frac{\varepsilon}{3}$

$\langle \tau_n f, g\rangle = \langle \tau_n(f - \tilde f + \tilde f), g - \tilde g + \tilde g\rangle =$

$= \langle \tau_n(f-\tilde f), g\rangle + \langle \tau_n(f-\tilde f), \tilde g\rangle + \langle \tau_n \tilde f, g - \tilde g\rangle$

$+ \langle \tau_n \tilde f, \tilde g\rangle$

since $\tilde f$ and $\tilde g$ have compact support, we know that for $n >> 1$ $\langle \tau_n \tilde f, \tilde g\rangle = 0$

$|\langle \tau_n f, g\rangle| \le |\langle \tau_n(f-\tilde f), g\rangle| + |\langle \tau_n(f-\tilde f), \tilde g\rangle|$

$+ |\langle \tau_n \tilde f, g - \tilde g\rangle|$

$\le \|\cancel{\tau_n}(f-\tilde f)\|_{\ell^p} \|g\|_{\ell^{p'}} + \|\cancel{\tau_n}(f-\tilde f)\|_{\ell^p} \|\tilde g\|_{\ell^{p'}}$

$+ \|\cancel{\tau_n}\tilde f\|_{\ell^p} \|g - \tilde g\|_{\ell^{p'}}$

$\le \frac{\varepsilon}{3} \|g\|_{\ell^{p'}} + \frac{\varepsilon}{3} \|\tilde g\|_{\ell^{p'}}$

$+ \|\tilde f\|_{\ell^p} \frac{\varepsilon}{3} \le \varepsilon \left(\|f\|_{\ell^p} + \|g\|_{\ell^{p'}} \right)$

here $\|\tilde g\|_{\ell^{p'}} \le \|g\|_{\ell^{p'}}$

$\|\tilde f\|_{\ell^p} \le \|f\|_{\ell^p}$

We have shown that $\forall \varepsilon > 0 \; \exists N_\varepsilon$

s.t. $n > N_\varepsilon \Rightarrow$

$|\langle \tau_n f, g\rangle| \le \varepsilon \left(\|f\|_{\ell^p} + \|g\|_{\ell^p} \right)$

$\Rightarrow \boxed{\lim_{n\to+\infty} \langle \tau_n f, g\rangle = 0}$

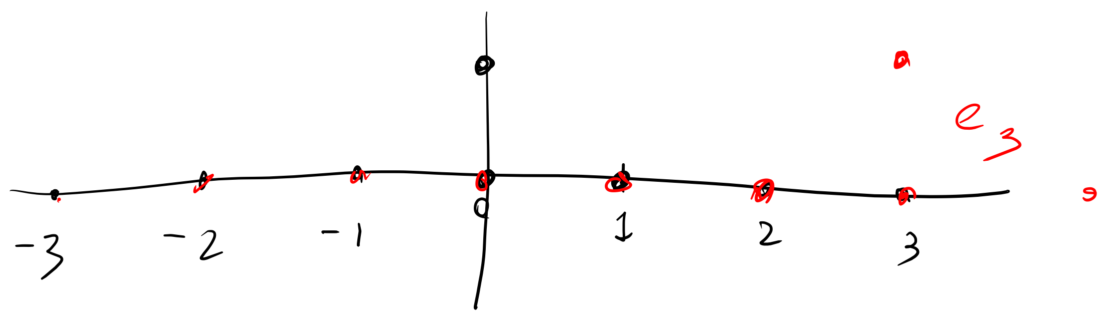
So for any $f \in \ell^p(\mathbb{Z})$ $\quad 1 < p < +\infty$

$\Rightarrow \tau_n f \longrightarrow 0$ for $\sigma(\ell^p, \ell^{p'})$

$f \in \ell^1(\mathbb{Z})$ $\qquad$ $(\ell^1(\mathbb{Z}))' = \ell^\infty(\mathbb{Z})$

$f(k) = e_0(k)$

$$e_0(k) = \begin{cases} 0 & \text{if } k \neq 0 \\ 1 & \text{if } k = 0 \end{cases}$$



$\tau_n e_0 = e_n$

$$e_n(k) = \begin{cases} 0 & k \neq n \\ 1 & k = n \end{cases}$$

$1 : \mathbb{Z} \longrightarrow \mathbb{R}$

$$\langle e_n, 1 \rangle = \sum_{k=-\infty}^{+\infty} e_n(k)\, 1(k) = \sum_{k=-\infty}^{+\infty} e_n(k) = 1$$

$\tau_n e_0 \cancel{\xrightarrow{n\to\infty} 0}$ for $\sigma(\ell^1, \ell^\infty)$

$1 \in \ell^\infty(\mathbb{Z})$

$\tau_n 1 = 1$ $\qquad$ $\tau_n 1 \longrightarrow 1 \neq 0$

Analogously in $L^p(\mathbb{R}^d)$

$\mathbb{R}^d$ acts as group on $L^p(\mathbb{R}^d)$

$x_0 \in \mathbb{R}^d \qquad f \in L^p(\mathbb{R}^d)$

$$\tau_{x_0} f(x) = f(x - x_0)$$

and if $1 < p < +\infty$ and if $\lim_{n \to +\infty} x_n = \infty$

in $\mathbb{R}^d$ $\Rightarrow$ $\tau_{x_n} f \longrightarrow 0$

$\mathbb{R}_+$ acts on $L^p(\mathbb{R}^d)$ by "scaling"

$\forall \lambda > 0$

$$S_\lambda f(x) = \lambda^{\frac{d}{p}} f(\lambda x)$$

$$\int_{\mathbb{R}^d} |S_\lambda f(x)|^p \, dx = \int_{\mathbb{R}^d} |\lambda^{\frac{d}{p}} f(\lambda x)|^p \, dx$$

$$= \lambda^d \int_{\mathbb{R}^d} |f(\lambda x)|^p \, dx =$$

$y = \lambda x \quad | \quad dy = \lambda^d \, dx$

$$= \int_{\mathbb{R}^d} |f(y)|^p \, dy = \|f\|^p_{L^p}$$

$$\|S_\lambda f\|_{L^p(\mathbb{R})} = \|f\|_{L^p(\mathbb{R})}$$

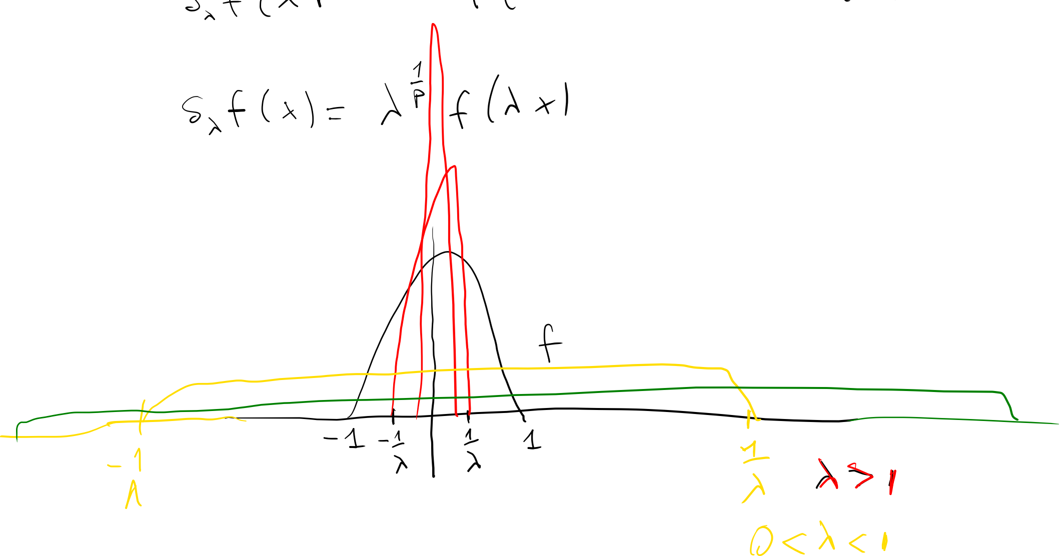
$\lim_{\lambda \to 0^+} S_\lambda f = 0$ in $\sigma(L^p(\mathbb{R}^d), L^{p'}(\mathbb{R}^d))$

$1 < p < +\infty$

$\lim_{\lambda \to +\infty} S_\lambda f = 0$ — — —

$S_\lambda f(x) = \lambda^{\frac{d}{p}} f(\lambda x)$ $\qquad d = 1$

$S_\lambda f(x) = \lambda^{\frac{1}{p}} f(\lambda x)$



$$S_\lambda f(x) = \lambda^{\frac{d}{p}} f(\lambda x) \xrightarrow{\lambda \to 0^+} 0$$

Need to show that $\forall\ f \in L^p(\mathbb{R}^d)$

$\forall\ g \in L^{p'}(\mathbb{R}^d)$ then

$$\langle S_\lambda f, g \rangle \xrightarrow{\lambda \to 0^+} 0$$

First let us assume $f, g \in C^0_c(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} \lambda^{\frac{d}{p}} f(\lambda x)\, g(x)\, dx$$

$\exists\ M > 0$

$f \in C^0_c(\mathbb{R}^d) \Rightarrow |f(x)| \le M \quad \forall\ x \in \mathbb{R}^d$

$$\left| \lambda^{\frac{d}{p}} f(\lambda x)\, g(x) \right| \le \lambda^{\frac{d}{p}} M\, |g(x)|$$

$$\lim_{\lambda \to 0^+} \int_{\mathbb{R}^d} \lambda^{\frac{d}{p}} f(\lambda x)\, g(x)\, dx = \int_{\mathbb{R}^d} \underbrace{\lim_{\lambda \to 0^+} \lambda^{\frac{d}{p}} f(\lambda x)\, g(x)}_{0}\, dx = 0$$

then $S_\lambda f \xrightarrow{\lambda \to 0^+} 0$ in $\sigma(L^p(\mathbb{R}^d), L^{p'}(\mathbb{R}^d))$

$1 < p < +\infty$

$$\|S_\lambda f\|_{L^p(\mathbb{R}^d)} = \|f\|_{L^p(\mathbb{R}^d)}$$

$$S_\lambda f \xrightarrow{\lambda \to +\infty} 0 \qquad \sigma(L^p(\mathbb{R}^d), L^{p'}(\mathbb{R}^d))$$

Need to show

$\lim_{\lambda \to +\infty} \langle S_\lambda f, g \rangle = 0 \quad \forall\ g \in L^{p'}(\mathbb{R}^d)$

$f \in L^p(\mathbb{R}^d)$

$$\langle S_\lambda f, g \rangle = \int_{\mathbb{R}^d} S_\lambda f(x)\, g(x)\, dx$$

$$= \lambda^{\frac{d}{p}} \int_{\mathbb{R}^d} f(\lambda x)\, g(x)\, dx =$$

$$= \lambda^{\frac{d}{p}} \lambda^{-d} \int_{\mathbb{R}^d} f(y)\, g(\lambda^{-1} y)\, dy \qquad y = \lambda x, \quad x = \frac{y}{\lambda}$$

$$= \lambda^{d(\frac{1}{p} - 1)} \int_{\mathbb{R}^d} f(y)\, g(\lambda^{-1} y)\, dy \qquad dy = \lambda^d\, dx, \quad dx = \lambda^{-d}\, dy$$

$$= \int_{\mathbb{R}^d} f(y) \left(\frac{1}{\lambda}\right)^{\frac{d}{p'}} g\left(\frac{1}{\lambda} y\right) dy \qquad 1 - \frac{1}{p} = \frac{1}{p'}, \quad p' = \frac{p}{p-1}$$

$$= \int_{\mathbb{R}^d} f(y)\, S_{\frac{1}{\lambda}} g(y)\, dy =$$

$$\langle S_\lambda f, g \rangle = \langle f, S_{\frac{1}{\lambda}} g \rangle \xrightarrow{\frac{1}{\lambda} \to 0^+} 0$$

$\Rightarrow S_\lambda f \rightharpoonup 0$ as $\lambda \longrightarrow +\infty$.

$\ell^1(\mathbb{Z})$

If $\{f_n\}$ is a sequence in $\ell^1(\mathbb{Z})$

with $f_n \xrightarrow{n\to+\infty} 0 \implies f_n \xrightarrow{n\to+\infty} 0$

(another reason why if $f \neq 0$ in $\ell^1(\mathbb{Z}$

then we do not have $\tau_n f \xrightarrow{n\to+\infty} 0$)

Banach

$E \longrightarrow F$ and $T : E \to F$

a linear operator. Then $T \in \mathcal{L}(E,F)$

iff $T : (E, \sigma(E,E')) \longrightarrow (F, \sigma(F,F'))$

is continuous.

Banach Alaoglu

E' $\sigma(E',E)$