November 14

The weak $\sigma(E', E)$ topology on $E'$.

$E$ B space, $E'$ B-space, $E''$
$\sigma(E', E'')$

Def Given a dual space $E'$ the weak $\sigma(E', E)$ topology is the topology generated by the seminorms $\{|\langle x, \cdot \rangle_{E \times E'}|\}_{x \in E}$

It is easy to see that for $f_0 \in E'$ a basis of nbhd of $f_0$ for the $\sigma(E', E)$ topology

$$V(f_0, x_1, \dots, x_n, \varepsilon) = \{f \in E' : |f(x_j) - f_0(x_j)| < \varepsilon \text{ for all } j = 1, \dots, n\}$$

for all finite families $x_1, \dots, x_n \in E$ and any $\varepsilon > 0$.

Lemma $E'$ with the $\sigma(E', E)$ topology is Hausdorff

Pf If $f_0 \neq f_1$ in $E'$ $\Rightarrow \exists\, x \in E$ s.t.
$f_0(x) \neq f_1(x)$ so we can assume $f_0(x) < f_1(x)$

$f_0(x) < \alpha < f_1(x)$ for some $\alpha \in \mathbb{R}$

It is easy to see $\{f \in E' : f(x) < \alpha\}$ is an open nbhd of $f_0$ in $\sigma(E, E')$

and $\{f \in E' : f(x) > \alpha\}$ ... — $f_1$

They are disjoint.

**Lemma** Let $\{f_n\}$ be a sequence in $E'$.

1) $f_n \xrightarrow{*} f \iff \forall x \in E$
$f_n(x) \longrightarrow f(x)$

2) If $f_n \longrightarrow f$ strongly in $E'$ $\Rightarrow$ $f_n \xrightarrow{*} f$
$\left( \|f_n - f\|_{E'} \xrightarrow{n \to +\infty} 0 \right)$

3) If $f_n \xrightarrow{*} f$ then $\sup\{\|f_n\|_{E'}\} < +\infty$
and $\|f\|_{E'} \leq \liminf_{n \to \infty} \|f_n\|_{E'}$

4) $f_n \xrightarrow{*} f$ and $x_n \longrightarrow x$ strongly in $E$, then
$f_n(x_n) \longrightarrow f(x)$

3) If $f_n \xrightarrow{*} f$ then $\sup\{\|f_n\|_{E'}\} < +\infty$

and $\|f\|_{E'} \leq \liminf_{n\to\infty} \|f_n\|_{E'}$

Fathou Lemma

Pf For $f_n \xrightarrow{*} f$ w by 1° statement

$\Rightarrow \lim_{n\to+\infty} f_n(x) = f(x) \quad \forall x \in E$

$\Rightarrow \{f_n(x)\}_{n\in\mathbb{N}}$ is bounded $\forall x$

$\Rightarrow$ by Banach Steinhaus $\exists$ $M>0$

s.t. $\|f_n\|_{E'} \leq M \quad \forall n \in \mathbb{N}.$

$$\|f\|_{E'} \leq \liminf_{n\to\infty} \|f_n\|_{E'}$$

Fatou Lemma

$$f_n \xrightarrow{*} f \iff f_n(x) \longrightarrow f(x)$$

$$|f(x)| = \lim_{n\to+\infty} |f_n(x)| = \lim_{k\to+\infty} |f_{n_k}(x)|$$

for any subsequence.

Let $f_{n_k}$ be a subsequence s.t. $\lim_{k\to+\infty} \|f_{n_k}\|_{E'} = \liminf_{n\to+\infty} \|f_n\|_{E'}$

$\|x\|_E = 1$

$$|f_{n_k}(x)| \leq \|f_{n_k}\|_{E'}$$

$$|f(x)| = \lim_{k\to+\infty} |f_{n_k}(x)| \leq \lim_{k\to+\infty} \|f_{n_k}\|_E = \liminf_{n\to+\infty} \|f_n\|_{E'}$$

$$|f(x)| \leq \liminf_{n\to+\infty} \|f_n\|_{E'}$$

$\forall$ $x$ with $\|x\|_E = 1$

$$\|f\|_{E'} \leq \liminf_{n\to+\infty} \|f_n\|_{E'} .$$

<u>Observation</u> Recall that

in $E$ given a convex set $C$

$C$ closed for the strong topology $\iff$

$C$ $\sigma(E,E')$ topology.

In the case of the $\sigma(E',E)$ in $E'$ what we said above is not true in general

<u>Example</u> $E = \ell^1(\mathbb{N})$ $E' = \ell^\infty(\mathbb{N})$

$c_0(\mathbb{N}) \in \ell^\infty(\mathbb{N})$ is closed subspace of $\ell^\infty$ for strong topology in $\ell^\infty(\mathbb{N})$.

But $c_0(\mathbb{N})$ is not closed for the $\sigma(\ell^\infty(\mathbb{N}), \ell^1(\mathbb{N}))$ top.

$\mathbb{N} \longrightarrow \mathbb{R}$ $\quad 1_{[0,N]}(k) = \begin{cases} 1 & 1 \le k \le N \\ 0 & k > N \end{cases}$

$\{1_{[0,N]}(k)\}_{N\in\mathbb{N}}$ in $c_0(\mathbb{N})$

$f \in \ell^1(\mathbb{N})$

$\langle 1_{[0,N]}, f\rangle_{\ell^\infty(\mathbb{N})\times\ell^1(\mathbb{N})} = \qquad \sum_{n=1}^{\infty} |f(n)| < 1$

$= \sum_{n=1}^{\infty} 1_{[0,N]}(n)\, f(n) = \sum_{n=1}^{N} f(n) \xrightarrow{N\to+\infty} \sum_{n=1}^{\infty} f(n)$

$= \langle 1, f\rangle_{\ell^\infty(\mathbb{N})\times\ell^1(\mathbb{N})}$

$1_{[0,N]} \xrightarrow{\not} 1$ in $\ell^\infty(\mathbb{N})$ in the sense of the $\sigma(\ell^\infty(\mathbb{N}), \ell^1(\mathbb{N}))$

but $1 \notin c_0(\mathbb{N})$

$c_0(\mathbb{N})$ $\quad \ell^\infty(\mathbb{N})$ $\quad \boxed{\ell^1(\mathbb{N}).}$ $\quad \ell^\infty(\mathbb{N})(\ell^\infty(\mathbb{N}))'$

$[0, N]$ $\quad n = 1, 2, \dots$

$\{f \in c_0(\mathbb{N}) : f(k) = 0 \ \forall k \ge n\} \simeq \mathbb{R}^{n-1} \supseteq \mathbb{Q}^{n-1}$

Theorem (Banach Alaoglu)

Let $E$ be a B-space and consider $E'$.
The $D_{E'}(0,1) = \{ f \in E' : \|f\|_{E'} \le 1 \}$
is compact for the $\sigma(E', E)$ topology.

Pf Consider $E' \xrightarrow{\Phi} \mathbb{R}^E = \{$ is the set of functions $E \to \mathbb{R} \}$

In $\mathbb{R}^E$ we consider the product topology
given $f_0 \in \mathbb{R}^E$ a basis of neighbs is of the
form $V(f_0, x_1, \dots, x_n, \varepsilon) = \{ f \in \mathbb{R}^E : |f(x_j) - f_0(x_j)| < \varepsilon \ \forall j = 1, \dots, n \}$
$x_1, \dots, x_n \in E$, $\varepsilon > 0$

Then $\phi : (E', \sigma(E', E)) \longrightarrow \phi(E') \subseteq \mathbb{R}^E$
is a homeomorphism.

$$\overline{D_{E'}(0,1)} \longrightarrow \underbrace{\phi\left(\overline{D_{E'}(0,1)}\right)}_{\text{need to show it is compact in } \mathbb{R}^E} = \overline{D_E(0,1)}$$

we show $\overline{D_E(0,1)}$ is compact in $\mathbb{R}$

$$\overline{D_E(0,1)} = K_1 \cap K_2$$

where $K_1$ is compact in $\mathbb{R}^E$ and $K_2$ is closed.

$$K_2 = \left( \bigcap_{x,y \in E} A_{xy} \right) \cap \left( \bigcap_{\substack{\lambda \in \mathbb{R} \\ x \in E}} B_{\lambda x} \right)$$

$$A_{xy} = \{ \omega : E \to \mathbb{R} : \underline{\omega(x+y) = \omega(x) + \omega(y)} \}$$

$$B_{\lambda x} = \{ \omega : E \to \mathbb{R} : \omega(\lambda x) = \lambda \omega(x) \}$$

All these sets are closed in $\mathbb{R}^E$

$A_{xy}$ $\quad \mathbb{R}^E \longrightarrow \mathbb{R}^3 \longrightarrow \mathbb{R}$

$\omega \longrightarrow (\omega(x+y), \omega(x), \omega(y)) \longrightarrow \omega(x+y) - \omega(x) - \omega(y)$

$A_{xy}$ is closed in $\mathbb{R}^E$ $\forall x, y \in E$
$B_{\lambda x}$ $\qquad \forall \lambda \in \mathbb{R} \ \forall x \in E$

$\Rightarrow K_2$ is closed in $\mathbb{R}^E$
is exactly the set of linear functions $E \to \mathbb{R}$.

$f \in \overline{D_{E'}(0,1)}$

$|f(x)| \leq \|x\|_E \quad \forall x \in E$

$K_1 = \prod_{x \in E} [-\|x\|_E, \|x\|_E] \subseteq \mathbb{R}^E$

$K_1$ is a compact subspace of $\mathbb{R}^E$ by T- theorem.

$\overline{D_{E'}(0,1)} \subseteq K_1 \cap K_2 = \{ f : E \to \mathbb{R} :$ f is linear and $|f(x)| \leq \|x\|_E \ \forall x \in E$

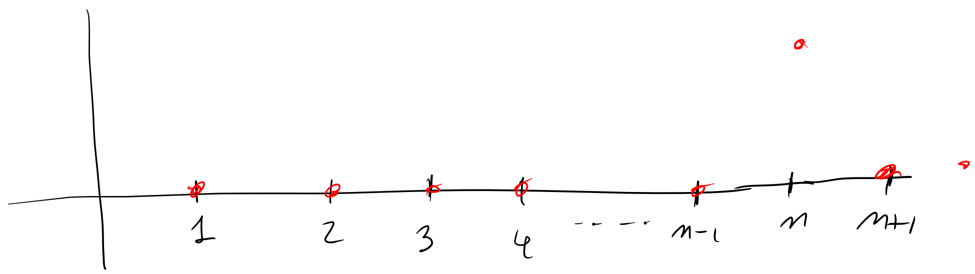
# Example

$E = \ell^\infty(\mathbb{N})$ $\qquad E'$.

$$e_n(k) = \begin{cases} 1 & \text{if } k = n \\ 0 & k \neq n \end{cases}$$

$\in$

$\ell^1(\mathbb{N})$




$$\|e_n\|_{E'} = \|e_n\|_{\ell^1(\mathbb{N})} = 1$$

It is a sequence in $D_{E'}(0,1)$ $\qquad \sigma(E', E)$

$\{e_n\}$ does not have convergent subsequences for the $\sigma(E', E)$ topology

Let $\{e_{n_k}\}$ be a subsequence and define

$\xi \in E = \ell^\infty(\mathbb{N})$ $\qquad \|\xi\|_{\ell^\infty(\mathbb{N})} = 1$

$$\xi(m) = \begin{cases} 0 & \text{if } m \neq n_k \quad \forall k \\ (-1)^k & \text{if } m = n_k \end{cases}$$

$$\langle e_{n_k}, \xi \rangle_{E' \times E} = \sum_{m=1}^{\infty} e_{n_k}(m)\, \xi(m) = \xi(n_k) = (-1)^k$$

is a non convergent sequence

$\sup\{|\xi(m)| : m \in \mathbb{N}\} = 1$

Def A B-space $E$ is reflexive if

$J : E \to E''$ is an isomorphism.

Theorem (Kakutani) $E$ is reflexive if and only if $D_E(0,1)$ is compact for the $\sigma(E,E')$ topology.