QFT III - Problem sheet

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February 4, 2026

1 Current and charge in QED

Consider the Lagrangian of QED, with one fermion of electric charge q:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \bar{\psi}\left(i\slashed{D} - m\right)\psi \tag{1.1}$$

with $D_\mu\psi = (\partial_\mu + i\,q\,e\,A_\mu)\psi$, invariant under the local transformation

$$\psi \mapsto e^{-iq\theta}\psi\,, \tag{1.2}$$

$$A_\mu \mapsto A_\mu + \frac{1}{e}\partial_\mu\theta\,. \tag{1.3}$$

1. Discuss why local gauge invariance forbids a mass term for the photon, although this is allowed by the corresponding global symmetry.

2. Find the equation of motion for the matter fields and the photon.

3. Show that the Noether current corresponding to the gauge symmetry transformation of Eq. (1.2) is

$$J^\mu = \frac{1}{e}\partial_\nu(F^{\nu\mu}\theta)\,. \tag{1.4}$$

4. Show that the associated conserved charge either vanishes or reduces to the electric charge associated with global U(1) invariance, provided θ approaches an angle-independent limit at spatial infinity. Thus, gauge invariance leads to no new conservation laws as compared to global invariance.

2 Covariant derivative of the gauge parameter θ

The parameter of gauge transformations $\theta = \theta^a t_R^a$ is a vector in the representation R of the algebra $\mathcal{G}$. Therefore, it transforms in the adjoint representation. Consider the vector $e^{i\theta}\phi \in V_R$. Under a global transformation, this transforms as

$$e^{i\theta}\phi \mapsto e^{i\beta}(e^{i\theta}\phi)\,. \tag{2.1}$$

From this transformation law, derive the transformation law of θ under an infinitesimal transformation

$$\theta^a \mapsto \theta^a - f^{abc}\beta^b\theta^c\,. \tag{2.2}$$

Since θ transforms in the adjoint representation, define $D_\mu\theta$.

3 Gauge transformations of $F^{a\,\mu\nu}$

Derive the transformation law of the components $F^{a\,\mu\nu}$ under an infinitesimal, local transformation.

4 θ-term

Show that the θ-term

$$I = \theta\,\epsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}(\mathbb{F}_{\mu\nu}\mathbb{F}_{\rho\sigma}) \tag{4.1}$$

is a 4-divergence:

$$I = 4\,\theta\,\partial_\rho[\epsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}(\mathbb{A}_\sigma\partial_\mu\mathbb{A}_\nu + \frac{2i}{3}\mathbb{A}_\sigma\mathbb{A}_\mu\mathbb{A}_\nu)]\,. \tag{4.2}$$

5 Current and charge in non-Abelian gauge theory

Consider the Lagrangian of a non-Abelian gauge theory with a single fermion of mass m, in a representation R of the gauge group G:

$$\mathcal{L} = -\frac{1}{4g^2}F^a_{\mu\nu}F^{a\,\mu\nu} + \bar\psi(i\slashed{D} - m)\psi \tag{5.1}$$

with $D_\mu\psi = (\partial_\mu + iA^a_\mu t^a_R)\psi$, invariant under the local transformation

$$\psi \mapsto e^{-i\theta^a t^a_R}\psi\,, \tag{5.2}$$

$$A^a_\mu \mapsto A^a_\mu + \partial_\mu\theta^a - f^{abc}A^b_\mu\theta^c\,. \tag{5.3}$$

1. Derive the equations of motion of the system in pure Yang-Mills (i.e. no fermions)

$$(D_\mu F^{\mu\nu})^a = \partial_\mu F^{a\,\mu\nu} - f^{abc}A^b_\mu F^{c\,\mu\nu} = 0\,. \tag{5.4}$$

2. Show that, in pure Yang-Mills the current is

$$j^a_\mu = f_{abc}F^{b\,\nu}_\mu A^c_\nu \tag{5.5}$$

   for a global symmetry transformation and

$$j_\mu = -\frac{2}{g^2}\,\mathrm{Tr}(\mathbb{F}^{\nu\mu}D_\nu\theta) \tag{5.6}$$

   for a local gauge transformation. Check that both lead to the same charge Q^a for $\theta \to$ const at infinity.

3. Find the trasformation law of Q^a for global symmetry transformations. Does Q^a transform under a representation of the gauge group? Under which condition on A_μ is the charge constant in time?

4. Add back the fermion ψ. Show that the current is

$$J_\mu = j_\mu + j^{a\,\nu}_\psi\theta^a \tag{5.7}$$

   with

$$j^{a\,\nu}_\psi = \bar\psi\,t^a_R\gamma^\nu\psi\,. \tag{5.8}$$

   We see from the above that the non-abelian charge associated with the fermionic current is not a constant of motion; since gauge fields are not neutral, their contribution must be included to find conserved charges.

5. Show that the equation of motion can be rewritten as

$$(D_\mu F^{\mu\nu})^a = \partial_\mu F^{a\,\mu\nu} - f^{abc} A^b_\mu F^{c\,\mu\nu} = j^{a\,\nu}_\psi \tag{5.9}$$

$$(i\not\partial - m)\psi = A^a_\mu t^a_R \gamma^\mu \psi \tag{5.10}$$

and that the current is

$$J^\mu = -\frac{1}{g^2} \partial_\nu(\theta^a F^{a\,\mu\nu}). \tag{5.11}$$

6. Show that $j^{a\,\nu}_\psi$ is covariantly conserved

$$D_\nu j^{a\,\nu}_\psi \tag{5.12}$$

Thus, gauge fields can only couple consistently to currents which are covariantly conserved.

3

6 BRST symmetry

Consider the Lagrangian

$$\mathscr{L} = -\frac{1}{4}(F_{\mu\nu}^a)^2 + \bar{\psi}(i\not{D} - m)\psi + \frac{\xi}{2}B^a B^a - B^a \partial^\mu A_\mu^a + \bar{c}^a(-\partial^\mu D_\mu)^{ab}c^b \tag{6.1}$$

and the infinitesimal BRST global transformation

$$\delta A_\mu^a = \epsilon(D_\mu c)^a = \epsilon(\partial_\mu c^a - g f^{abc} A_\mu^b c^c)$$
$$\delta\psi = -ig\epsilon c^a t_R^a \psi$$
$$\delta c^a = \frac{1}{2}g\epsilon f^{abc} c^b c^c$$
$$\delta\bar{c}^a = -\epsilon B^a$$
$$\delta B^a = 0. \tag{6.2}$$

1. Check the invariance of the Lagrangian under BRST. It is convenient to consider at once the variation of the non gauge-invariant piece $\delta(-B^a \partial^\mu A_\mu^a + \bar{c}^a(-\partial^\mu D_\mu)^{ab}c^b)$ and connect it to the variation of $\delta(D_\mu c)^a$, and then compute the variation of the latter.

2. Show that BRST is nilpotent when applied to the elementary fields appearing in the Lagrangian, $\delta_1\delta_2\phi = 0$. During the lectures, we assumed this result and derived $\delta^2 F(\phi) = 0$ for any function F. This completes the proof of the nilpotence of the BRST operator.

3. Find the correct redefinition of the fields that eliminates g from the transformation law and the covariant derivative. How does g enter the Lagrangian in this notation?

7 Faddeev-Popov matrix

Compute the FP matrix

$$M_G^{a,b}(x,y) \equiv \left.\frac{\delta(G^\mu A_\mu^{\theta\,a}(x))}{\delta\theta^b(y)}\right|_{\theta=0} \tag{7.1}$$

with $A_\mu^{\theta\,a} = A_\mu^a + \frac{1}{g}\partial_\mu\theta^a - f^{abc}A_\mu^b\theta^c$ for the following choices of the *gauge-fixing* term $G^\mu A_\mu^a$:

1. $G^\mu = (0, \vec{\nabla})$ (Coulomb gauge)

2. $G^\mu = \partial^\mu$ (Lorenz gauge, aka covariant gauge)

3. $G^\mu = n^\mu$ with constant space-like $n^\mu = (0, \vec{n})$ (spatial axial gauge)

4. $G^\mu = (1, \vec{0})$ (temporal axial gauge)

8 Decoupling of ghosts in axial gauges

Consider the axial gauge choice $G^\mu \mathbb{A}_\mu = n^\mu \mathbb{A}_\mu$. We want to convince ourselves that ghosts decouple in this gauge.

1. The gauge fixing term enforces $G^\mu \mathbb{A}_\mu = 0$ on the S matrix. Show that, if we impose $G^\mu \mathbb{A}_\mu = 0$ on the Lagrangian, the ghosts decouple from other fields in $\mathcal{L}$.

We now want to get rid of the $\bar{c}Ac$ coupling in the path integral, in the FP construction we saw in the lecture. This can be obtained with a change of integration variables

$$c^b(x) \to W^{bc}(x)\chi^c(x), \qquad \bar{c}^a(x) \to \bar{\chi}^d(x)(W(x)^{-1})^{da} \tag{8.1}$$

such that

$$\bar{c}(x)(-n^\mu D_\mu)c(x) = \bar{\chi}(x)(-n^\mu \partial_\mu)\chi(x). \tag{8.2}$$

where, for α in the adjoint rep,

$$(D_\mu \alpha)^a = (\partial_\mu \delta^{ab} - f^{adb}A_\mu^d)\alpha^b \tag{8.3}$$

With such a change of variables, the ghost fields completely decouple from the other fields of the Lagrangian and their action act as a normalization factor that can be simply integrated out in the path integral.

2. Find the differential equation that defines $W(x)$

3. Show that this equation is solved by the Wilson line operator

$$W(y) = \mathcal{P}\exp\left\{-i\int_{-\infty}^{s_y} ds f'(s)\,n^\mu \mathbb{A}_\mu\right\} \tag{8.4}$$

   where, for a point y^μ, we have defined a straight path that goes through it and is parallel to the vector n^μ:
   $$x^\mu(s) = x^\mu(0) + f(s)n^\mu \tag{8.5}$$

   with $x^\mu(s_y) = y^\mu$, and $f(s)$ is any monotonically growing function.

4. Complete the proof by showing that W does not affect the path integral measure, using the fact that $\forall x\ (\det W(x))(\det W(x)^{-1}) = 1$

The axial gauge is often used in perturbative QCD calculation, with the advantage of the ghost decoupling. The price to pay is the loss of Lorentz-invariance, which is nevertheless restored in the calculation of gauge-invariant S-matrix elements.

9   Perturbative calculation of scattering amplitudes

In a theory of massless vector bosons such as YM or QED, gauge invariance is imposed in order to get rid of the two unphysical polarizations of the photon. In QED this implies that, for any amplitude involving a photon with momentum k_μ in the final state,

$$M^\mu k_\mu = 0 \tag{9.1}$$

where $M^\mu \epsilon_\mu(\vec{k})$ is the original amplitude, and ϵ^μ is a polarization vector with $\epsilon^2 = 1$. It is enough to have one non-transverse photon for the amplitude to vanish. This implies that, when summing over photon polarizations (eg. in the computation of the $e^+e^- \to \gamma\gamma$ cross section), one can replace the sum over photon polarizations as

$$\sum \epsilon_\mu^* \epsilon_\nu \to -g_{\mu\nu} \tag{9.2}$$

Things are more complicated in non-Abelian gauge theory, as we we will show in this exercise.

1. In comoving gauge, compute the amplitude at tree level for the fermion – anti-fermion annihilation process

$$\psi_i(\vec{p})\,\bar{\psi}_j(\vec{q}) \to g_a(\vec{k}_1)\,g_b(\vec{k}_2)\,, \tag{9.3}$$

where i, j and a, b are $\mathcal{G}$ indices. Include in the calculation *only* the two diagrams with $\bar{\psi} A \psi$ couplings (no gluon self-interaction). Clearly the external gluons need to be on-shell for this calculation. You can use that fermions are *on-shell* as well, ie they satisfy the Dirac equation, and you can neglect their masses. Write the amplitude in the form

$$\epsilon_\mu^*(\vec{k}_1)\,M_{ab\,ij}^{\mu\nu}\,\epsilon_\nu^*(\vec{k}_2) \tag{9.4}$$

where $\epsilon(\vec{k}_i)$ are the polarization four-vectors of the final state gluons. Compute $k_{1\mu} M_{ab\,ij}^{\mu\nu}$ and show that it doesn't vanish, but is proportional to f^{abc} (which vanishes in QED).

2. Now add the triple gluon vertex and compute the amplitude, leaving ξ as a free parameter (it should disappear from the amplitude when you put external particles on shell: check it!) . Show that, summing the three diagrams,

$$k_{1\mu}\,M_{ab\,ij}^{\mu\nu}\,\epsilon_\nu^*(\vec{k}_2) = 0 \tag{9.5}$$

iff the second gluon is transverse, ie. $k_2 \cdot \epsilon(\vec{k}_2) = 0$. Your result should look like

$$k_{1\mu}\,M_{ab\,ij}^{\mu\nu}\,\epsilon_\nu^*(\vec{k}_2) \propto k_2 \cdot \epsilon^*(\vec{k}_2)\,. \tag{9.6}$$

This result can be interpreted by saying that, in a non-Abelian gauge theory, amplitudes are gauge-invariant *iff* all gluons are physical. Since $k^2 = 0$, this result implies as well

$$k_{1\mu}\,M_{ab\,ij}^{\mu\nu}\,k_{2\mu} = 0\,. \tag{9.7}$$

3. For transverse gluons, the sum over polarizations gives

$$\Sigma_{\mu\nu}(k) \equiv \sum_{\text{pol}} \epsilon_\mu^* \epsilon_\nu = -g_{\mu\nu} + \frac{1}{k \cdot \eta}\,(k_\mu \eta_\nu + \eta_\mu k_\nu) \tag{9.8}$$

where η is defined such that, if $k_\mu = k_0(1, 0, 0, 1)$, $\eta_\mu = k_0(1, 0, 0, -1)$.

Compute the squared matrix element (summing over all three diagrams), using Eq. (9.8) for the polarizations:

$$|M_T|^2 = \sum_{\text{pol}} M^{\mu\nu}(M^{\rho\sigma})^*\,\epsilon(\vec{k}_1)_\mu^*\,\epsilon(\vec{k}_2)_\nu\,\epsilon(\vec{k}_1)_\rho\,\epsilon(\vec{k}_2)_\sigma^*$$

$$= M^{\mu\nu}(M^{\rho\sigma})^*\,\Sigma(k_1)_{\mu\rho}\,\Sigma(k_2)_{\sigma\nu} \tag{9.9}$$

and show that

$$|M_T|^2 = M^{\mu\nu}(M^{\rho\sigma})^*(-g_{\mu\rho})(-g_{\sigma\nu}) - 2\left(\frac{g^2}{2\,p \cdot q}\right)^2 \left|f^{abc} t_{ij}^a \bar{v}(q) \slashed{k}_1 u(p)\right|^2 \tag{9.10}$$

The first term is what we would have found had we replaced $\sum \epsilon_\mu^* \epsilon_\nu \to -g_{\mu\nu}$ as in QED.

Suggestions: Start expanding only one of the two Σ's, and using the result (9.7) you can rewrite $\Sigma_{\mu\rho}\Sigma_{\sigma\nu} \to \Sigma_{\mu\rho}(-g_{\sigma\nu})$. Then expand the other Σ and rewrite $-g_{\sigma\nu}$ using Eq. (9.8), and see what other terms cancel. You can also use Eq. (9.6) replacing $\epsilon(\vec{k}_2) \to \eta_2$ for computing $k_{1\mu}\,M_{ab\,ij}^{\mu\nu}\,\eta_{2\nu}$.

4. Compute the amplitudes $M_{ab\,ij}^{\text{ghost}}$ and $M_{ba\,ij}^{\text{ghost}}$ for the *virtual* processes

$$\psi_i(\vec{p})\,\bar{\psi}_j(\vec{q}) \to c_a(\vec{k}_1)\,\bar{c}_b(\vec{k}_2)\,, \qquad \psi_i(\vec{p})\,\bar{\psi}_j(\vec{q}) \to \bar{c}_a(\vec{k}_1)\,c_b(\vec{k}_2) \tag{9.11}$$

respectively, and compare $|M_{ab\,ij}^{\text{ghost}}|^2 + |M_{ba\,ij}^{\text{ghost}}|^2$ with Eq. (9.10). The result can be interpreted by saying that the ghosts exactly cancel the (unphysical) longitudinal polarizations. As a consequence, in perturbative calculations one can, instead of summing over physical polarizations using Eq. (9.8), simplify the calculation using $\sum \epsilon_\mu^* \epsilon_\nu \to -g_{\mu\nu}$, and *subtract* the analogous unphysical diagram with two ghosts in the final state.

5. Discuss (no need to do calculations) why this is related to the optical theorem, and in particular to the diagrams shown below.

10 Slavnov-Taylor

11 One-loop calculation of the $\bar{\psi}A\psi$ vertex
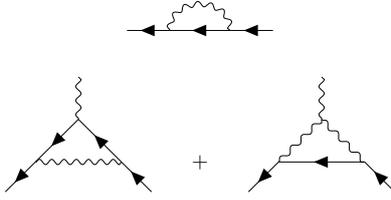
Compute the divergent part of the 1-loop correction to the fermion propagator and to the $\bar{\psi}A\psi$ vertex, and determine the corresponding counter-terms.

12 Scheme dependence of the beta function

1. Consider the beta function of a non-abelian gauge theory in two different *mass independent* renormalization schemes, at the same scale μ:

$$\beta \equiv \mu \frac{dg_R(\mu)}{d\mu}\,, \qquad \beta' \equiv \mu \frac{dg'_R(\mu)}{d\mu}\,. \tag{12.1}$$

The two constants are related by a finite renormalization constant

$$g_R = \tilde{Z}_g\, g'_R = (1 + \tilde{z}_0 g'^{\,2}_R + \tilde{z}_1 g'^{\,4}_R + \dots)g'_R\,. \tag{12.2}$$

Expanding

$$\beta = -\beta_0\, g_R^3 - \beta_1\, g_R^5 - \beta_2\, g_R^7 + \dots \tag{12.3}$$

Assuming $\mu \gg m_R$, the constant $\tilde{Z}_g$ depends on μ only through g'_R, ie $g_R(\mu) = g_R(g'_R(\mu))$ and $g'_R(\mu) = g'_R(g_R(\mu))$. Verify that the first two terms in the perturbative expansion of β (obtained from a calculation up to 2 loops) are scheme independent, ie

$$\beta' = -\beta'_0\, g'^{\,3}_R - \beta'_1\, g'^{\,5}_R - \beta'_2\, g'^{\,7}_R + \dots \qquad \text{with} \qquad \beta'_0 = \beta_0\,,\ \beta'_1 = \beta_1\,. \tag{12.4}$$

13 Wilson loop and gauge transformations

1. Define the non-Abelian Wilson line as

$$W(x,y) = \mathcal{P}\exp\left\{-i\int_{\tau_x}^{\tau_y} d\tau\, \frac{dx^\mu}{d\tau}\mathbb{A}_\mu\right\} \tag{13.1}$$

Show explicitly that

$$W(x \to y)W(y \to z) = W(x \to z) \tag{13.2}$$

where on the RHS we consider the path obtained as the union of the two paths on the LHS. This property is a consequence of the definition of the parallel transport: here you should work it out from the $\mathcal{P}$-ordered exponential.

2. Using the above property, divide the path $x \to y$ in infinitesimal steps, and derive the transformation law of $W(x,y)$.

3. In QED, the Wilson loop can be defined simply as

$$W_{\mathrm{QED}} = \exp\left(-ie \oint A_\mu dx^\mu\right).$$

(13.3)

Why is the path ordering is not necessary?