CHAPTER 3

Hyperbolic Conservation Laws

A *first order system of conservation laws* is a PDE of the form

$$u_t + \operatorname{div}_x f(u) = 0, \quad u : \mathbb{R}^d \to \mathbb{R}^m, f : \mathbb{R}^m \to \mathbb{R}^{d\times m}. \tag{0.1}$$

More precisely

$$\partial_t u_i + \sum_j \partial_j f_{ij}(u) = 0.$$

Moreover, sometimes we have *a system of balance laws*

$$u_t + \operatorname{div}_x f(u) = g(u),$$

with also in cases the *flux* f and the *source* g depend on t, x. These PDE are usually supplemented with an initial condition

$$u(t = 0) = u_0.$$

The *weak solution* is obtained by integrating by testing with a $C_c^\infty([0,+\infty)\times\mathbb{R}^d,\mathbb{R}^m)$-function ϕ:

$$\int_0^\infty u \cdot \phi_t + \sum_{ij} f_{ij}\partial_j\phi_i dxdt + \int u_0 \cdot \phi(t = 0)dx = 0,$$

where it is customary to assume that

$$u, f \in L^1_{\mathrm{loc}}([0,+\infty)\times\mathbb{R}^d), \quad u_0 \in L^1_{\mathrm{loc}}(\mathbb{R}^d).$$

Remark 0.1. The PDE is invariant for the *hyperbolic scaling* $(t, x) \mapsto \epsilon(t, x)$.

0.1. Classification. This is mainly for historical reasons, since at the end every PDE has different techniques and results (e.g. Compressible euler is very different than traffic flow and crowd dynamics). In the next sections we will explain why these classes arise.

Conservation: The PDE is in divergence form, so that

$$\frac{d}{dt}\int_\Omega u\mathscr{L}^d = \int_{\partial\Omega} f(u)\cdot \mathbf{n}\mathscr{H}^{d-1},$$

$\mathbf{n}$ being the outer normal: the variation of the quantity in a domain is due to the flux across the boundary.

Hyperbolic: For every $u \in U$, U being the region of admissible values, $\xi \in \mathbb{S}^{d-1}$ it holds

$$\sum_{j=1}^d \xi_j D u f_j(u) \quad \text{diagonalizable with real eigenvalues,}$$

where $f(u) = (f_1(u), \dots, f_d(u))$ is the flux.

Symmetric hyperbolic: There exists a symmetric strictly definite positive matrix $A_0(u)$ such that

$$A_0(u)Df_j(u) \quad \text{is symmetric.}$$

Clearly this condition implies the previous one.

Entropy: There exists a uniformly convex function $\eta(u)$ (entropy) and a vector valued function $q(u) = (q_1(u), \dots, q_d(u))$ (entropy flux) such that

$$Dq_j(u) = D\eta(u)Df_j(u).$$

We will see that this condition implies the previous one.

Commuting: It holds

$$Df_i(u)Df_j(u) = Df_j(u)Df_i(u) \quad \forall i, j.$$