

1. Linear analysis

We first consider the case of a linear system:

$$u_t + \sum_j A_j \partial_j u = 0, \quad A_j \in \mathbb{R}^{m \times m}. \quad (1.1)$$

Taking the Fourier transform

$$\hat{u}(t, \xi) = \mathcal{F}(u(t))(\xi) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int u(t, x) e^{-i\xi \cdot x} dx,$$

we have

$$\frac{d}{dt} \hat{u}(t, \xi) - i \left(\sum_j A_j \xi \right) \hat{u}(t, \xi) = 0.$$

Hence a solution is such that the coefficients satisfy the ODE system above:

$$\hat{u}(t, \xi) = e^{i \sum_j \xi_j A_j t} \hat{u}(0, \xi),$$

Note that if $\alpha \in \mathbb{R}$ then by Remark 0.1

$$\hat{u}(t, \alpha \xi) = e^{i \sum_j \alpha \xi_j A_j t} \hat{u}(0, \alpha \xi) = e^{i \sum_j \xi_j A_j (\alpha t)} \hat{u}(0, \alpha \xi) = \hat{u}(\alpha t, \eta/\alpha),$$

so that if there is a frequency ξ such that $e^{i \sum_j \xi_j A_j t} \rightarrow \infty$ as $t \rightarrow \infty$, then no integrable solution u is possible: indeed the frequency $\alpha \xi$ diverges as $(e^{i \sum_j \alpha \xi_j A_j t})^\alpha$.

DEFINITION 1.1. The PDE (0.1) is *hyperbolic* if for all $\xi \in \mathbb{S}^{d-1}$ the matrix $\sum_j A_j \xi_j$ is diagonalizable with real eigenvalues.

A condition for hyperbolicity is the following.

DEFINITION 1.2. The PDE (1.1) is *symmetric hyperbolic* if there is a symmetric strictly positive matrix A_0 such that $A_0 A_j$ are symmetric.

LEMMA 1.3. *A symmetric hyperbolic system is hyperbolic.*

PROOF. By $A_0 = B^2 > 0$, $B = B^T$, we obtain that $v = Bu$ satisfies

$$v_t + \sum_j B A_j B^{-1} \partial_j v = 0,$$

and

$$B A_j B^{-1} = B^{-1} (A_0 A_j) B^{-1}$$

is symmetric, in particular diagonalizable. \square

1.1. Well-posedness for linear equation in integral spaces. For the L^2 -norm, we have that if the PDE is symmetric hyperbolic that

$$\frac{d}{dt} \|u\|_2^2 = -2 \int \sum_j u^T A_j \partial_j u = - \sum_j \int \partial_j (u^T A_j u) = 0,$$

so that the L^2 -norm is conserved. In particular the operator

$$u_0 \mapsto u(t) = \mathcal{F}^{-1}(\hat{u}(\xi)) = \mathcal{F}^{-1}(e^{i \sum_j \xi_j A_j t}) * u_0$$

is an L^2 -multiplied, according to the following

DEFINITION 1.4. A L^p -multiplier λ is a tempered distribution such that for all $f \in \mathcal{S}$ it holds

$$\|\lambda * f\|_p \leq M_p(\lambda) \|f\|_p,$$

the constant $M_p(\lambda)$ being the minimal one.

In the L^1 -case in particular $M_2(\mathcal{F}^{-1}(e^{i \sum_j \xi_j A_j t})) = 1$. Recall also that this requires that

$$e^{i \sum_j \xi_j A_j t} \in L^\infty,$$

as we have observed above.

We have the following theorem.

THEOREM 1.5. *Consider a symmetric hyperbolic linear system. If $e^{i\sum_j \xi_j A_j}$ is a p -multiplier for $p \neq 2$, then the matrices A_j commute.*

Thus the requirement is that

$$\|u(t)\|_p \leq M\|u(0)\|_p, \quad \hat{u}(t, \xi) = e^{i\sum_j A_j \xi_j} \hat{u}(0, \xi).$$

We can assume that the system is of the form (1), i.e. the matrices A_j are symmetric. Note that if $M(t)$ is not uniformly bounded, then by the hyperbolic scaling one concludes that the solution blows immediately up.

PROOF. First we prove that if the matrices A_j commute, then it is a p -multiplier: indeed, since they commute there is a common diagonalization: for example, for two matrices

$$A_2 A_1 r = A_1 A_2 r = \lambda A_1 r,$$

i.e. A_1 maps eigenspaces of A_2 into eigenspaces of A_2 : thus we just diagonalize A_1 restricted to each eigenspace of A_2 .

If P is the common diagonalization matrix

$$P A_j P^{-1} = \text{diag}(\lambda_{j,i}),$$

we have

$$(Pv)_t + \sum_j P A_j P^{-1} (Pv)_{x_j} = 0,$$

and in components of $Pv = w$

$$\partial_t w_i + \sum_j \lambda_{j,i} \partial_j w_i = 0, \quad w_i(t, x) = w(0, x - \vec{\lambda}_i t), \quad \vec{\lambda}_i = (\lambda_{i,1}, \dots, \lambda_{i,d}).$$

Clearly the L^p -norm is preserved, being just a translation.

For the necessary part, observe that when evaluated at $t = m$ this implies that if

$$T_\lambda u = \mathcal{F}^{-1}(\lambda \mathcal{F}(u)),$$

then we have that

$$\|T_{e^{im\sum_i \xi_i A_i}}\| \leq M.$$

Consider the scaling

$$u_\alpha(x) = \alpha^{\frac{d}{p}} u(\alpha x), \quad \|u_\alpha\|_p = \|u\|_p, \quad \hat{u}(\xi) = \alpha^{\frac{d}{p}-d} \hat{u}\left(\frac{\xi}{\alpha}\right).$$

In particular we can concentrate the Fourier transform of u about a point ξ_0 and preserve the L^p -norm.

Expanding about a point where the eigenvalues are analytic as well as the eigenprojectors

$$\sum_i A_i \xi_i = \sum_i \lambda_i(\xi) r_i(\xi) \otimes r_i^\perp(\xi),$$

we can test with the L^p -function

$$\hat{u}_i(\xi) = \psi(\xi) r_i(\xi),$$

obtaining that

$$\mathcal{F}^{-1}(e^{i\sum_i \xi_i A_i} \psi(\xi) r_i(\xi)) = \mathcal{F}^{-1}(e^{i\lambda_i(\xi)} \psi(\xi) r_i(\xi)).$$

In particular, also $e^{im\lambda_i(\xi)}$ is a Fourier multiplier, and $\lambda_i(\xi)$ is analytic in a small ball.

We thus have for the scaling and expansion about one point

$$\begin{aligned} \mathcal{F}^{-1}\left(e^{im(\lambda_0 + \lambda_1 \xi + \frac{1}{2} \xi A \xi + \mathcal{O}(\xi^3))} \alpha^{-\frac{d}{p}} \hat{u}\left(\frac{\xi}{\alpha}\right)\right)(x) &= e^{im\lambda_0} \mathcal{F}^{-1}\left(e^{im(\frac{1}{2} \xi A \xi + \mathcal{O}(\xi^3))} \alpha^{-\frac{d}{p}} \hat{u}\left(\frac{\xi}{\alpha}\right)\right)(x + m\lambda_1) \\ &= \alpha^{\frac{d}{p}} e^{im\lambda_0} \mathcal{F}^{-1}(e^{im\alpha^2 \xi A \xi + \mathcal{O}(m\alpha^3)} \hat{u}(\xi))(\alpha(x + m\lambda_1)). \end{aligned}$$

Hence

$$\left\| \mathcal{F}^{-1}\left(e^{im(\lambda_0 + \lambda_1 \xi + \frac{1}{2} \xi A \xi + \mathcal{O}(\xi^3))} \alpha^{\frac{d}{p}} \hat{u}\left(\frac{\xi}{\alpha}\right)\right) \right\|_p = \left\| \mathcal{F}^{-1}(e^{im\alpha^2 \xi A \xi + \mathcal{O}(m\alpha^3)} \hat{u}(\xi)) \right\|_p,$$

so that letting $\alpha \searrow 0$ with $m\alpha^2 = m'$ constant, and observing that we have the uniform convergence in compact sets we deduce that $e^{i\frac{\xi A \xi}{2}}$ is Fourier multiplier for all functions with Fourier transform with bounded support, i.e. for all:

$$\|T_{e^{in'\xi A \xi}}\| \leq C.$$

Being A symmetric, we can take

$$A = \text{diag}(\lambda_i),$$

and consider the action on Gaussians

$$\varpi^{-\frac{d}{p}} e^{-\frac{x^2}{2\varpi^2}}.$$

The Fourier transform is

$$\mathcal{F}\left(\varpi^{-\frac{d}{p}} e^{-\frac{x^2}{2\varpi^2}}\right)(\xi) \sim \varpi^{\frac{d}{p'}} e^{-\frac{\varpi^2 \xi^2}{2}},$$

and then the action of the multiplier is

$$\begin{aligned} \mathcal{F}^{-1}\left(e^{i\frac{1}{2}m'\xi A \xi} \varpi^{\frac{d}{p'}} e^{-\frac{\varpi^2 \xi^2}{2}}\right) &= \prod_i \int_{\mathbb{R}} \varpi^{\frac{1}{p'}} e^{i\xi_i x_i - \frac{\xi_i^2}{2}(\varpi^2 - im'\lambda_i)} d\xi_i \\ &= \prod_i \int_{\mathbb{R}} \varpi^{\frac{1}{p'}} e^{-\frac{1}{2}(\xi_i \sqrt{\varpi^2 - im'\lambda_i} - i \frac{x_i}{\sqrt{\lambda^2 - im'\lambda_i}})^2 - \frac{x_i^2}{2(\varpi^2 - im'\lambda_i)}} d\xi_i \\ &\sim \prod_i \frac{\varpi^{\frac{1}{p'}}}{\sqrt{\varpi^2 - im'\lambda_i}} e^{-\frac{x_i^2}{2(\varpi^2 - im'\lambda_i)}}. \end{aligned}$$

Computing the L^p -norm we get

$$\begin{aligned} \|\mathcal{F}^{-1}\left(e^{i\frac{1}{2}m'\xi A \xi} \varpi^{\frac{d}{p'}} e^{-\frac{\varpi^2 \xi^2}{2}}\right)\| &\sim \prod_i \frac{\varpi^{\frac{1}{p'}}}{\sqrt{|\varpi^2 - im'\lambda_i|}} \left(\int e^{-\frac{px_i^2}{2} \frac{\varpi^2}{\varpi^4 + \lambda_i^2}} dx_i \right)^{\frac{1}{p}} \\ &\sim \prod_i \frac{\varpi^{\frac{1}{p'}}}{(\varpi^4 + (m')^2 \lambda_i^2)^{\frac{1}{4}}} \frac{(\varpi^4 + (m')^2 \lambda_i^2)^{\frac{1}{2p}}}{\varpi^{\frac{1}{p}}} \\ &= \prod_i \frac{(\varpi^4 + (m')^2 \lambda_i^2)^{\frac{1}{2p} - \frac{1}{4}}}{\varpi^{\frac{1}{p} - \frac{1}{p'}}}. \end{aligned}$$

If $m' \rightarrow \infty$, then we conclude the the only way of being bounded is either $p = 2$ or $\lambda_i = 0$.

The statement thus becomes that $y \mapsto \lambda_i(y)$ is linear: then

$$\sum_i y_i A_i = \sum_j (p_j \cdot y) r_j(y) \otimes r_j(y) = \sum_i y_i \sum_j p_{ji} r_j(y) \otimes r_j(y),$$

i.e.

$$A_j(y) = p_{ji} r_j(y) \otimes r_j(y)$$

which means they commute, and also $r_i(y)$ is independent on y . □