2. Borel isomorphisms and Polish spaces

In this section we want to reduce several computations in Polish spaces to computations on a G_δ-subset of $\mathbb{R}$.

DEFINITION 2.1. A *Polish space* is a separable complete metric space (X, d).
If $(X_1, d_1), (X_2, d_2)$ are two Polish spaces, a continuous map $\phi : X_1 \to X_2$ such that ϕ is a homeomorphism on $\phi(X_1)$ is an *embedding.*

Note that while the topology of X is determined, there are several equivalent metrics: thus *completion* means that there is a distance d such that (X, d) is complete. Clearly the identity map is an embedding (X, d) into (X, d'), with d, d' equivalent metrics.

DEFINITION 2.2. A set is *clopen* if it is open and close.
The *Borel σ-algebra* is the σ-algebra generated by the topology.

2.1. Extension of continuous functions and embeddings. The property of having a complete metric is a topological property, so we are interested in finding which subsets of a complete metric space are Polish (i.e. they have a complete metric generating the induced topology): a trivial example is closed sets.

LEMMA 2.3. *If $O \subset X$ is open, then there is an equivalent distance d_O such that (O, d_O) is complete.*

PROOF. Just define

$$d_O(x, x') = d(x, x') + \left| \frac{1}{\mathrm{dist}(x, \partial O)} - \frac{1}{\mathrm{dist}(x', \partial O)} \right|.$$

The second term is the inverse of the Euclidean distance by the map $x \mapsto \mathrm{dist}(x, \partial O)$, which is continuous in O. Clearly a Cauchy sequence must have $\frac{1}{d(x_n, \partial O)}$ converging, with means that $d(x, \partial O) > 0$, $x \in O$. □

PROPOSITION 2.4. *A G_δ-subset G of X Polish is Polish.*

PROOF. We have to define a complete metric d' on G leaving the topology the same: set

$$d'(x, x') = \sum_n 10^{-n} \max\{1, d_{O_n}(x, x')\}, \quad G_\delta = \bigcap_n O_n.$$

Then if x_n is Cauchy for d', it follows that x_k is Cauchy for all d_{O_n}, so that in particular it converges to some x which belongs to all O_n because $\frac{1}{d(x_k, O_n)}$ has to converge: this gives that $x \in \bigcap_n O_n = G$. It is immediate to see that the series is converging, because every terms is converging and the sequence is uniformly summable. □

PROPOSITION 2.5. *A continuous function from $A \subset X$ subset of metric space (X, d_X) to a Polish space (Y, d_Y) can be extended to a G_δ-set G.*

PROOF. Define for $x \in A$ the oscillation

$$\mathrm{Osc}(x) = \lim_{r \to 0} \sup\{d_Y(f(x'), f(x')), x', x'' \in A \cap B^X_r(x)\}.$$

This function is defined on $\bar{A}$, and moreover of $x_n \to x$, then

$$\sup\{d_Y(f(x'), f(x')), x', x'' \in A \cap B^X_r(x)\} \geq \sup\{d_Y(f(x'), f(x')), x', x'' \in A \cap B^X_{r/2}(\tilde{x})\} \geq \mathrm{Osc}(\tilde{x})$$

if $d_X(x, \tilde{x}) < \frac{r}{2}$. This means that

$$\mathrm{Osc}(x) \geq \limsup_{\tilde{x} \to x} \mathrm{Osc}(f(\tilde{x})).$$

The map

$$\bar{A} \ni x \mapsto \mathrm{Osc}(x)$$

is then u.s.c., which means that the set

$$G = \{\mathrm{Osc}(x) = 0\} = \bigcap_n \{\mathrm{Osc}(x) < 2^{-n}\}$$

is a G_δ.

By definition the extension of f to G defined by

$$f(x) = \lim_{y \to x} f(y)$$

is well defined and continuous. $\square$

THEOREM 2.6 (Lavrentiev). *If f is an embedding of $A \subset X$ Polish into the Polish space Y, then it can be extended as embedding to a G_δ-set containing A.*

PROOF. The assumptions are that $f(A) = B$ and f is bi-continuous there, with inverse $g = f^{-1}$, $g : B \to A$.

Let $\bar{f}$ be the continuous extension of f to a G_δ-set $A \subset G_A \subset X$. By definition this means that the graph

$$\mathrm{Graph}(\bar{f}) = \big\{(x, \bar{f}(x)), x \in G_A\big\}$$

is closed in the product distance in the set $G_A \times Y$.

Consider now the function $g = f^{-1}$: in the same way it can be extended to a continuous function $\bar{g}$ to a G_δ-set $B \subset G_B \subset Y$ such that $\mathrm{Graph}(\bar{g})$ is closed in $X \times G_B$.

Let now

$$F = \mathrm{Graph}(\bar{f}) \cap \mathrm{Graph}(\bar{g}).$$

This is a closed set in $G_A \times G_B$, and then it is a G_δ set in the product $X \times Y$.

Being $\bar{f}$ continuous on G_δ, it follows that $(\mathrm{id}, \bar{f})^{-1}(F)$ is closed in G_A, in particular it is a G_δ set $\hat{A}$ of X. The same property for $\bar{B} = (g, \mathrm{id})^{-1}(F)$, closed in G_B and thus G_δ-set $\hat{B}$ of Y.

By construction

$$F = \mathrm{Graph}(\bar{f}\llcorner_{\hat{A}}) = \mathrm{Graph}(\bar{g}\llcorner_{\hat{B}}),$$

so that on $\hat{A} \times \hat{B}$ it holds $g = f^{-1}$ and being relatively closed both f, g are continuous. $\square$

COROLLARY 2.7. *The embedding of a Polish space into a Polish space is a G_δ-set. In particular if $A \subset X$ Polish is Polish, then it is a G_δ-set.*

PROOF. By Lavrentiev's Theorem, the embedding : $X \to Y$ can be extended to a G_δ-set G_X containing X, but this can only be X. $\square$

2.2. Emebdding of a Polish space in $([0,1], |\cdot|)$. Among all maps between Polish spaces, we will consider the following one. Let $\{B_n = B_{r_n}(x_n)\}_{n \in \mathbb{N}}$ be a family of open balls generating the topology of X, and define

$$X \ni x \to \phi(x) = \sum_{n=1}^{\infty} 10^{-n} \mathbf{1}_{B_n}(x) \in [0,1] \tag{2.1}$$

LEMMA 2.8. *The map $\phi : X \to [0,1]$ is l.s.c., with continuous inverse. In particular it is a Borel map.*

PROOF. Indeed $x \mapsto \mathbf{1}_O(x)$ is l.s.c. for O open, and by writing

$$\phi(x) = \sup_N \sum_{n=1}^{N} 10^{-n} \mathbf{1}_{B_n}(x)$$

we conclude since the supremum of l.s.c functions is l.s.c..

Clearly given $z \in \phi(X)$, the expansion in decimals of $z \in \mathbb{R}$ determines a unique point $x \in X$ (since $\liminf_n \mathrm{diam}\, B_n \searrow 0$), and moreover if $\phi(x_n) \to \phi(x)$ then $x_n \to x$ for the same reason: if z_n, z coincide in the n-th decimal entry, then $x_n, x \in B_n$. $\square$

In particular, the image of open sets is relatively open in $\phi(X)$, but the counterimage of open sets is only Borel.

PROPOSITION 2.9. *There exists a metric d' on X with a base made of clopen sets such that the Borel σ-algebra $\mathcal{B}(X)$ remains the same and ϕ is an embedding.*

PROOF. If $\{B_n\}_n$ is a base for the topology, define the new distance

$$d'(x,x') = d_\phi(x,x') = |\phi(x) - \phi(x')|,$$

where ϕ is given by (2.1). Clearly this new topology is contained in the Borel σ-algebra, being closed by complementation.

By definition ϕ is a isometry, in particular an isomorphism. $\square$

In particular from the point of view of the structure as a measurable space, X remains invariant, but we have lost the completeness. Let

$$\mathrm{T}_\phi = \phi^{-1}(\mathrm{T}_{\mathbb{R},|\cdot|})$$

be the counterimage of the standard topology on $\mathbb{R}$. This topology is finer than the original one, but the completeness is more related to a *uniformity*, i.e. a way of evaluating how far are couple of points in $X \times X$.

PROPOSITION 2.10. *There exists a metric d'' on X such that (X, d'') is Polish and $\phi : X \to [0,1]$ is an embedding.*

PROOF. Recall the distance used Lemma 2.3

$$d_O(x,x') = d(x,x') + \left| \frac{1}{d(x,\partial O)} - \frac{1}{d(x',\partial O)} \right|.$$

Define for every B_n the distance

$$d_n(x,x') = \begin{cases} \min\{1, d_{B_n}(x,x')\} & x,x' \in B_n, \\ \min\{1, d(x,x')\} & x,x' \notin B_n, \\ 1 & x \in B_n, x' \notin B_n \text{ or } x \notin B_n, x \in B_n. \end{cases}$$

To verify it is a distance, being $\min\{1,\cdot\}$ subadditive, it is enough to check that if $x \in B_n, x' \notin B_n$ then their distance is 1 so that the triangle inequality is verified. It is also complete: indeed a Cauchy sequence x_n must belongs definitely to B_n or to $X \setminus B_n$, and the distances there are complete.

Set now

$$d''(x,x') = d(x,x') + \sum_n 10^{-n} d_n(x,x').$$

Clearly the space is metric separable. Let x_k, $k \in \mathbb{N}$, be a Cauchy sequence for d'': then being d complete there exists x such that $d(x_k,x) \to 0$. Moreover, it follows from the convergence of $d_n(x_k,x) \to 0$ that definitely either x_k belongs to B_n or to $X \setminus B_n$: in particular x is also the d_{B_n}-limit of the sequence x_k. Thus (X, d'') is Polish.

We need to prove that ϕ is an embedding. First one can observe that by construction

$$|\phi(x) - \phi(x')| \leq d''(x,x'),$$

so that the map ϕ is now 1-Lipschitz.

On the other hand, if $\phi(x_k) = y_k \to y = \phi(x)$, then the convergence of the decimals given that for every B_m such that $x \in B_m$ the points x_k belong definitely to B_m. If $x, x_k \in B_m$, let $B_{n'}$ and $r > 0$ be such that

$$B_{m'} + B_r(0) \subset B_m,$$

where we have used the fact that $\{B_n\}_n$ is a base for the topology. Then d_m restricted by $B_{m'} \times B_{m'}$ is equivalent to d, and then we can use Lemma 2.8 to deduce that $x_k \to x$ w.r.t. d_m. If $x \notin B_m$, then also $x_k \notin B_m$, otherwise there will be $\geq 9 \cdot 10^{-m}$ distance in the their images. Hence $x_k \to x$ also in d_m.

In particular each term of the series defining d'' converges, and we thus conclude that ϕ^{-1} is continuous. $\square$

COROLLARY 2.11. *The set $\phi(X)$ is a G_δ-set of $((0,1), |\cdot|)$.*

2.3. Exercises.

(1) Let $(X, \mathcal{B})$ be a measurable space, and assume that $\mathcal{B}$ is countably generated by the family of sets $\{B_n\}_n$ which is separating: for all $x \neq y$ there is $B \in \mathcal{B}$ such that $x \in B$, $y \in X \setminus B$. Define a separable metric d such that the Borel set of the topology are $\mathcal{B}$. (This exercise is just to show that one can play around with a countable family of sets in order to build suitable metric/Polish/Borel structures.)