4. Disintegration Theorem

The Disintegration Theorem can be in some sense interpreted as the inverse of the integration by parts.

Let X, Y be Polish, $m \in \mathcal{P}(Y)$ and

$$Y \ni y \mapsto \mu_y \in \mathcal{P}(X)$$

be a map with the following properties: for all $\phi \in C_b(X)$ it holds

$$y \mapsto \int \phi(x)\mu_y(dx) \quad \text{is } m\text{-measurable.} \tag{4.1}$$

LEMMA 4.1. *The function*

$$C_b(X) \ni \psi \mapsto \int \bigg[\int \psi(x)\mu_y(dx) \bigg] m(dy)$$

defines a Borel probability μ on X, written as

$$\mu = \int \mu_y m(dy) \quad \text{or} \quad \mu_y \otimes m.$$

PROOF. By the Lebesgue Convergence Theorem one deduces immediately that for all A Borel

$$y \mapsto \mu_y(A)$$

is m-measurable. If we use the Borel isomorphism ϕ_X, ϕ_Y, we can thus assume that $X, Y = [0, 1]$. In this case,

$$\psi \mapsto \int \bigg[\int \psi(x)\mu_y(dx) \bigg] m(dy)$$

is clearly a continuous function on $C_b(X)$ by Lebesgue Convergence Theorem, and hence it is a Borel measure. □

REMARK 4.2. By considering

$$y \mapsto (\mu_y, \delta_y) \in \mathcal{P}(X \times Y),$$

we can also define the measure on the product

$$\tilde{\mu} = \int (\mu_y, \delta_y) m(dy).$$

DEFINITION 4.3 (Conditional probabilities). If the measures μ_y are concentrated on a family of disjoint subsets A_y of X, then $\{\mu_y\}_y$ are called the *conditional probabilities of μ w.r.t. the partition A_y*. The measure μ_y are a decomposition of μ w.r.t. the weight m.

The Disintegration Theorem is the inverse of the above construction.

DEFINITION 4.4 (Disintegration). Let $\mu \in \mathcal{P}(X)$ and $f : X \to Y$ be a μ-measurable map. We say that $y \to \mu_y$ is a *(consistent) disintegration of μ w.r.t. f* if

$$\mu = \int \mu_y m(dy), \quad m = f_\sharp m, \quad \mu_y(f^{-1}(\{y\})) = 1 \ m\text{-a.e.}.$$

REMARK 4.5. Note that if we consider the disjoint partition $A_y = f^{-1}(\{y\})$, then there is a hidden regularity here: we are requiring that the quotient map $X/\{A_y\}_y$ is a μ-measurable map with values in a Polish space.

THEOREM 4.6 (Disintegration Theorem). *There exists a disintegration as in Definition 4.4, and it is unique up to m-negligible sets.*

PROOF. We can reduce to $X = Y = [0, 1]$ by the Borel isomorphism.

For all $\psi \in C([0, 1])$ define

$$m_\psi = f_\sharp(\psi\mu).$$

Since $m_\psi \ll m$, by Radon-Nichodym Theorem we can write

$$m_\psi = h_\psi m, \quad \|h_\psi\|_\infty \le \|\psi\|_\infty.$$

It is clear that in $L^\infty(m)$

$$h_{a\psi+b\psi'} = ah_\psi + bh_{\psi'}, \quad \|h_\psi\|_\infty \leq \|\psi\|_{C([0,1])},$$

i.e.

$$C(X) \ni \psi \mapsto h_\psi \in L^\infty$$

is a linear continuous map.

Considering a countable dense sequence ψ_n in $C([0,1])$, we can thus find a set D of full m-measure where pointwise

$$h_{a\psi_n+b\psi_m}(y) = ah_{\psi_n}(y) + bh_{\psi_m}(y), \quad |h_{\psi_n}(y)| \leq \|\psi_n\|_{C([0,1])}.$$

In particular for all $y \in D$ the map $\psi_n \mapsto h_{\psi_n}(y)$ is a linear with norm 1. It thus follows by passing to the limit that the functional

$$h_\psi(y) = \lim_m h_{\psi_n}, \quad \psi = \lim_n \psi_n, \ y \in D,$$

can be used as a representative in the equivalence class of the Radon-Nychodym derivative $\frac{dm_\psi}{dm}$.

The map

$$\psi \to h_\psi(y)$$

is linear and bounded, thus it is a measure:

$$h_\psi(y) = \int \psi(x)\mu_y(dx).$$

Since $h_1(y) = 1$ being μ a probability, we conclude that μ_y is a probability.

We need to prove that μ_y is concentrated on $f^{-1}(y)$. Passing to the limit we obtain that for all Borel sets $A \subset X, B \subset Y$ it holds

$$\mu(A \cap f^{-1}(B)) = \int_B \mu_y(A)m(dy) = \mu_y(A \cap f^{-1}(B))m(dy).$$

In particular

$$\mu(f^{-1}(B)) = m(B) = \int \mu(f^{-1}(B))m(dy).$$

Hence we deduce that

$$\mu_y(f^{-1}(B)) = \mathbf{I}_B(y) \quad m\text{-a.e. } y.$$

If B_n is a base of the topology in Y, then there is a set of full m-measure such that

$$\mu_y(f^{-1}(B_n)) = 1 \quad \Leftrightarrow \quad y \in B_n.$$

Taking $B_n \searrow \{y\}$ we obtain $\mu_y(f^{-1}(\{y\})) = 1$.

In the construction the only tool is Radon-Nychodym theorem, which gives that $\psi \mapsto h_\psi$ is unique, and then this gives that two disintegration must coincide. □

4.0.1. *Exercises.*

(1) Assume that $f : X \to Y$ is any map. Define the σ-algebra

$$M(Y) = \big\{B \subset Y : f^{-1}(B) \in M(X)\big\},$$

where $M(X)$ is the σ-algebra of X. Repeat the disintegration theorem proof to prove that there is a unique decomposition of $\mu \in \mathcal{P}(X)$, but in general $\mu_y(f^{-1}(y)) = 1$ (consistency) cannot be deduced.

(2) Prove that if $f : [0,1] \to [0,1]$ is the quotient map of of the Vitali set, then
- the image σ-algebra is made only of sets of m-measure 1 or 0;
- the unique disintegration is the trivial one.

$$\mathscr{L}^1 = \int \mathscr{L}^1 m(dy).$$

(3) Prove that the map (4.1) is a Borel map when $\mathcal{P}(X)$ is equipped with the distance $\tilde{d}$ of Corollary 3.11.