7. Existence and uniqueness for Sobolev vector fields

We first address the existence problem for the PDE

$$(\rho u)_t + \operatorname{div}(\rho u \mathbf{b}) = 0, \quad 0 \leq \rho,\ \rho, u \in L^\infty(\mathbb{R}^{d+1}). \tag{7.1}$$

In general there may be different ways to prove existence of a solution, the next sections considers a quite general situation.

7.1. Existence of solutions. Assume that

$$\frac{\mathbf{b}}{1+|x|} \in L^1((0,T)\times\mathbb{R}^d)L^\infty((0,T)\times\mathbb{R}^d), \quad \operatorname{div}\mathbf{b} \in L^1((0,T),L^\infty(\mathbb{R}^d)), \tag{7.2}$$

where the first condition means that

$$\mathbf{b} = \mathbf{b}_1 + \mathbf{b}_2, \quad \frac{\mathbf{b}_1}{1+|x|} \in L^1((0,T)\times\mathbb{R}^d),\ \frac{\mathbf{b}_2}{1+|x|} \in L^\infty((0,T)\times\mathbb{R}^d).$$

PROPOSITION 7.1. *Assume that* $\mathbf{b}$ *satisfies* (7.2) *and it is smooth. For every initial data* $\rho u \in L^1 \cap L^\infty(\mathbb{R}^d)$ *there exists a unique solution to the transport equation which is tight.*

PROOF. First of all the compute the Jacobian:

$$\frac{d}{dt}J(t,y) = \operatorname{div}(\mathbf{b}(t,X(t,y))J(t,y),$$

so that from the L^∞-bound on the divergence one obtains

$$J(t,y) \in \left[\frac{1}{\kappa_1},\kappa_2\right], \quad \kappa_1 \leq e^{\int_0^T \|\operatorname{div}\mathbf{b}(t)\|_\infty dt}.$$

Hence the solution remains in L^∞.

Next, let

$$\kappa_2 = \left\|\frac{b_2}{1+|x|}\right\|_\infty,$$

so that

$$\frac{d}{dt}\big(e^{-\kappa_2 t}(1+|X(t,y)|)\big) \leq e^{-\kappa_2 t}\big(|\mathbf{b}| - \kappa_2(1+|X|)\big) \leq e^{-\kappa_2 t}|\mathbf{b}_1|.$$

Compute the measure of trajectories with large displacement:

$$\mathscr{L}^d\big(\big\{\|e^{-\kappa_2 t}(1+|X(t,y)|\|_\infty > R\big\}\big) \leq \frac{1}{R}\int\int_0^T e^{-\kappa_2}|\mathbf{b}_1(t,X(t,y))|dtdy \leq \frac{\kappa_1\|\mathbf{b}_1\|_1}{R}.$$

Hence the measure of trajectories starting in $B_r(0)$ not contained in $B_{e^{\kappa_2 T}(\frac{\kappa_1\|\mathbf{b}_1\|_1}{\epsilon}+r)}(0)$ is less that ϵ, so that the solutions ρu are tight if $\rho_0 u_0$ are. □

By passing to the limit we obtain the following

COROLLARY 7.2. *For every initial data* $\rho_0 u_0 \in L^\infty \cap L^1(\mathbb{R}^d)$ *there exists a solution* $\rho u \in L^\infty \cap L^1((0,T)\times\mathbb{R}^d)$*, and moreover*

$$\rho_0 u_0 \geq 0 \to \rho u \geq 0, \int \rho(t)u(t)\mathscr{L}^d = \int \rho_0 u_0 \mathscr{L}^d.$$

PROOF. The last part is dude to the fact that positive solutions remains positive for $\mathbf{b}$ smooth. □

7.2. Uniqueness of solutions. We will prove the renormalization property of $\mathbf{b} \in L^1((0,T),W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d))$.

THEOREM 7.3. *Assume that*

$$\mathbf{b} \in L^1((0,T),W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d)), \quad i.e.\ \forall r\left(\int_0^T \|\mathbf{b}(t)\|_{W^{1,1}(B_r(0))}dt < \infty\right).$$

Then the renormalization property holds for L^∞*-solutions to* (7.1)*: if* $\rho u \in L^1\cap L^\infty(\mathbb{R}^{d+1})$ *is a solution, then also* $\rho\beta(u)$ *is a solution for all* $\beta \in C^1(\mathbb{R})$ *(and hence also for Lipschitz functions).*

REMARK 7.4. Since we use only local properties, by a cutoff argument the renormalization holds for L^∞_{loc}-solution.

PROOF. We have to prove that if

$$u_t + \mathbf{b} \cdot \nabla u = 0, \quad (u_t + \operatorname{div}(u\mathbf{b}) = u \operatorname{div} \mathbf{b}),$$

then also

$$\beta(u)_t + \mathbf{b} \cdot \nabla \beta(u) = 0, \quad (\beta(u)_t + \operatorname{div}(\mathbf{b}\beta(u)) = \beta(u) \operatorname{div} \mathbf{b}).$$

Let g^ϵ be a space convolution kernel with support in $B_\epsilon(0)$: then for $u^\epsilon = g^\epsilon * u$

$$u^\epsilon_t + \mathbf{b} \cdot \nabla u^\epsilon = \mathbf{b} \cdot \nabla u^\epsilon - (\mathbf{b} \cdot \nabla u)^\epsilon = [\mathbf{b} \cdot \nabla(\cdot), (\cdot)^\epsilon]u. \tag{7.3}$$

the last term is the commutator between the operator $\mathbf{b} \cdot \nabla$ and the convolution: it is explicitly given by

$$\begin{aligned}
[\mathbf{b} \cdot \nabla(\cdot), (\cdot)^\epsilon]u &= \int \mathbf{b}(t,x) \cdot \nabla g^\epsilon(x-y) u(t,y) - \int \mathbf{b}(t,y) \cdot \nabla g^\epsilon(x-y) u(t,y) dy + \int \operatorname{div} \mathbf{b}(t,y) g^\epsilon(x-y) u(t,y) dy \\
&= \int (\mathbf{b}(t,x) - \mathbf{b}(t,y)) \cdot \nabla g^\epsilon(x-y) u(t,y) dy + \int \operatorname{div} \mathbf{b}(t,y) g^\epsilon(x-y) u(t,y) dy \\
&= \int \frac{\mathbf{b}(t,x) - \mathbf{b}(t,x-\epsilon z)}{\epsilon} \cdot \nabla g(z) u(t,x-\epsilon z) dz + \int \operatorname{div} \mathbf{b}(t,y) g^\epsilon(x-y) u(t,y) dy.
\end{aligned}$$

In particular we have that

$$|u_t| \leq \frac{\|u\|_\infty |\mathbf{b}|}{\epsilon} \in L^1_{\mathrm{loc}}(\mathbb{R}^{d+1}),$$

so that $t \mapsto u(t,x)$ is a.c. for a.e. x.

Multiplying (7.3) by $\beta'(u^\epsilon)$ and applying the Chain Rule (here it is valid being u^ϵ smooth in x and a.c. in time)

$$\begin{aligned}
\beta'(u^\epsilon) u^\epsilon_t = \beta(u^\epsilon)_t, &\quad \beta'(u^\epsilon)\mathbf{b} \cdot \nabla u^\epsilon = \mathbf{b} \cdot \nabla \beta(u^\epsilon), \\
\beta(u^\epsilon)_t + \mathbf{b} \cdot \nabla \beta(u^\epsilon) &= \beta(u^\epsilon)[\mathbf{b} \cdot \nabla(\cdot), (\cdot)^\epsilon]u.
\end{aligned} \tag{7.4}$$

Next, as $\epsilon \to 0$ we have for every z fixed

$$\frac{\mathbf{b}(t,x) - \mathbf{b}(t,x-\epsilon z)}{\epsilon} \to_{L^1_{\mathrm{loc}}} \nabla \mathbf{b}(t,x) z, \quad u(t,x-\epsilon z) \to_{L^1_{\mathrm{loc}}} u(t,x),$$

so that a.e. z

$$\frac{\mathbf{b}(t,x) - \mathbf{b}(t,x-\epsilon z)}{\epsilon} \cdot \nabla g(z) u(t,x-\epsilon z) \to (\nabla \mathbf{b}(t,x) z) \cdot \nabla g(z) u(x).$$

Since

$$\begin{aligned}
&\int |\nabla g(z)| \int_{B_r(0)} \left| \frac{\mathbf{b}(t,x) - \mathbf{b}(t,x-\epsilon z)}{\epsilon} - \nabla \mathbf{b}(t,x) z \right| dx dz \\
&\leq \int |z| |\nabla g(z)| \int_{B_r(0)} \int_0^1 |\nabla \mathbf{b}(t, x + (s-1)\epsilon z) - \nabla \mathbf{b}(t,x)| ds dx dz,
\end{aligned}$$

and for all z, s

$$\int_{B_r(0)} |\nabla \mathbf{b}(t, x + (s-1)\epsilon z) - \nabla \mathbf{b}(t,x)| dx \leq 2\|\nabla \mathbf{b}\|_{L^1(B_{x+\epsilon}(0))},$$

$$\lim_{\epsilon \to 0} \int_{B_r(0)} |\nabla \mathbf{b}(t, x + (s-1)\epsilon z) - \nabla \mathbf{b}(t,x)| dx = 0,$$

we obtain that

$$\lim_{\epsilon \to 0} \int |\nabla g(z)| \int_{B_r(0)} \left| \frac{\mathbf{b}(t,x) - \mathbf{b}(t,x-\epsilon z)}{\epsilon} - \nabla \mathbf{b}(t,x) z \right| dx dz = 0,$$

i.e. in L^1 the commutator converges to

$$[\mathbf{b} \cdot \nabla(\cdot), (\cdot)^\epsilon]u \to \int \nabla \mathbf{b}(t,x) z \cdot \nabla g(z) u(t,x) dz + \operatorname{div} \mathbf{b}(t,x) u(t,x),$$

where we have used the strong convergence of convolutions. Integrating by parts the first term

$$\begin{aligned}
\int \nabla \mathbf{b}(t,x) z \cdot \nabla g(z) u(t,x) dz &= u(t,x) \nabla \mathbf{b}(t,x) : \int z \nabla g(z) dz \\
&= u(t,x) \nabla \mathbf{b}(t,x) : (-\mathrm{id}) \\
&= -u(t,x) \operatorname{div} \mathbf{b}(t,x),
\end{aligned}$$

and then the commutator converges strongly to 0. Being $u \in L^\infty$), the same clearly holds for

$$\beta'(u^\epsilon)[\mathbf{b} \cdot \nabla(\cdot), (\cdot)^\epsilon]u,$$

so that we deduce that the r.h.s. of (7.4) goes to 0. The r.h.s. converges distributionally to

$$\beta(u)_t + \operatorname{div}(\mathbf{b}\beta(u)) = \beta(u) \operatorname{div} \mathbf{b},$$

and then the renormalization property holds. □

COROLLARY 7.5. *The renormalization property holds for solutions*

$$u_t + \mathbf{b} \cdot \nabla u = 0, \quad \mathbf{b} \in L^q_{\mathrm{loc}}, \ \nabla \mathbf{b} \in L^p_{\mathrm{loc}}, \ u \in L^{\max\{p',q'\}}_{\mathrm{loc}},$$

where p', q' are the conjugate exponents of p, q respectively.

PROOF. By exercise. □