8. The mixing estimate

Aim of this section is to prove that the unique flow $X(t,y)$ generated by a Sobolev vector field enjoys a weak differentiability property of this form: fixing $t \geq 0$, for every $\epsilon > 0$ there exists Lipschitz function $\tilde{X}_t(y)$ with Lipschitz constant $\sim \frac{\|\nabla \mathbf{b}\|_{W^{1,p}}}{\epsilon}$ coinciding with $X(t,y)$ outside a set of measure ϵ. In particular the flow is a.e. approximatively differentiable.

8.1. Some preliminaries on the Maximal Function.

Definition 8.1. The *Maximal Function of* $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ is defined as

$$M_f(x) = \sup_{0<r<R} \fint_{B_r(x)} |f| \mathscr{L}^d.$$

In general the maximal function depends on R, but clearly

$$\left| \fint_{B_r(x)} |f| \mathscr{L}^d - \inf_{B_{r+h}(x)} |f| \mathscr{L}^d \right| \leq \frac{\|f\|_{L^1(B_{r+h}(x))}}{r^d},$$

so that the dependence on R is Lipschitz outside $r = 0$. Our estimates will be local, so that we will assume R to be sufficiently large.

Lemma 8.2. *It holds*

$$\fint_{B_r(0)} |f(x+y) - f(x)| dy \leq C(d) M_{\nabla f}(x) r.$$

Proof. We have for smooth functions (by translation)

$$\begin{aligned}
\fint_{B_r(0)} |f(y) - f(0)| dy &\leq \fint_{B_r(0)} \int_0^1 |\nabla f(ty) \cdot y| dt dy \\
&= \fint_{B_r(0)} \left| \nabla f(z) \cdot z \right| \int_{|z|}^r \frac{dt}{t} \frac{dz}{t^d} \\
&= \frac{C(d)}{r^d} \int_{B_r(0)} |\nabla f(z)| \left(\frac{1}{|z|^{d-1}} - \frac{|z|}{r^d} \right) dz \\
&\leq C(d) \sum_n \frac{2^{-n}}{r^{d-1}} \fint_{B_{2^{-n}r}(0)} |\nabla f(z)| dz \\
&\leq C(d) M_{|\nabla f|}(0) \sum_n 2^{-n} r \leq C(d) M_{\nabla f}(0) r.
\end{aligned}$$

□

Corollary 8.3. *It holds*

$$|f(x) - f(y)| \leq C(d) \big(M_{\nabla f}(x) + M_{\nabla f}(y) \big) |x - y|.$$

Proof. Just consider for $z \in B_{|x-y|}(x) \cap B_{|x-y|}(y)$

$$\begin{aligned}
|f(x) - f(y)| &\leq \fint_{B_{|x-y|}(x) \cap B_{|x-y|}(y)} \big(|f(x) - f(z)| + |f(z) - f(y)| \big) dz \\
&\leq C(d) M_{\nabla f}(x) |x-y| + C(d) M_{\nabla f}(y) |x-y|.
\end{aligned}$$

□

We recall the following fundamental result: recall that *weak-L^p* is the (Lorenz) space of functions ($L^{p,\infty}$) such that

$$\mathscr{L}^d(\{|f| > a\}) \leq \frac{C}{a^p}.$$

Theorem 8.4. *If $p > 1$ and $f \in L^p_{\mathrm{loc}}$, then $M_f \in L^p_{\mathrm{loc}}$, if $p = 1$ of f is a measure, then M_f belongs to weak-L^1.*

Proof. Consider the set $E_\alpha = \{M_f > \alpha\}$. By definition, for each point in E_α there is a closed ball such that

$$\int_{\bar{B}_r(x)} |f| > \alpha |\bar{B}_r(x)|.$$

By Besicovitch covering theorem, there are at most $C(d)$ countable disjoint families of balls covering the whole set: we thus have

$$\alpha|\{M_f > \alpha\}| \leq \sum_i^{C(d)} \sum_n \alpha |B_{n,i}| \leq C(d) \int |f|.$$

Thus we obtain the weak-L^1 estimate

$$|\{M_f > \alpha\}| \leq \mathcal{O}(1) \frac{\|f\|_1}{\alpha}.$$

Moreover, we can write

$$f_t = |f| \mathbf{1}_{\{|f|>t/2\}},$$

so that

$$⨍_{B_r(x)} |f| \mathscr{L}^d \leq \frac{t}{2} + ⨍_{B_r(x)} |f| \mathbf{1}_{\{|f|>t/2\}} \mathscr{L}^d.$$

We thus deduce that

$$\{M_f > t\} \leq \frac{C(d) \int_{|f|>t/2} |f| dt}{t}.$$

Now if $|f| \in L^p$, then by the formula

$$\|f\|_p^p = \int_0^\infty \frac{t^{p-1}}{p} \mathscr{L}^d(\{|f| > t\}) dt$$

we obtain

$$\begin{aligned}
\|M_f\|_p^p &= \int_0^\infty \frac{t^{p-1}}{p} \mathscr{L}^d(\{M_f > t\}) dt \\
&\leq \frac{C(d)}{p} \int t^{p-2} \int_{|f|>t/2} |f| dx dt \\
&= \frac{C(d)}{p} \int |f| \int_0^{|f|} t^{p-2} = \frac{C(d)}{p(p-1)} \|f\|_p^p.
\end{aligned}$$

This concludes the proof. □

8.2. Mixing estimates for Sobolev vector fields. Assume now that $X(t,x)$ is a flow for $\mathbf{b}$ with the following properties:

(1) $\|b\| \in L^1_{\mathrm{loc}} \cap L^\infty(\mathbb{R}^{d+1})$ smooth, and there is an $L^p + L^\infty$, $p > 1$, function $M(x)$ such that
$$|\mathbf{b}(t,x) - \mathbf{b}(t,y)| \leq \big(M(x) + M(y)\big)|x - y|,$$
(2) $JX(t) \in [\frac{1}{C}, C]$.

We then compute

$$\begin{aligned}
\frac{d}{dt}|X(t,y_1) - X(t,y_2)| &\leq \big|\mathbf{b}(t, X(t,y_1)) - \mathbf{b}(t, X(t,y_2))\big| \\
&\leq \big(M(t, X(t,y_1)) + M(t, X(t,y_2))\big)|X(t,y_1) - X(t,y_2)|,
\end{aligned}$$

$$\frac{d}{dt} \log\left(1 + \frac{|X(t,y_1) - X(t,y_2)|}{|y_1 - y_2|}\right) \leq M(t, X(t,y_1)) + M(t, X(t,y_2)).$$

$$\log\left(1 + \frac{|X(t,y_1) - X(t,y_2)|}{|y_1 - y_2|}\right) \leq \log(2) + \int_0^t \big(M(s, X(s,y_1)) + M(s, X(s,y_2))\big) ds,$$

$$\sup_r ⨍_{B_r(0)} \log\left(1 + \frac{|X(t,y+h) - X(t,y)|}{|h|}\right) dh \leq \log(2) + \int_0^t \big(M(s, X(s,y)) + M_M(s, X(s,y))\big) ds.$$

We have used the estimate

$$⨍_{B_r(0)} M(s, X(s,y+h)) dh \leq ⨍_{B_{Cr}(0)} M(s, X(s,y) + h) dh \leq M_M(s, X(s,y)).$$

Hence

$$\int \sup_r \fint_{B_r(0)} \log\left(1 + \frac{|X(t,y+h) - X(t,y)|}{|h|}\right) dhdy \leq \mathcal{O}(1)(1 + \|M\|_1 + \|M_M\|_1)$$
$$\leq \mathcal{O}(1)(1 + \|M\|_p).$$

being $M \in L^p$ and using Theorem 8.4.

If the same computation is applied to a plan (exercise), then one gets

$$\fint_{B_r(0)} \int \left[\int \log\left(1 + \frac{|\gamma(t) - \gamma'(t)|}{|y|}\right) \eta_x(d\gamma')\eta_{x+y}(\gamma)\right] dxdy \leq \mathcal{O}(1)(1 + \|M\|_p).$$

Letting $r \searrow 0$ we obtain that η_y is a Dirac delta, i.e. a flow. We have proved the following

THEOREM 8.5. *If $\mathbf{b} \in W^{1,p}_{\mathrm{loc}} \cap L^\infty(\mathbb{R}^d)$, then the unique flow $X(t,y)$ satisfies*

$$\int \sup_r \fint_{B_r(0)} \log\left(1 + \frac{|X(t,y+h) - X(t,y)|}{|h|}\right) dhdy < \mathcal{O}(1)(1 + \|\nabla \mathbf{b}\|_p).$$

8.2.1. *Lusin Lipschitz property.* The previous computations shows that the set where

$$\sup_r \frac{1}{|B_r(0)|} \int_{B_r(0)} \log\left(1 + \frac{|X(t,y+h) - X(t,y)|}{|h|}\right) dh > A$$

has measure

$$\mathscr{L}^d\left(\left\{y : \sup_r \frac{1}{|B_r(0)|} \int_{B_r(0)} \log\left(1 + \frac{|X(t,y+h) - X(t,y)|}{|h|}\right) dh > A\right\}\right) < \frac{C(1 + \|\nabla b\|_p)}{A}.$$

Hence, the set of points y such that there exists r with

$$\mathcal{L}^d\left(\left\{h \in B_r(0) : \frac{|X(t,y+h) - X(t,y)|}{|h|} > L\right\}\right) \geq \frac{1}{K}|B_r(0)|$$

is bounded by

$$\mathscr{L}^d\left(\left\{y : \exists r\left(\mathscr{L}^d\left(h \in B_r(0) : \frac{|X(t,y+h) - X(t,y)|}{|h|} > L\right\} \geq \frac{1}{K}|B_r(0)|\right)\right\}\right) \leq \frac{CK(1 + \|\nabla b\|_p)}{\log(1 + L)}.$$

Assume that K is chosen such that

$$|B_r(0) \cap B_r(r\varepsilon_1)| > 2K|B_r(0)|.$$

Then if two points y, y' are in the set

$$R = \left\{y : \exists r\left(\mathscr{L}^d\left(h \in B_r(0) : \frac{|X(t,y+h) - X(t,y)|}{|h|} > L\right\} < \frac{1}{K}|B_r(0)|\right)\right\},$$

in the intersection

$$B_{|y-y'|}(y) \cap B_{|y-y'|}(y')$$

there is a common point y'' such that

$$|X(t,y'') - X(t,y)| \leq L|y'' - y|, \quad |X(t,y'') - X(t,y')| \leq L|y'' - y'|,$$

so that it holds

$$|X(t,y') - X(t,y)| \leq 2L|y' - y|.$$

We thus obtain the following Lusin Lipschitz property:

PROPOSITION 8.6. *By removing a set of measure ϵ, the flow X is Lipschitz with Lipschitz constant*

$$e^{C\frac{\|\nabla \mathbf{b}\|_p}{\epsilon}}.$$

8.3. Mixing. There are two definition of mixing: for vector fields with $\operatorname{div} \mathbf{b} = 0$, $\|b\|_\infty = 1$ (i.e. measure preserving)

Functional mixing, the decrease of the $\mathcal{H}^{-1}$-norm: determine the speed with which the norm converges to 0 for an initial data with 0 means

$$\|\rho(t)\|_{H^{-1}} \to 0;$$

Geometric mixing, the amount of mass inside balls of radius r: the time it takes to mix up to scale ϵ, i.e. indie every ball of radius ϵ the average is $< 1/2$.

The two notions are not equivalent, but usually a vector fields mixing one can be deformed in order to get mixing also for the other.

Exercise 1.
Prove that L^p is contained in weak-L^p, but the opposite is false.