2. Entropy condition

Assume that there exists a uniformly convex *entropy* η: this means that there exists an *entropy flux* $q(u) = (q_1(u), \dots, q_d(u))$ such that

$$Dq_j(u) = D\eta(u)Df_j(u), \quad \partial_\ell q_j = \sum_k \partial_k \eta(u) \partial_\ell f_{kj}(u).$$

This condition follows by requiring that the additional conservation laws holds

$$\eta_t + \operatorname{div} q = 0,$$

$$\begin{aligned} 0 &= \sum_k \partial_k \eta \partial_t u_k + \sum_{j\ell} \partial_\ell q_j \partial_j u_\ell \\ &= -\sum_k \partial_k \eta \sum_{j\ell} \partial_\ell F_{kj} \partial_j u_\ell + \sum_{j\ell} \partial_\ell q_j \partial_j u_\ell \\ &= \sum_{j\ell} \left(\partial_\ell q_j - \sum_k \partial_k \eta \partial_\ell f_{kj} \right) \partial_j u_\ell. \end{aligned}$$

We have proved the following

LEMMA 2.1. *If η is an entropy (not necessarily convex), then for smooth solutions*

$$\eta_t + \operatorname{div} q = 0.$$

REMARK 2.2. For non smooth solutions the above lemma is false. The *dissipation condition* (which gives also a time direction) is that

$$\eta_t + \operatorname{div} q \leq 0$$

in distributions, is the entropy is dissipated.

LEMMA 2.3. *It holds*

$$D^2 \eta D f_j \quad \textit{is symmetric.}$$

PROOF. Indeed

$$\partial_{\ell m} q_j = \sum_k \partial_{km} \eta \partial_\ell f_{kj} + \sum_k \partial_k \eta \partial_{\ell m} f_{kj}.$$

$$D^2 \eta D f_j = D^2 q_j - D\eta D^2 f_j.$$

Since the r.h.s. is symmetric, then we conclude. □

COROLLARY 2.4. *If there exists a convex entropy, then the system is symmetric hyperbolic with diagonalizing matrix $A_0 = D^2\eta$.*

2.1. Entroy-entropy flux variables.

Define the new unknown

$$v = D\eta(u), \quad u = D\eta^*(v),$$

where η^* is the Legendre transform. Then

$$\begin{aligned} 0 &= \partial_t (D\eta^*(v)) + \sum_j \partial_j f_j(D\eta^*(v)) \\ &= (D\eta^*)_t + \sum_j D f_j D^2 \eta * \partial_j v \\ &= (D\eta^*)_t + \sum_j D^2 \eta^* (D^2 \eta D f_j) D^2 \eta^* \partial_j v \\ &= A_0 v_t + \sum_j A_j \partial_j v, \end{aligned}$$

with $A_0(v) = D^2\eta^*$, $A_j(v)$ symmetric. We have proved the following

LEMMA 2.5. *In the entropy variable $v = D\eta(u)$, the system is written as*

$$A_0 v_t + \sum_j A_j \partial_j v = 0,$$

with A_0, A_j symmetric.