3. Scalar conservation laws

The equation is

$$u_t + \mathrm{div}(f(u)) = 0, \quad u \in \mathbb{R}, f : \mathbb{R} \to \mathbb{R}^d. \tag{3.1}$$

Since the entropy relation

$$\frac{dq_i}{du} = \frac{d\eta}{du}\frac{df_i}{du}$$

has always a solution, we conclude that

Lemma 3.1. *Every convex function η is an entropy for (3.1) with flux*

$$q(u) = \int^u \eta'(u')f'(u')du'.$$

In particular, writing every convex function as whose second derivative has compact support as

$$\eta(u) = \alpha + \beta(u - \bar{u}) + \frac{1}{2}\int_{\mathbb{R}} |u - u'|\eta''(u')du',$$

$$\alpha = \eta(\bar{u}) - \frac{1}{2}\int_{\mathbb{R}} |\bar{u} - u'|\eta''(u')du', \ \beta = \eta'(\bar{u}) - \frac{1}{2}\int_{\mathbb{R}} \mathrm{sign}(\bar{u} - u')\eta''(u')du'',$$

we obtain that

$$\begin{aligned} q'(u) &= \left(\beta + \frac{1}{2}\int \mathrm{sign}(u - u')\eta''(u')du'\right) f'(u) \\ &= \beta f'(u) + \frac{1}{2}\int \mathrm{sign}(u - u')f'(u)\eta''(u')du' \\ &= \beta f'(u) + \frac{1}{2}\int \partial_u[\mathrm{sing}(u - u')(f(u) - f(u'))]\eta''(u')du', \end{aligned}$$

so that

$$q(u) = \beta f(u) + \frac{1}{2}\int_0^u \mathrm{sign}(u - u')(f(u) - f(u'))\eta''(u')du'$$

is an associated flux.

Corollary 3.2. *To test whether a solution is an entropy solution, it is enough to check for the Kruzkhov entropies*

$$\eta_k(u) = |u - k|, \quad q_k(u) = \mathrm{sign}(u - k)(f(u) - f(k)).$$

3.1. Existence of an entropy solution. We consider the parabolic approximation

$$u_t + \mathrm{div}(f(u)) = \epsilon\Delta u, \quad u_0 \in L^\infty. \tag{3.2}$$

By the scaling

$$(t, x) \mapsto \left(\frac{t}{\epsilon}, \frac{x}{\epsilon}\right)$$

the PDE becomes

$$u_t + \mathrm{div}(f(u)) = \Delta u,$$

with initial data

$$u_0(\epsilon x.$$

Lemma 3.3. *It holds*

$$\|u_0(\epsilon\cdot)\|_\infty = \|u_0\|_\infty, \quad \|\nabla u_0(\epsilon\cdot)\|_1 = \|\nabla u_0\|_1.$$

The proof is elementary. The above lemma gives the quantities which are invariant for the scaling, so that we expect that $\|u(t)\|_\infty, \mathrm{Tot.Var.}(u(t))$ will play an important role in the computations for (3.1).

Proposition 3.4. *The solution is $C^{1,k}$ if f is in $C^{2(k+1)}$.*

PROOF. We consider mild solution, i.e.

$$u(t) = G(t-s) * u(s) + \int \nabla G(t-\tau) \cdot f(u(\tau)) d\tau.$$

Recall the estimates for the Heat Kernel

$$G(t,x) = \frac{e^{-\frac{|x|^2}{2t}}}{(2\pi t)^{\frac{d}{2}}}, \quad \nabla G = -\frac{x}{2t(2\pi t)^{\frac{d}{2}}} e^{-\frac{|x|^2}{2t}}, \quad \|\nabla G\|_1 \leq \frac{1}{\sqrt{\pi t}}.$$

We have first

$$\begin{aligned}\|u(t)\|_\infty &\leq \|G(t)\|_1 \|u_0\|_\infty + \int_s^t \|\nabla G(t-s)\|_1 \|f(u(s)) - f(0)\|_\infty ds \\ &\leq \|u_0\|_\infty + \int_0^t \frac{1}{2\sqrt{\pi(t-\tau)}} \|f'\|_\infty \|u(\tau)\|_\infty d\tau,\end{aligned}$$

so that assuming the apriori estimate

$$\|u(t)\|_\infty \leq 2\|u_0\|_\infty$$

we obtain

$$\begin{aligned}&\|u_0\|_\infty + \int_s^t \frac{1}{2\sqrt{\pi(t-\tau)}} \|f'\|_\infty \|u(\tau)\|_\infty d\tau \\ &\leq \|u_0\|_\infty \left(1 + 2\int_s^t \frac{1}{2\sqrt{\pi(t-\tau)}} \|f'\|_\infty d\tau\right) \\ &= \|u_0\|_\infty \left(1 + 2\|f'\|_\infty \sqrt{\frac{t}{\pi}}\right) \leq 2\|u\|_\infty.\end{aligned}$$

if

$$t \leq \frac{\pi}{4\|f'\|_\infty^2}.$$

Hence the L^∞ norm is controlled: the same reasoning can be applied to every initial time where $\|u(s)\|_\infty$ is bounded, giving that it cannot blow up faster than an exponential

$$\|u(t)\|_\infty \leq \mathcal{O}(1) 2^{\frac{4\|f'\|_\infty^2 t}{\pi}}.$$

We repeat for the derivative:

$$\nabla u(t) = \nabla G(t) * u_0 - \int_0^t \nabla G(t-s) * f'(u(s)) \nabla u(s) ds,$$

$$\|\nabla u(t)\|_\infty \leq \|\nabla G(t)\|_1 \|u_0\|_\infty + C_0 \int_0^t \sqrt{t-s} \|\nabla u(s)\|_\infty ds.$$

The constant C_0 depends only linearly on $\|f'\|_\infty$. Using the apriori bound

$$\|\nabla u(s)\|_\infty \leq \frac{C_1 \|u_0\|_\infty}{\sqrt{t-s}},$$

we obtain

$$\begin{aligned}&\|\nabla G(t)\|_1 \|u_0\|_\infty + C_0 \int_0^t \frac{1}{\sqrt{t-s}} \|\nabla u(s)\|_\infty ds \\ &\leq \frac{C_0 \|u_0\|_\infty}{\sqrt{t}} + C_0 C_1 \|u_0\|_\infty \int_0^t \frac{1}{\sqrt{(t-s)s}} ds \\ &\leq \frac{C_1 \|u_0\|_\infty}{\sqrt{t}} \left(\frac{C_0}{C_1} + C_0 \pi \sqrt{t}\right) < \frac{C_1 \|u_0\|_\infty}{\sqrt{t}}\end{aligned}$$

if

$$C_1 = 2C_0, \quad t \leq t \leq \frac{1}{4C_0^2 \pi^2}.$$

Repeating the same starting with the time s, we thus deduce that in the interval where $\|u(s)\|_\infty$ is bounded, obtaining that

$$\|\nabla u(t+\tau)\|_\infty \leq \frac{C_1}{\sqrt{\tau}} \|u(t)\|_\infty.$$

Next, we write the PDE for the derivative

$$(\nabla u)_t + \nabla \operatorname{div}(f(u)) = \Delta \nabla u,$$

whose mild solution is

$$\nabla u(t) = G(t-s) * \nabla u(s) - \int_s^t G(t-\tau) * \nabla(f'(u(\tau))\nabla u(\tau)) d\tau.$$

Hence with the same estimates

$$\nabla^2 u(t) = \nabla G(t) * \nabla u(s) + \int_s^t \nabla G(t-\tau) * \nabla(f'(u(\tau))\nabla u(\tau)) d\tau,$$

$$\begin{aligned}
\|\nabla^2 u(t)\|_\infty &\leq \frac{C_0 \|\nabla u(s)\|_\infty}{\sqrt{t-s}} + C_0 \int_s^t \frac{1}{\sqrt{t-s}} \big(\|\nabla u(\tau)\|_\infty^2 + \|\nabla^2 u(\tau)\|_\infty \big) d\tau \\
&\leq \frac{C_0 \|\nabla u(s)\|_\infty}{\sqrt{t-s}} + C_0 \sqrt{t-s} \|\nabla u\|_\infty^2 + \int_s^t \frac{1}{\sqrt{t-s}} \|\nabla^2 u(\tau)\|_\infty d\tau \\
&\leq \frac{C_1 \|\nabla u(s)\|_\infty}{\sqrt{t-s}},
\end{aligned}$$

with the same apropri assumption

$$\|\nabla^2 u\|_\infty \leq \frac{C_1 \|\nabla u(s)\|_\infty}{\sqrt{t-s}}, \quad C_1 = 2C_0, \quad 0 < t-s \ll 1.$$

Hence we obtain that the solution has bounded second derivative.

We can proceed in this way to obtain the desired regularity, and for th time derivative we can use the PDE to obtain u_t. □

As a corollary we obtain that the maximum principle holds, which in particular shows that the solution is smooth for all times.

COROLLARY 3.5. *The maximum principle holds in a parabolic domain. In particular if* $u_1(0) \leq u_2(0)$ *then* $u_1(t) \leq u_2(t)$.

PROOF. Consider the equation written as

$$u_t = \Delta u + b(t,x) \cdot \nabla u$$

It is well known that the maximum principle holds for the above PDE for C^2 functions, which implies that a local maximum/minimum cannot belong to any interior point of $\mathbb{R}^+ \times \mathbb{R}^d$: in particular

$$\|u(t)\|_\infty \leq \|u_0\|_\infty.$$ □

Clearly by scaling back the estimates becomes

$$\|\nabla^k u(t)\|_\infty \leq \frac{C\|u\|_\infty}{\epsilon^k},$$

so they are not uniform w.r.t. to ϵ. However the L^∞-bound is preserved.

The next proposition shows that also the total variation is preserved.

PROPOSITION 3.6. *If* u_1, u_2 *are two solutions then for* $s < t$

$$\|u_1(t) - u_2(t)\|_1 \leq \|u_1(s) - u_2(s)\|_1.$$

In particular the TV of any solution is decreasing.

Proof. We can assume that $s = 0$ and $u_1(0) \leq u_2(0)$, so that

$$\|u_1(t) - u_2(t)\|_1 = \int_{\mathbb{R}^d} (u_2(t,x) - u_1(t,x))dx. \tag{3.3}$$

The difference fo two solutions satisfies

$$\partial_t(u_1 - u_2) + \mathrm{div}(f(u_1) - f(u_2)) = \Delta(u_1 - u_2,$$

which implies that (3.3) is constant. Since

$$\|u_1(0) - u_2(0)\| = \int \big(\max\{u_1, u_2\} - \min\{u_1, u_2\}\big)dx = \int (\tilde{u}_2(0) - \tilde{u}_1(0))dx,$$

then by maximum principle

$$\tilde{u}_1(t,x) \leq u_1(t,x), u_2(t,x) \leq \tilde{u}_2(t,x),$$

and then

$$\|u_1(t) - u_2(t)\| \leq \|\tilde{u}_1(t) - \tilde{u}_2(t)\|_1 = \|u_1(0) - u_2(0)\|_1.$$

This gives the L^1-stability.

In particular, being the PDE invariant for translation we get

$$\int |u(t, x+h) - u(t,x)|dx \leq \int |u(0, x+h) - u(0,x)|dx.$$

Dividing by h and letting $h \to 0$ we obtain

$$\int |\nabla u(t,x) \cdot h|dx \leq \int |\nabla u(0,x) \cdot h|dx,$$

which gives the L^1-estimate

$$\int \sum_i |\partial_i u(t,x)|dx \leq \int \sum_i |\partial_i u(0,x)|dx.$$

Observe also that $v = \partial_j u$ satisfies

$$v_t + \sum_i \partial_{x_i}(f_i'(u)v) = \Delta v,$$

The equation for a convex function $h(v)$ is then

$$\begin{aligned} h(v) + \sum_i \partial_i(f'(u)h(v)) &= \Delta h(v) - h''|\nabla v|^2 + \sum_i \partial_i(f_i h(v)) - \nabla h \cdot \partial_i(f_i'(u)v) \\ &\leq \Delta h(v) + \mathrm{div}(f_i')(h - h'v). \end{aligned}$$

If we take

$$h(v) = \begin{cases} \frac{v^2+\delta^2}{2\delta} & |v| < \delta, \\ |v| & |v| \geq \delta, \end{cases}$$

then

$$h - h'v = \frac{\delta^2 - v^2}{2\delta} \mathbb{I}_{|v|<\delta} \leq \frac{\delta}{2} \mathbb{I}_{|v|<\delta},$$

so that the last term disappears in L^1_{loc}, so that

$$|v|_t + \mathrm{div}(f'|v|) \leq \Delta|v|,$$

in distribution, which is the previous estimate. □

The next estimate gives a uniform in time dependence in L^1 if the total variation is bounded: it is perfect coherent with the conservation law PDE. First we observe that the same estimate for the L^1 norms gives

Corollary 3.7. *It holds*

$$\|\nabla^2 u\|_1 \leq C\|\nabla u\|_1.$$

Proof. Using the same estimate for mild solutions of the equation

$$(\nabla u)_t + \mathrm{div}(f'(u)\nabla u) = \Delta \nabla u$$

$$\begin{aligned}\|\nabla^2 u(t)\|_1 &\le \|\nabla G(t)\|_1 \|\nabla u(0)\|_1 + C_0 \|f'\|_\infty \int_0^t \|\nabla G(t-s)\|_1 \big(\|\nabla u(s)\|_2^2 + \|\nabla^2 u(s)\|_1\big) ds \\ &\le \frac{C_0 \|\nabla u(0)\|_1}{\sqrt{t}} + C_0 \int_0^t \frac{1}{\sqrt{t-s}} \|\nabla u(s)\|_\infty \|\nabla u\|_1 \sqrt{t} + C_0 \int_0^t \frac{1}{\sqrt{t-s}} \|\nabla^2 u(s)\|_1 ds,\end{aligned}$$

so that again the apriori bound

$$\|\nabla^2 u(t)\|_1 \le \frac{C_1 \|\nabla u(0)\|_1}{\sqrt{t}}$$

can be applied. One concludes by using the uniform control on the total variation. □

Lemma 3.8. *If* Tot.Var. $\le C$*, the solutions are uniformly Lipschitz in time.*

Proof. Scaling back the previous estimate

$$\|\nabla^2 u(t)\|_1 \le \frac{C}{\epsilon} \|\nabla u_0\|_1,$$

which gives that

$$\|u_t\|_1 \le \|\epsilon \Delta u\|_1 + \|f'\|_\infty \|\nabla u\|_1 \le C \|\nabla u(0)\|_1.$$ □

This holds for $t > \mathcal{O}(\epsilon)$ in the rescaled variable.

For the time $[0, \epsilon]$ we use the estimate

$$\|\Delta u\|_1 \le \frac{C \|\nabla u_0\|_1}{\epsilon \sqrt{t/\epsilon}},$$

to get

$$\|u(t) - u_0\|_1 \le C \|\nabla u_0\| \sqrt{\frac{t}{\epsilon}}.$$

Clearly the family of functions

$$\begin{cases} \sqrt{\frac{t}{\epsilon}} & 0 \le t < \epsilon, \\ \sqrt{\epsilon} + (t-\epsilon) & t \ge \epsilon. \end{cases}$$

are uniformly continuous, and Lipschitz after ϵ.

We thus have a family of Lipschitz semigroups

$$\mathbb{R}^+ \times L^1 \cap \{\text{Tot.Var.}(u) \le C\} \ni (t, u_0) \mapsto S_t^\epsilon u_0 \in L^1$$

with the property of being 1-Lipschitz for fixed t

$$\|S_t^\epsilon u_0 - S_t^\epsilon v_0\|_1 \le \|u_0 - v_0\|_1,$$

and uniformly continuous in L^1

$$\|S_t^\epsilon u_0 - S_s^\epsilon u_0\|_1 \le \omega(t-s).$$

By Ascoly-Arzelá, there is a limiting subseuqnec converging to a semigroup $S_t u_0$, which is now Lipschitz in t, u_0.

Corollary 3.9. *If* Tot.Var.$(u_0) < C$*, then up to subsequence the semigroup generated by the parabolic approximations converge to a semigroup of entropy solutions to the scalar conservation law:*

$$\|u(t) - v(s)\|_1 \le C \text{Tot.Var.}(u_0)|t-s| + \|u_0 - v_0\|_1.$$

Corollary 3.10. *The semigroup can be extended to a continuous in time semigroup,* 1*-Lipschitz in* L^1*.*

Proof. If $u_{0,n} \to u_0$, the continuous function $t \mapsto u(t)$ is converging uniformly. □

3.2. Entropy estimate. The last part is the uniqueness: we are going to use the Kruzkhov's entropies.

We use the double integration estimate

$$\int\int |u(t,x) - v(t,y)|(\partial_t - \partial_s)\phi(t,s,x,y)dxdt + \int\int \mathrm{sign}(u-u')(f-f')\cdot\nabla\phi \geq 0.$$

Indeed the entropy condition for $u(t,x)$ with $v(s,y)$ as parameter

$$\int |u-v|\phi_t + \mathrm{sign}(u-v)(f(u)-f(v))\cdot\nabla_x\phi dxdt + \int |u_0-v_0|\phi dxdt \geq 0,$$

for $v(s,y)$ with $u(t,x)$ as parameter

$$\int |u-v|\phi_s + \mathrm{sign}(u-v)(f(u)-f(v))\cdot\nabla_y\phi dyds + \int |u_0-v_0|\phi dyds \geq 0,$$

adding

$$\int |u-v|(\phi_t+\phi_s) + \mathrm{sign}(u-v)(f(u)-f(v))\cdot(\nabla_x\phi+\nabla_y\phi)dxdtdyds + \int |u_0-v_0|\phi dxdtdyds \geq 0.$$

Taking

$$\phi(t,x,s,y) = \varsigma(t-s,x-y)\varphi\left(\frac{t+s}{2},\frac{x+y}{2}\right)$$

we obtain

$$\int |u-v|\varphi_\tau + \mathrm{sign}(u-v)(f(u)-f(v))\cdot\nabla_z\varphi)\varsigma dxdtdyds + \int |u_0-v_0|\phi dxdtdyds \geq 0,$$

Since the expression does not contain derivatives of ς, then we can let $\varsigma \to \delta_0(dtdx)$ (it is also better to take ς as a product), and being $u(t)$ continuous in L^1 and translations continuous in L^1 we obtain the weak formulation of

$$\partial_t|u-u'| + \mathrm{div}(\mathrm{sign}(u-u')(f(u)-f(u'))) \leq 0,$$

and integrating we have the L^1 stability estimate for entropy solutions

$$\|u(t)-v(t)\|_1 \leq \|u(s)-v(s)\|_1,$$

which gives uniqueness.

We collect all the results in this section into the following

Theorem 3.11. *For every initial data u_0 in L^∞ there exists a unique entropy solution to the PDE (3.1), which generates a semigroup S_t with the properties*

(1) 1-Lipschitz in L^1:

$$\|S_tu - S_tv\|_1 \leq \|u-v\|_1;$$

(2) continuous in times

$$\lim_{t\to s}\|S_tu - S_su\|_1 = 0;$$

(3) if $\mathrm{Tot.Var.}(u) \leq C$, then

$$\|S_tu - S_su\|_1 \leq C\mathrm{Tot.Var.}(u)|t-s|;$$

(4) it is the limit of the parabolic approximations (3.2).

(5) it has a finite speed of propagation,

$$\int_{B_r(x)} |S_tu - S_tv|\mathscr{L}^d \leq \int_{B_{r+\|f'\|_\infty t}(x)} |u-v|\mathscr{L}^d.$$

Proof. The only missing part are the finite speed of propagation and the limit of the parabolic approximations for initial data which are not BV. The first implies the second because we can assume u to have compact support and by taking an approximation of BV function

$$\mathrm{BV}\cap L^\infty \ni u_n(0) \to u(0) \in L^\infty \cap L^1$$

the L^1-stability implies

$$\|u_n^\epsilon(t) - u^\epsilon(t)\|_1 \leq \|u_n(0)-u(0)\|_1,$$

so that $u^\epsilon(t)$ must be a Cauchy sequence because

$$\lim_\epsilon \sup_t \|u_n^\epsilon(t) - u^\epsilon(t)\| = \lim_\epsilon \sup_t \|S_t u_n(0) - u^\epsilon(t)\|_1 \leq \|u_n - u\|_1 \to_n 0.$$

To prove the finite speed of propagation, consider the balance for entropy solutions

$$\partial_t |u - v| + \operatorname{div}(\operatorname{sing}(u - v) f(u) - f(v)) \leq 0,$$

and test with a Lipschitz function of the form

$$h(|x| - Mt) \geq 0.$$

It holds

$$\begin{aligned}\partial_t(|u - v|h) + \operatorname{div}(\operatorname{sing}(u - v)(f(u) - f(v))h) &\leq |u - v|h_t + \operatorname{sing}(u - v)(f(u) - f(v)) \cdot \nabla h \\ &\leq |u - v|h' \bigg(- Mt + \sin(u - v) \frac{f(u) - f(v)}{|u - v|} \cdot \frac{x}{|x|} \bigg).\end{aligned}$$

If

$$M > \|f'\|_\infty,$$

then the r.h.s. is negative, which means that

$$\int |u(t, x) - v(t, x)| h(|x| - Mt) dx \leq \int |u_0(x) - v_0(x)| h(x) dx.$$

Now letting

$$h \to \mathbb{1}_{|x| < R - Mt}$$

we obtain the statement. □