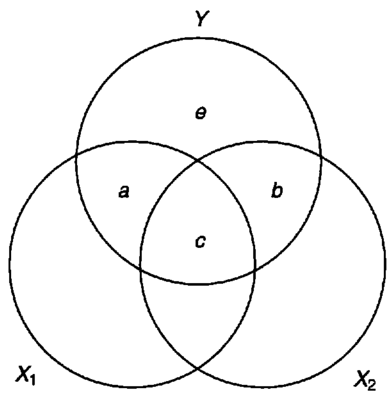
3.3.2 Semipartial Correlation Coefficients and Increments to R^2

One of the important problems that arises in MRC is that of defining the contribution of each IV in the multiple correlation. We shall see that the solution to this problem is not so straightforward as in the case of a single IV, the choice of coefficient depending on the substantive reasoning underlying the exact formulation of the research questions. One answer is provided by the semipartial correlation coefficient sr and its square, sr^2. To understand the meaning of these coefficients, it is useful to consider the "ballantine." Recall that in the diagrammatic representation of Fig. 2.6.1 the variance of each variable was represented by a circle of unit area. The overlapping area of two circles represents their relationship as r^2. With Y and two IVs represented in this way, the total area of Y covered by the X_1 and X_2 circles represents the proportion of Y's variance accounted for by the two IVs, $R^2_{Y.12}$.

Figure 3.3.1 shows that this area is equal to the sum of areas designated a, b, and c. The areas a and b represent those portions of Y overlapped uniquely by IVs X_1 and X_2, respectively, whereas area c represents their simultaneous overlap with Y. The "unique" areas, expressed as proportions of Y variance, are squared semipartial correlation coefficients, and each equals the increase in the squared multiple correlation that occurs when the variable is added to the other IV.[7] Thus

(3.3.7)
$$a = sr_1^2 = R^2_{Y.12} - r^2_{Y2},$$
$$b = sr_2^2 = R^2_{Y.12} - r^2_{Y1}.$$



$$r^2_{Y1} = a + c$$
$$r^2_{Y2} = b + c$$
$$R^2_{Y.12} = a + b + c$$

FIGURE 3.3.1 The ballantine for X_1 and X_2 with Y.

[7] Throughout the remainder of the book, whenever possible without ambiguity, partial coefficients are subscripted by the relevant independent variable only, it being understood that Y is the dependent variable and that all other IVs have been partialed. In this expression (i) indicates that X_i is not included in the variables X_1 to X_k that are being partialed. Thus, $sr_i = r_{Y(i.12\ldots(i)\ldots k)}$, the correlation between Y and X_i from which all other IVs in the set under consideration have been partialed. Similarly, R without subscript refers to $R_{Y.12\ldots k}$.

A formula for sr for the two IV case may be given as a function of zero-order rs as

(3.3.8)
$$sr_1 = \frac{r_{Y1} - r_{Y2}r_{12}}{\sqrt{1 - r_{12}^2}}$$
and
$$sr_2 = \frac{r_{Y2} - r_{Y1}r_{12}}{\sqrt{1 - r_{12}^2}}.$$

For our running example (Table 3.2.1), these values are

$$sr_1 = \frac{.710 - .588(.657)}{\sqrt{1 - .657^2}} = .430,$$
$$sr_1^2 = .1850$$

or, by Eq. (3.3.7)

$$sr_1^2 = .5305 - .3455 = .1850.$$

For X_2,

$$sr_2 = \frac{.588 - .710(.657)}{\sqrt{1 - .657^2}} = .161$$
$$sr_2^2 = .0258,$$

or, by Eq. (3.3.7),

$$sr_2^2 = .5305 - .5047 = .0258.$$

The semipartial correlation sr_1 is the correlation between all of Y and X_1 from which X_2 has been partialed. It is a *semi*partial correlation because the effects of X_2 have been removed from X_1 but not from Y. Recalling that in this system "removing the effect" is equivalent to subtracting from X_1 the X_1 values estimated from X_2, that is, to be working with $X_1 - \hat{X}_{1.2}$, we see that another way to write this relationship is

(3.3.9)
$$sr_1 = r_{Y(X_1 - \hat{X}_{1.2})}.$$

Another notational form of sr_1 used is $r_{Y(1.2)}$, the 1.2 being a shorthand way of expressing "X_1 from which X_2 has been partialed," or $X_1 - \hat{X}_{1.2}$. It is a convenience to use this dot notation to identify which is being partialed from what, particularly in subscripts, and it is employed whenever necessary to avoid ambiguity. Thus $i \cdot j$ means *i from which j is partialed.* Note also that in the literature the term *part* correlation is sometimes used to denote semipartial correlation.

In Table 3.3.1 we present the $X_2 - \hat{X}_{2.1}$ (residual) values for each case in the example in which salary was estimated from publications and time since Ph.D. The correlation between these residual values and Y is seen to equal .4301, which is sr_1; and $.4301^2 = .1850 = sr_1^2$, as before.

To return to the ballantine (Fig. 3.3.1) we see that for our example, area $a = .1850$, $b = .0258$, and $a + b + c = R_{Y.12}^2 = .5305$. It is tempting to calculate c (by $c = R_{Y.12}^2 - sr_1^2 - sr_2^2$) and interpret it as the proportion of Y variance estimated jointly or redundantly by X_1 and X_2. However, any such interpretation runs into a serious catch—there is nothing in the mathematics that prevents c from being a negative value, and a negative proportion of variance hardly makes sense. Because c is not necessarily positive, we forgo interpreting it as a proportion of variance. A discussion of the circumstances in which c is negative is found in Section 3.4. On the other hand, a and b can never be negative and are appropriately considered proportions of variance; each represents the increase in the proportion of Y variance accounted for by the addition of the corresponding variable to the equation estimating Y.

3.3.3 Partial Correlation Coefficients

Another kind of solution to the problem of describing each IV's participation in determining R is given by the *partial* correlation coefficient pr_1, and its square, pr_1^2. The squared partial correlation may be understood best as that proportion of sd_Y^2 *not* associated with X_2 that *is* associated with X_1. Returning to the ballantine (Fig. 3.3.1), we see that

(3.3.10)
$$pr_1^2 = \frac{a}{a+e} = \frac{R_{Y.12}^2 - r_{Y2}^2}{1 - r_{Y2}^2}$$
$$pr_2^2 = \frac{b}{b+e} = \frac{R_{Y.12}^2 - r_{Y1}^2}{1 - r_{Y1}^2}.$$

The a area or numerator for pr_1^2 is the squared semipartial correlation coefficient sr_1^2; however, the base includes not all the variance of Y as in sr_1^2 but only that portion of Y variance that is not associated with X_2, that is, $1 - r_{Y2}^2$. Thus, this squared partial r answers the question, How much of the Y variance that is not estimated by the other IVs in the equation is estimated by this variable? Interchanging X_1 and X_2 (and areas a and b), we similarly interpret pr_2^2. In our faculty salary example, we see that by Eqs. (3.3.10)

$$pr_1^2 = \frac{.5305 - .3455}{1 - .3455} = \frac{.1850}{.6545} = .2826$$

$$pr_2^2 = \frac{.5305 - .5046}{1 - .4312} = \frac{.0259}{.5688} = .0455$$

Obviously, because the denominator cannot be greater than 1, partial correlations will be larger than semipartial correlations, except in the limiting case when other IVs are correlated 0 with Y, in which case $sr = pr$.

pr may be found more directly as a function of zero-order correlations by

(3.3.11)
$$pr_1 = \frac{r_{Y1} - r_{Y2}r_{12}}{\sqrt{1 - r_{Y2}^2}\sqrt{1 - r_{12}^2}}$$
$$pr_2 = \frac{r_{Y2} - r_{Y1}r_{12}}{\sqrt{1 - r_{Y1}^2}\sqrt{1 - r_{12}^2}}.$$

For our example

$$pr_1 = \frac{.710 - .588(.657)}{\sqrt{1 - .3455}\sqrt{1 - .4312}} = .5316$$

and $pr_1^2 = .5316^2 = .2826$, as before;

$$pr_2 = \frac{.588 - .710(.657)}{\sqrt{1 - .5047}\sqrt{1 - .4312}} = .2133$$

and $pr_2^2 = .2133^2 = .0455$, again as before.

In Table 3.3.1 we demonstrate that pr_2 is literally the correlation between X_2 from which X_1 has been partialed (i.e., $X_2 - \hat{X}_{2.1}$) and Y from which X_1 has also been partialed (i.e., $Y - \hat{Y}_1$). Column 6 presents the partialed X_2 values, the residuals from $\hat{X}_{2.1}$. Column 7 presents the residuals from Y_1 (given in column 2). The simple correlation between the residuals in columns 6 and 7 is $.2133 = pr_2$ (the computation is left to the reader, as an exercise). We thus see that the partial correlation for X_2 is literally the correlation between Y and X_2, each similarly residualized from X_1. A frequently employed form of notation to express the partial r is $r_{Y2.1}$, which conveys that X_1 is being partialed from both Y and X_2 (i.e., $r_{(Y.1)(2.1)}$), in contrast to the semipartial r, which is represented as $r_{Y(2.1)}$.

Before leaving Table 3.3.1, the other correlations at the bottom are worth noting. The r of Y with $\hat{Y}_1$ of .710 is identically r_{Y1} and necessarily so, since $\hat{Y}_1$ is a linear transformation of X_1 and therefore must correlate exactly as X_1 does. Similarly, the r of Y with $\hat{Y}_{12}$ of .728 is identically $R_{Y.12}$ and necessarily so, by definition in Eq. (3.3.3). Also, $Y - \hat{Y}_1$ (that is, $Y \cdot X_1$) correlates zero with $\hat{Y}_1$, because when a variable (here X_1) is partialed from another (here Y), the residual will correlate zero with any linear transformation of the partialed variables. Here, $\hat{Y}_1$ is a linear transformation of X_1 (i.e., $\hat{Y}_1 = B_1X_1 + B_0$).

Summarizing the results for the running example, we found $sr_1^2 = .1850$, $pr_1^2 = .2826$ and $sr_2^2 = .0258$, $pr_2^2 = .0522$. Whichever base we use, it is clear that number of publications (X_2) has virtually no *unique* relationship to salary, that is, no relationship beyond what can be accounted for by time since doctorate (X_1). On the other hand, time since doctorate (X_1) is uniquely related to salary (sr_1) and to salary holding publications constant (pr_1) to a quite substantial degree. The reader is reminded that this example is fictitious, and any resemblance to real academic departments, living or dead, is mostly coincidental.

3.4 PATTERNS OF ASSOCIATION BETWEEN *Y* AND TWO INDEPENDENT VARIABLES

A solid grasp of the implications of all possible relationships among one dependent variable and two independent variables is fundamental to understanding and interpreting the various multiple and partial coefficients encountered in MRC. This section is devoted to an exposition of each of these patterns and its distinctive substantive interpretation in actual research.

3.4.1 Direct and Indirect Effects

As we have stated, the regression coefficients $B_{Y1\cdot2}$ and $B_{Y2\cdot1}$ estimate the causal effects of X_1 and X_2 on Y in the causal model given in Fig. 3.4.1, Model A. These coefficients, labeled f and g in the diagram, are actually estimates of the *direct effects* of X_1 and X_2, respectively. Direct effects are exactly what the name implies—causal effects that are not mediated by any other variables in the model. All causes, of course, are mediated by some intervening mechanisms. If such an intervening variable is included, we have Model B shown in Fig. 3.4.1. In this diagram X_1 is shown as having a causal effect on X_2. Both variables have direct effects on Y. However, X_1 also has an *indirect* effect on Y via X_2. Note that the difference between Models A and B is not in the mathematics of the regression coefficients but in the understanding of the substantive causal process.

The advantage of Model B, if it is valid, is that in addition to determining the *direct* effects of X_1 and X_2 on Y, one may estimate the *indirect* effects of X_1 on Y as well as the effect of X_1 on X_2. This latter (h) in Model B is, of course, estimated by the regression coefficient of X_2 on X_1, namely B_{21}. The direct effects, f and g, are the same in both Models A and B and