Nov 21

A topological space is separable if it contains a countable dense set.

$C^0([0,1]) \supseteq \underline{\mathbb{R}[x]}$ is dense.

$\Uparrow$

is separable

<u>Lemma</u> E B space, E' separable $\Rightarrow$ E separable

(E separable $\not\Rightarrow$ E' separable
$L^1(\mathbb{R})$ $\quad L^\infty(\mathbb{R}) = (L^1(\mathbb{R}))'$)

<u>Pf</u> Let $\{f_n\}$ be dense in E' and consider $\{x_n\}$ in E with $\|x_n\|_E = 1$ s.t.
$$\underline{f_n(x_n) \geq \frac{1}{2}\|f_n\|_{E'}}$$

$L = \overline{\text{Span}\{x_n : n \in \mathbb{N}\}}$

L is separable. we will show $L = E$

Suppose $L \subsetneq E \Rightarrow \exists f \in E' \quad f \neq 0$
s.t. $f|_L = 0 \quad f(x_n) = 0 \ \forall n \in \mathbb{N}.$

There exists $f_{n_k} \xrightarrow{k\to+\infty} f$ in E'.

$$\underbrace{\|f_{n_k} - f\|_{E'}}_{\downarrow k\to+\infty \ 0} \geq (f_{n_k} - f)(x_{n_k}) = f_{n_k}(x_{n_k}) \geq \frac{1}{2}\|f_{n_k}\|_{E'}$$

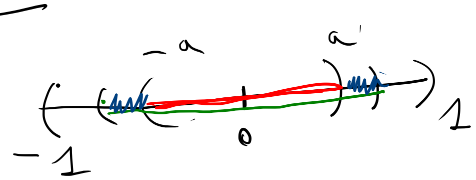
$$\downarrow \quad \frac{1}{2}\|f\|_{E'}$$

$$0 \geq \frac{1}{2}\|f\|_{E'} > 0$$

E B space

<u>Exercise</u> E reflexive and separable $\Leftrightarrow$ E' reflexive and separable.

Lemma $L^\infty((-1,1))$ is non separable



$a > 0$ $\quad \{ 1_{(-a,a)} : a \in (0,1) \}$

$$\left\{ \underbrace{D_{L^\infty((-1,1))}\left(1_{(-a,a)}, \tfrac{1}{2}\right)}_{B_a} \right\}_{a\in(0,1)}$$

there are mutually disjoint ✓

$0 < a < b$ $\quad$ if $\exists\, f \in D_{L^\infty(-1,1)}\left(1_{(-a,a)}, \tfrac12\right) \cap D_{L^\infty(-1,1)}\left(1_{(-b,b)}, \tfrac12\right)$

$$\Rightarrow \quad |1_{(-b,b)} - 1_{(-a,a)}|_{L^\infty} \leq |f - 1_{(-a,a)}|_{L^\infty} + |f - 1_{(-b,b)}|_{L^\infty}$$

$$< \frac12 + \frac12 = 1$$

$$1 = |1_{[a,b)} + 1_{(-b,-a]}|_{L^\infty} < 1$$

If $X \subseteq L^\infty(-1,1)$ is a countable dense subset

$X \cap B_a \neq \phi \quad \forall\, 0 < a < 1$

$X \leftarrow (0,1)$

$X \cap B_a \ni x_a \longleftarrow a$ $\quad$ injective : impossible.

$\dim E = +\infty$

$E \qquad E'$ $\qquad X \subset E'$ dense ~~card X~~ $\operatorname{card} X = \operatorname{card} \mathbb{N}$

$(E, \sigma(E, E'))$ is not metrizable.

If E' is separable then

$(\overline{D_E(0,1)}, \sigma(E, E'))$ is metrizable

$\{|f|\}_{f \in E'}$ $\qquad \{|f|\}_{f \in X}$

Lemma Let $\{x_n\}$ be bounded in a reflexive B space E. Then $\exists \{x_{n_k}\}$ weakly convergent in $\sigma(E,E')$.

Pf Let $F = \overline{\text{span}\{x_n : n \in \mathbb{N}\}}$

F is separable and reflexive

F' $\qquad \|$ $\qquad \|$

$(F, \sigma(F,F'))$ Bounded subspaces of F are metrizable for $\sigma(F,F')$

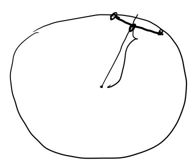
$\Rightarrow \{x_{n_k}\}$ s.t. $x_{n_k} \to \bar{x}$ in $\sigma(F,F')$

same as $(F, \sigma(E,E'))$

Uniformly convex spaces

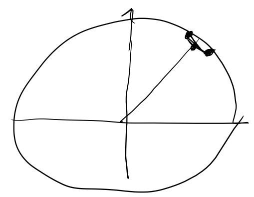
Def A B-space $(E, \|\cdot\|_E)$ is uniformly convex if for any $\varepsilon > 0$ $\exists$ $\delta > 0$ s.t. for any pair $x, y \in \overline{D_E(0,1)}$ the following holds

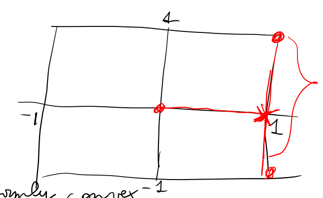
$$\|x-y\|_E \geq \varepsilon \Rightarrow \left\|\frac{x+y}{2}\right\|_E < 1-\delta$$



$\mathbb{R}^2$ $\quad \|(x_1, y_1)\|_2 = \sqrt{x_1^2 + y_1^2}$

$\|(x,y)\|_\infty = \sup\{|x|, |y|\}$





Thm (Milman–Pettis) A uniformly convex B space E is reflexive

Pf Let $\xi \in E''$ $\|\xi\|_{E''} = 1$. We will show that $\xi_0 \in J\,\overline{D_E(0,1)}$

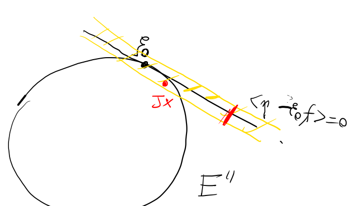
It is enough to prove that

$\forall \varepsilon > 0$ $\exists$ $x \in \overline{D_E(0,1)}$ s.t. $\|\xi_0 - Jx\|_{E''} \leq \varepsilon$

Fix $\varepsilon > 0$ and consider $\delta > 0$.

Let $f \in E'$ s.t. $\|f\|_{E'} = 1$

$\langle \xi_0, f \rangle_{E'' \times E'} > 1 - \frac{\delta}{2}$



$\sigma(E'', E')$

$V = \{\eta \in E'' : |\langle \xi - \eta, f \rangle_{E'' \times E'}| < \frac{\delta}{2}\}$

There exists $x \in D_E(0,1)$ s.t. $Jx \in V$

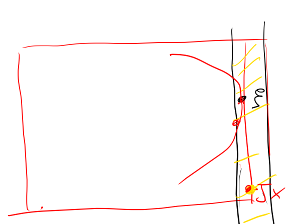
$V \cap D_{E''}(0,1)$ is non empty open set in $D_{E''}(0,1)$

$t \mapsto t\xi_0$ $\quad \mathbb{R} \to E''$

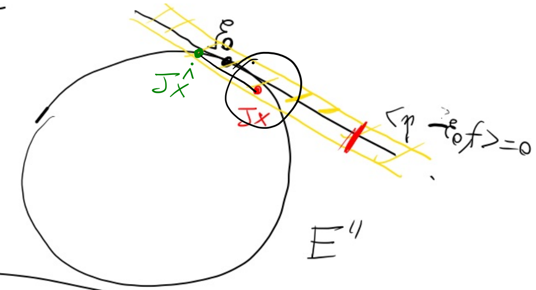
$J\,D_E(0,1)$ is dense in $D_{E''}(0,1)$ for the $\sigma(E'', E')$ topology)

Goldstine

$\to$ x



Let $f \in E'$ s.t. $\|f\|_{E'} = 1$



$\langle \xi_0, f \rangle_{E'' \times E'} > 1 - \frac{\delta}{2}$

$\sigma(E'', E')$

$V = \{ \eta \in E'' : |\langle \xi - \eta, f \rangle_{E'' \times E'}| < \frac{\delta}{2} \}$

There exists $x \in D_E(0,1)$ s.t. $Jx \in V$

Suppose $\xi_0 \notin \overline{Jx + \varepsilon D_{E''}(0,1)}$

Let W be the region outside

W is open for $\sigma(E'', E')$ and $\xi_0 \in V \cap W$

Then $\exists$ $\hat{x} \in D_E(0,1)$ s.t $J\hat{x} \in V \cap W$

Jx and $J\hat{x}$ ~~then~~ $\|x\|_E < 1$, $\|\hat{x}\|_E < 1$ $\|J\hat{x} - Jx\|_{E''} = \|\hat{x} - x\|_E > \varepsilon$

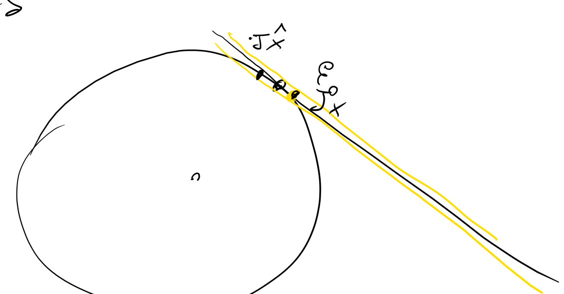
$J \frac{x + \hat{x}}{2} \in V$

$-\frac{\delta}{2} < \langle J \frac{x + \hat{x}}{2}, f \rangle_{E'' \times E'} - \langle \xi_0, f \rangle < \frac{\delta}{2}$

$\underbrace{\langle \xi_0, f \rangle}_{> 1 - \frac{\delta}{2}} - \frac{\delta}{2} < \langle \frac{x + \hat{x}}{2}, f \rangle_{E \times E'}$ $\|x - \hat{x}\|_E > \varepsilon$

$1 - \delta < \langle \frac{x + \hat{x}}{2}, f \rangle_{E \times E'} \leq \left\| \frac{x + \hat{x}}{2} \right\|_E < 1 - \delta$

$1 - \delta < 1 - \delta$



L^p $\qquad (X, \mu)$

$1 \leq p < +\infty$

$L^p(X, d\mu) = \{ f \text{ measurable} : |f|^p \in L^1(X, d\mu) \}$

$L^\infty(X, d\mu) = \{ f \quad '' \quad : \text{ a.e. } |f(x)| \leq C < +\infty \}$

$$\left(\int_X |f(x)|^p d\mu \right)^{\frac{1}{p}}$$

$$\|f\|_{L^\infty(X)} \doteq \sup\{ c \geq 0 : \mu(\{x : |f(x)| \geq c\}) > 0 \}$$

<u>Exercise</u> $f_n \xrightarrow{n \to +\infty} f$ in $L^{1}(0,1)$ ~~$1 \leq p < \infty$~~

then it is not true that $f_n(x) \xrightarrow{n \to +\infty} f(x)$ a.e.

$I_n \qquad \left(\frac{j-1}{n}, \frac{j}{n}\right) \quad j = 1, \dots, n; \quad n \in \mathbb{N}$

$\{ 1_{I_n} \} \qquad 1_{I_n} \longrightarrow 0 \qquad \| 1_{I_n} \|_{L^1} = |I_n| \longrightarrow$