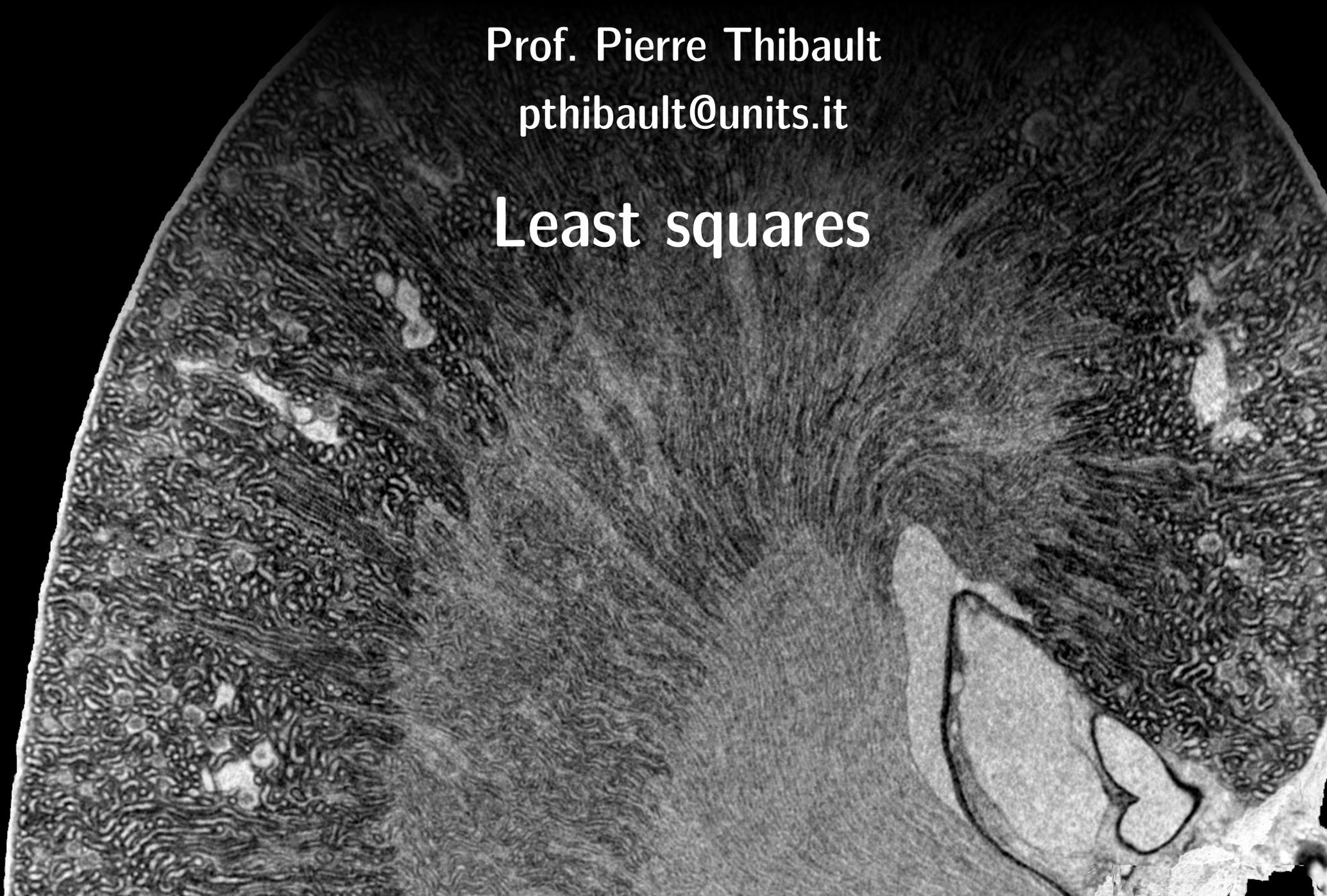


# Image Processing for Physicists

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Least squares



# Overview

- General remarks on optimization
- Least squares principle
  - Application examples
- Lagrange multipliers
  - Application examples

# Image Processing Problems

- Image processing problems can be formulated as linear/nonlinear equations
- In many cases “true” solution does not exist (random noise!) or is hard to calculate (inverse problem)
- Find “best-guess” approximation
- Need understanding of “approximation”
- Need understanding of “best” approximation



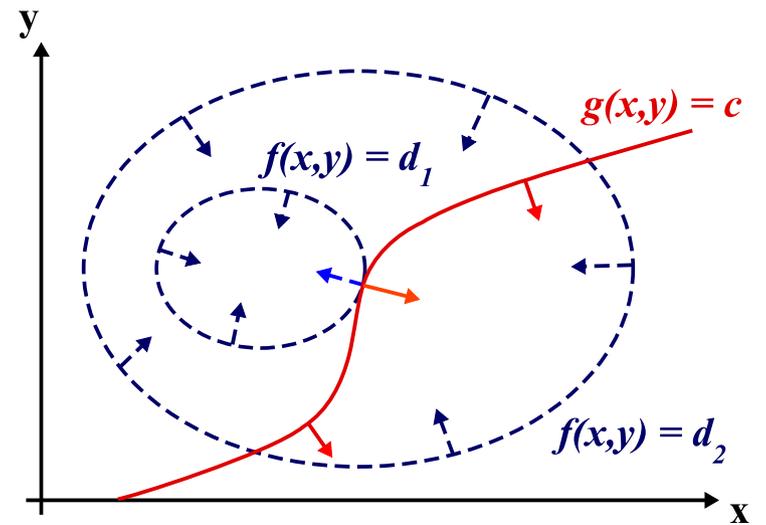
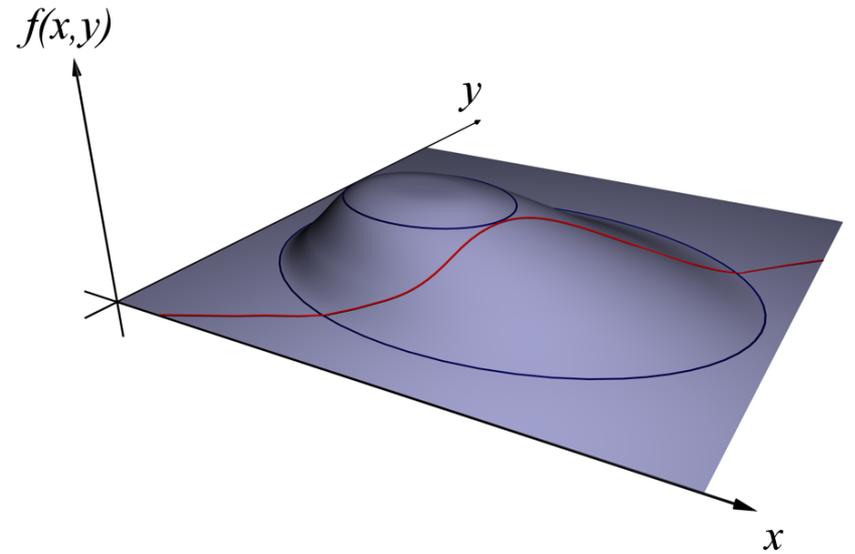


# Optimization

- Find minimum/maximum of objective function (in our case: the cost function)
- Inequality constraints
- Equality constraints
- Standard: minimization problem (negation of maximization problem)

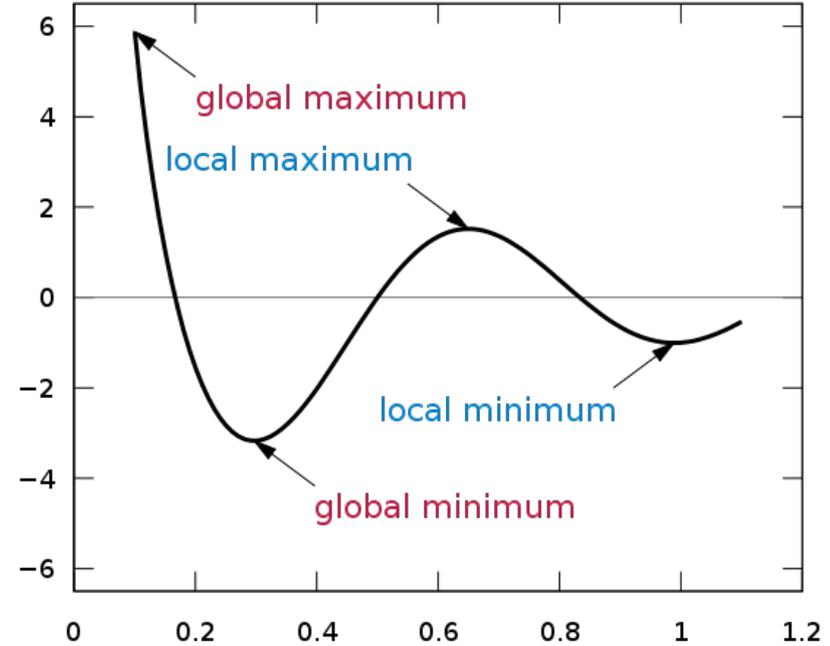
# Lagrange multipliers

- Optimization under equality constraints



# Global/Local Minima/Maxima

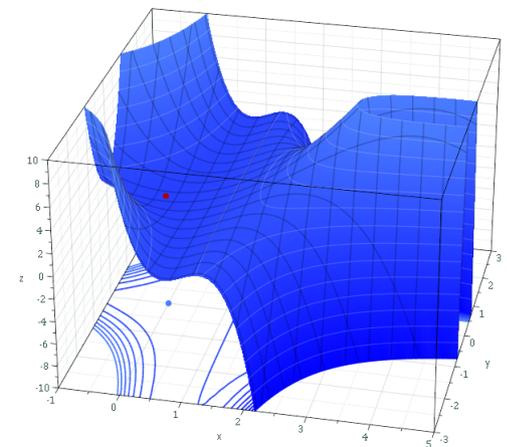
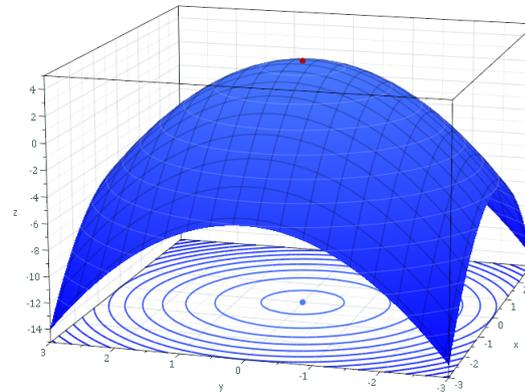
- Find extremal point of function



- Convex problems:

→ local minimum is also global minimum

- All linear problems are convex!



# Linear least squares

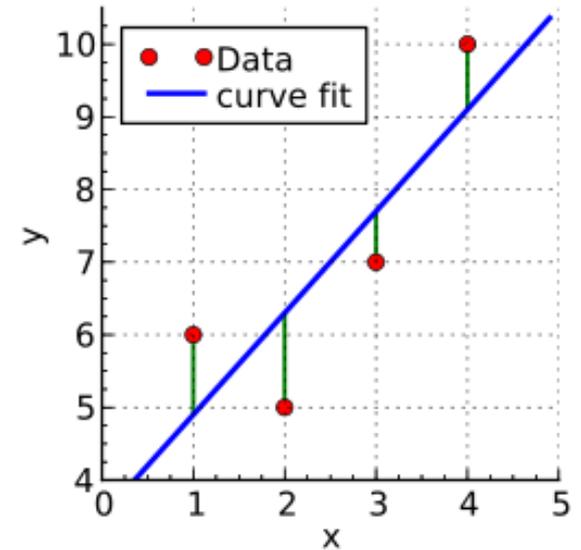
- Problem formulation
- Minimize cost function

# Example: Expectation value

- Given a set of random numbers, find an estimate for the expectation value of the underlying probability distribution

# Example: Linear regression

- Given a set of measurements, find the parameters of a linear regression model

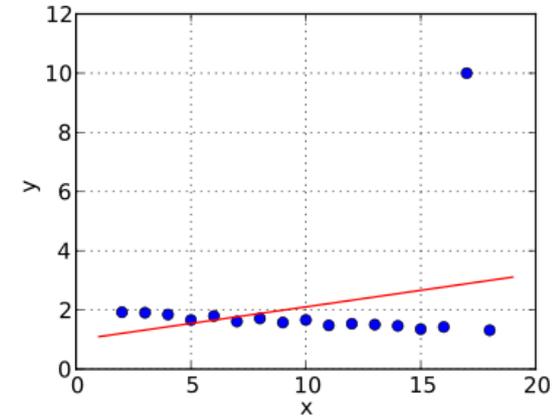


# General linear least squares

# General linear least squares

# Weighted least squares

- Problem: sensitivity to outliers



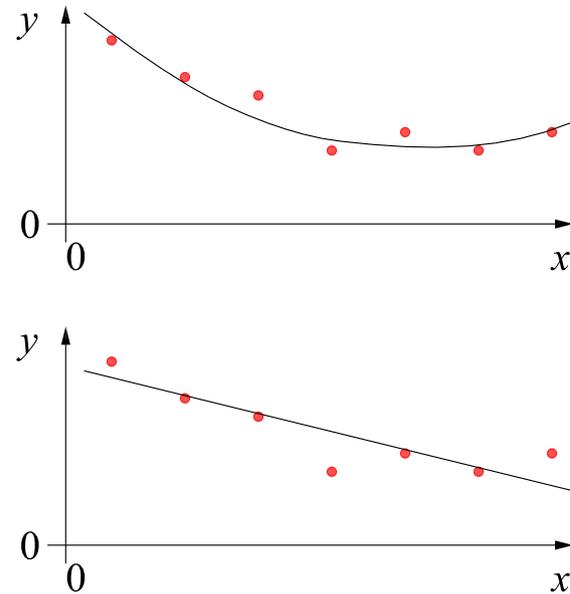
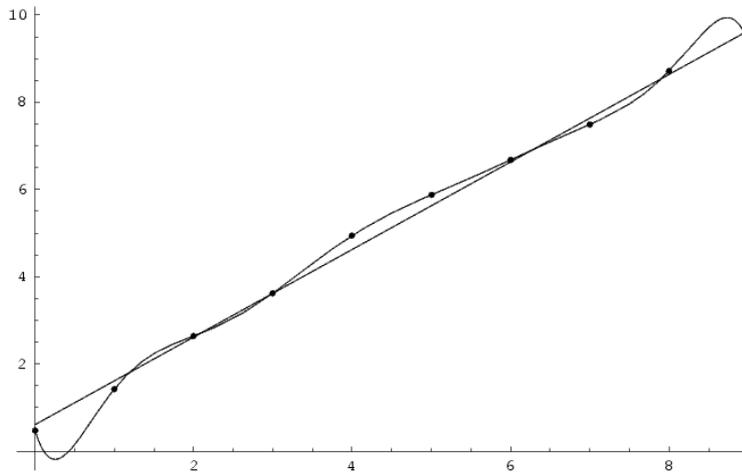
- Solution: penalize problematic values using weights

# Solving least squares problems

- Many approaches to solution exist
  - Pseudo inverse
  - Singular value decomposition (SVD)
  - QR decomposition
  - Iterative methods
  - ...
  
- Choice depends on
  - Robustness
  - Speed
  - Memory consumption
  - ...

# Overfitting & ill-defined problems

- Guess can only be as good as the underlying model
- Too complicated models can lead to too complicated solutions



- Simultaneous optimization of model and its parameters
- Need *regularization*

# Tikhonov Regularization



# Example: Image registration

- Problem formulation: estimate the parameters of a transform s.t. the difference between original and distorted image is minimal

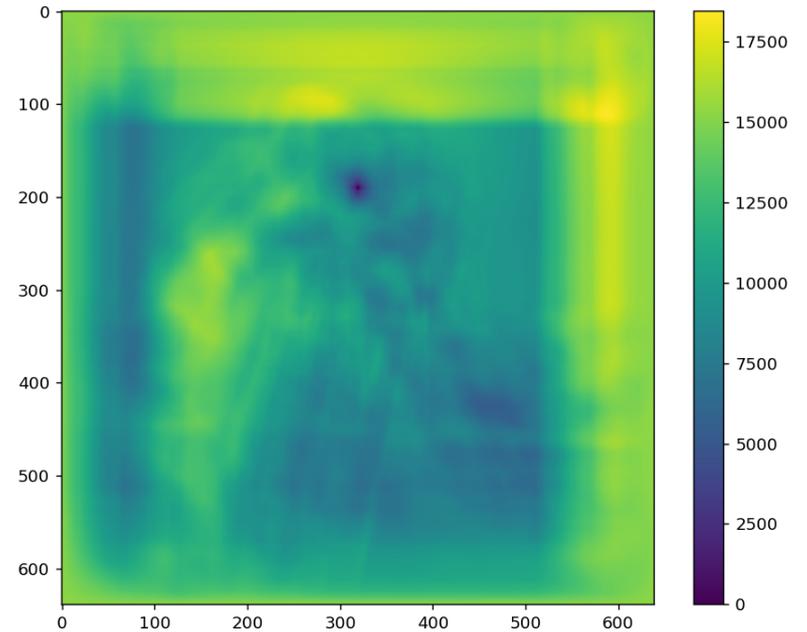
Base image



Template



Distance map

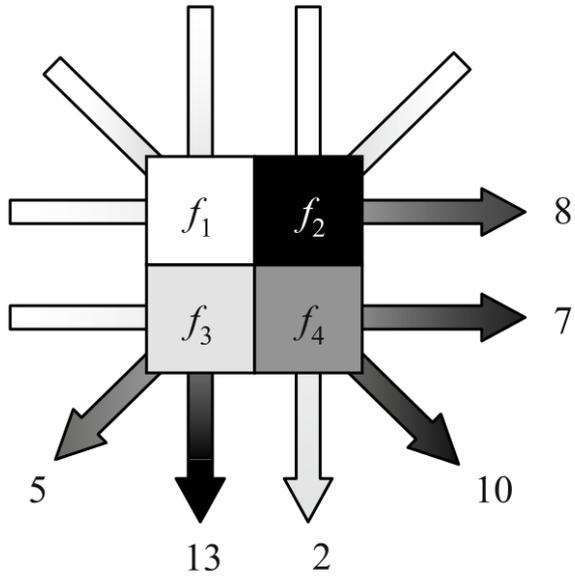


# Example: Image registration

# Iterative solutions

- Gradient-based methods:  
Gradient descent (steepest descent), Stochastic Gradient Descent, Adaptive methods (ADAM, ...)
- Newton and quasi-Newton methods  
Newton's method, Gauss-Newton, Levenberg-Marquardt, Quasi-Newton (BFGS, ...)
- Conjugate Gradients
- Proximal / Operator-splitting  
ADMM, Douglas-Rachford, Primal-Dual, ...
- Derivative-free methods  
Nelder-Mead, Powell, Genetic algorithms, ...

# Tomography revisited



# Algebraic reconstruction techniques

# Summary

- Approximate solutions can be found using estimation
- Approximation quality is quantified by cost function
- Optimum solution is found by minimizing the cost function
- Least square estimator minimizes squared residues
- Lagrange multipliers can be used to implement additional constraints
- Iterative schemes are often required for harder problems