Image Processing for Physicists

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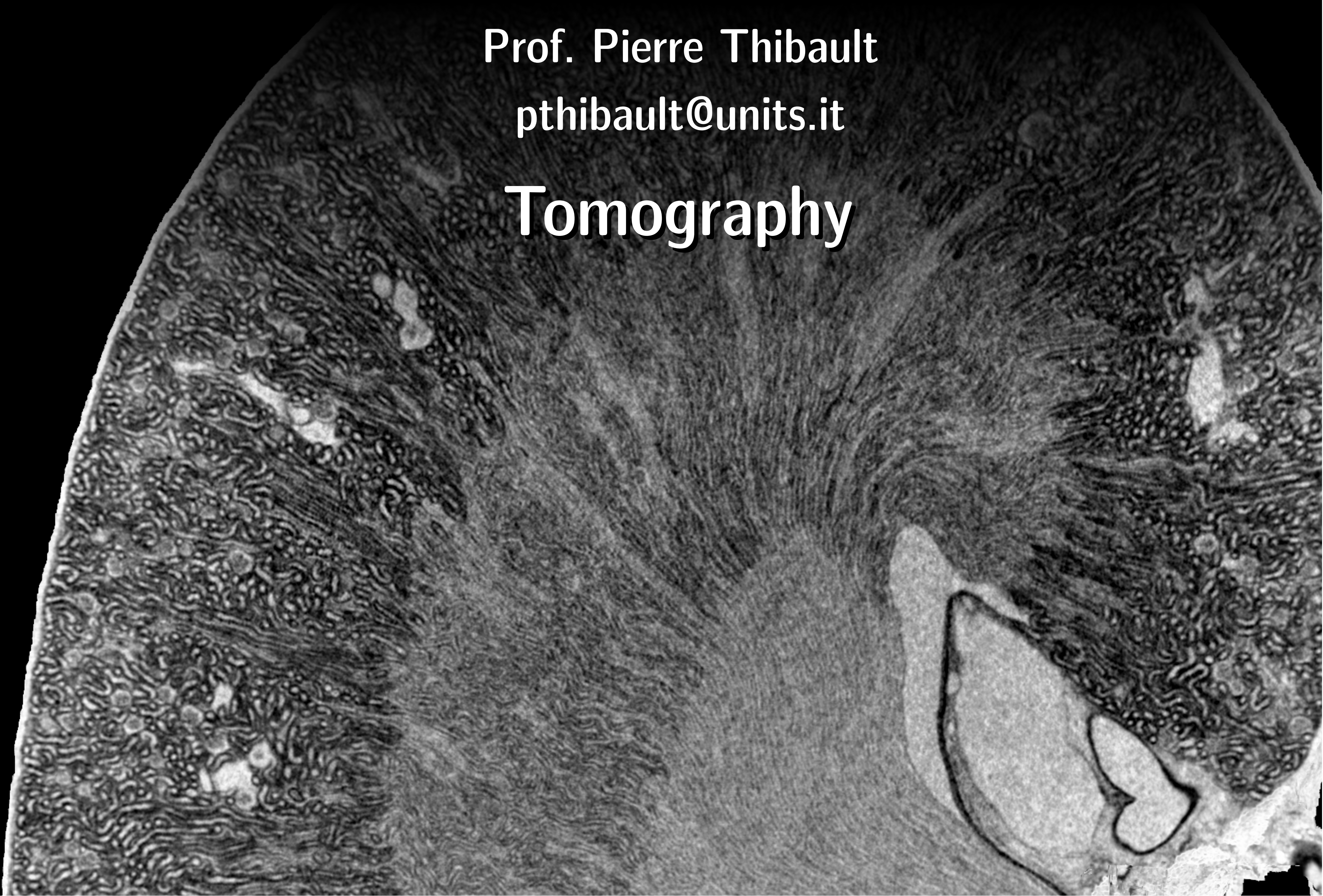
Tomography

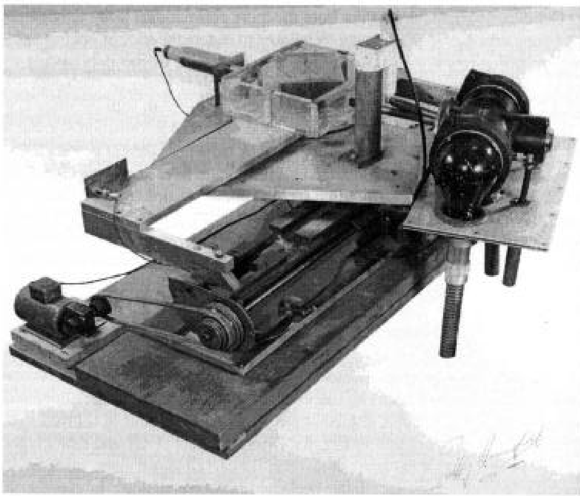


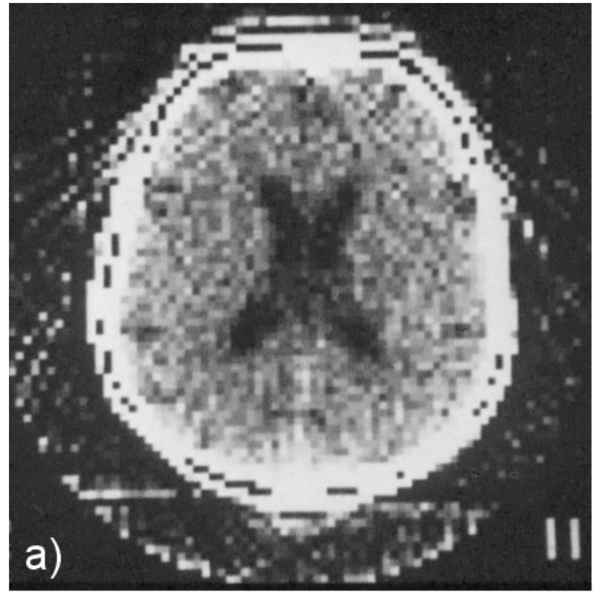
Overview

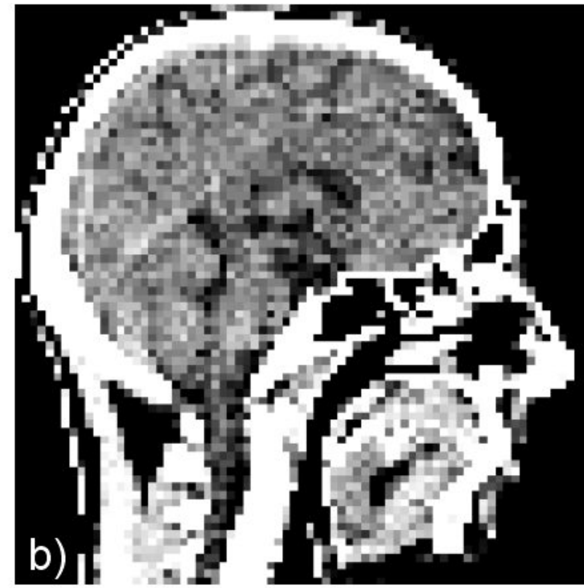
- Fundamentals of tomography
  - Physics & geometry
- Analytic formulation
  - Radon transform
  - Filtered back-projection
- Algebraic formulation
- Some comments on common artifacts and visualization

Examples of tomographic imaging

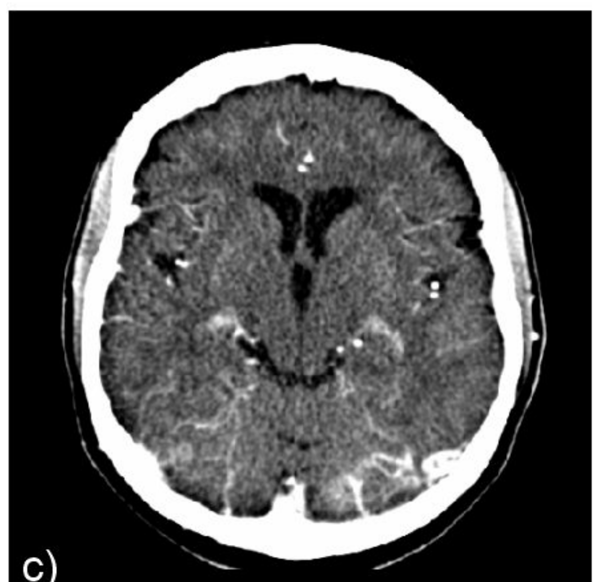
Computed (X-ray) Tomography (CT)

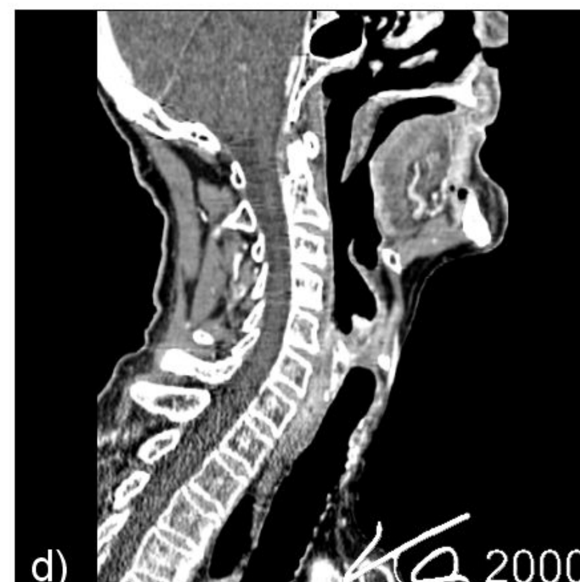






1974, 80x80 pixels



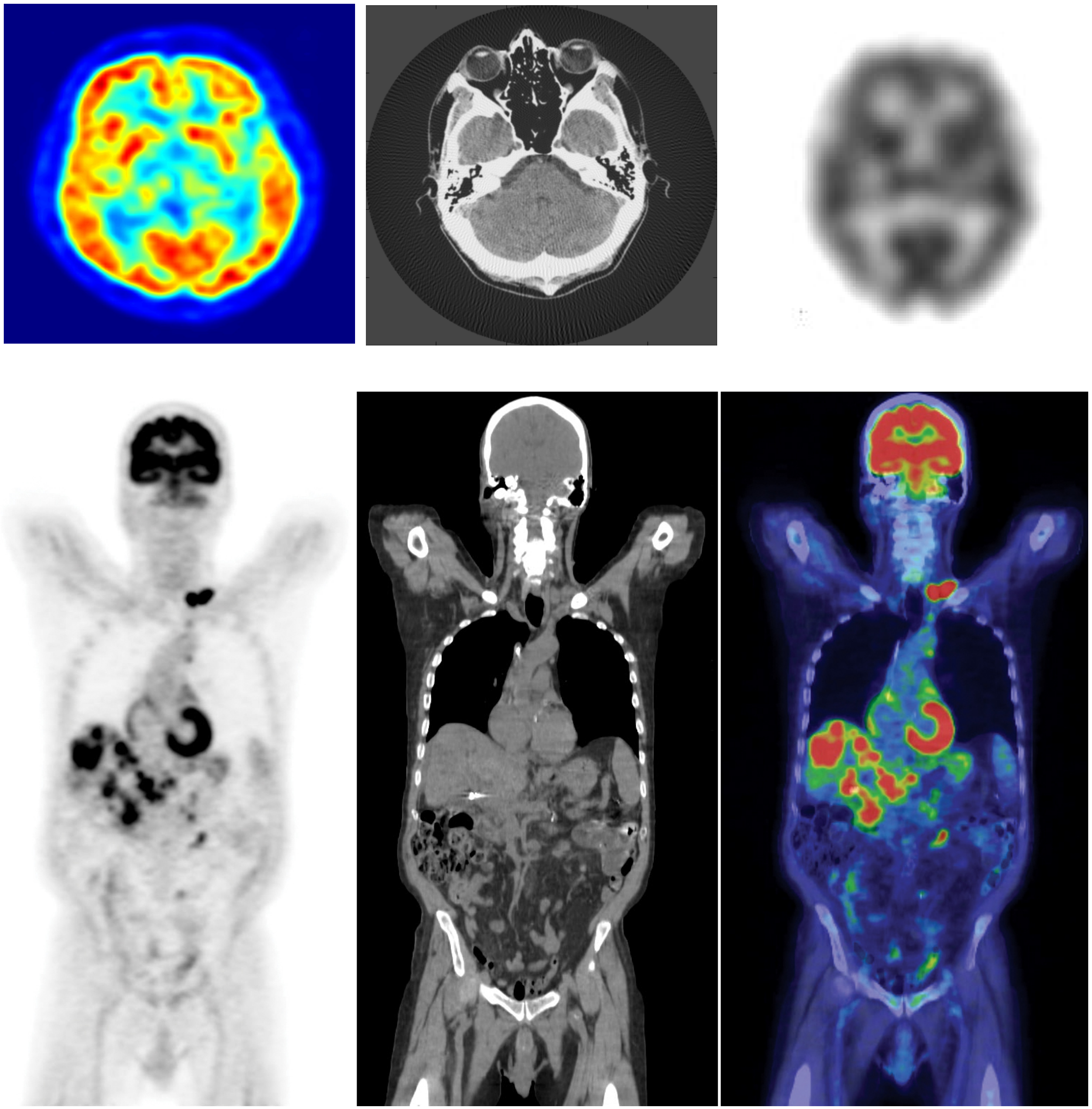


2000, 512x512 pixels, spiral CT

source: W. Kalender, Publicis, 3rd ed. 2011

Examples of tomographic imaging

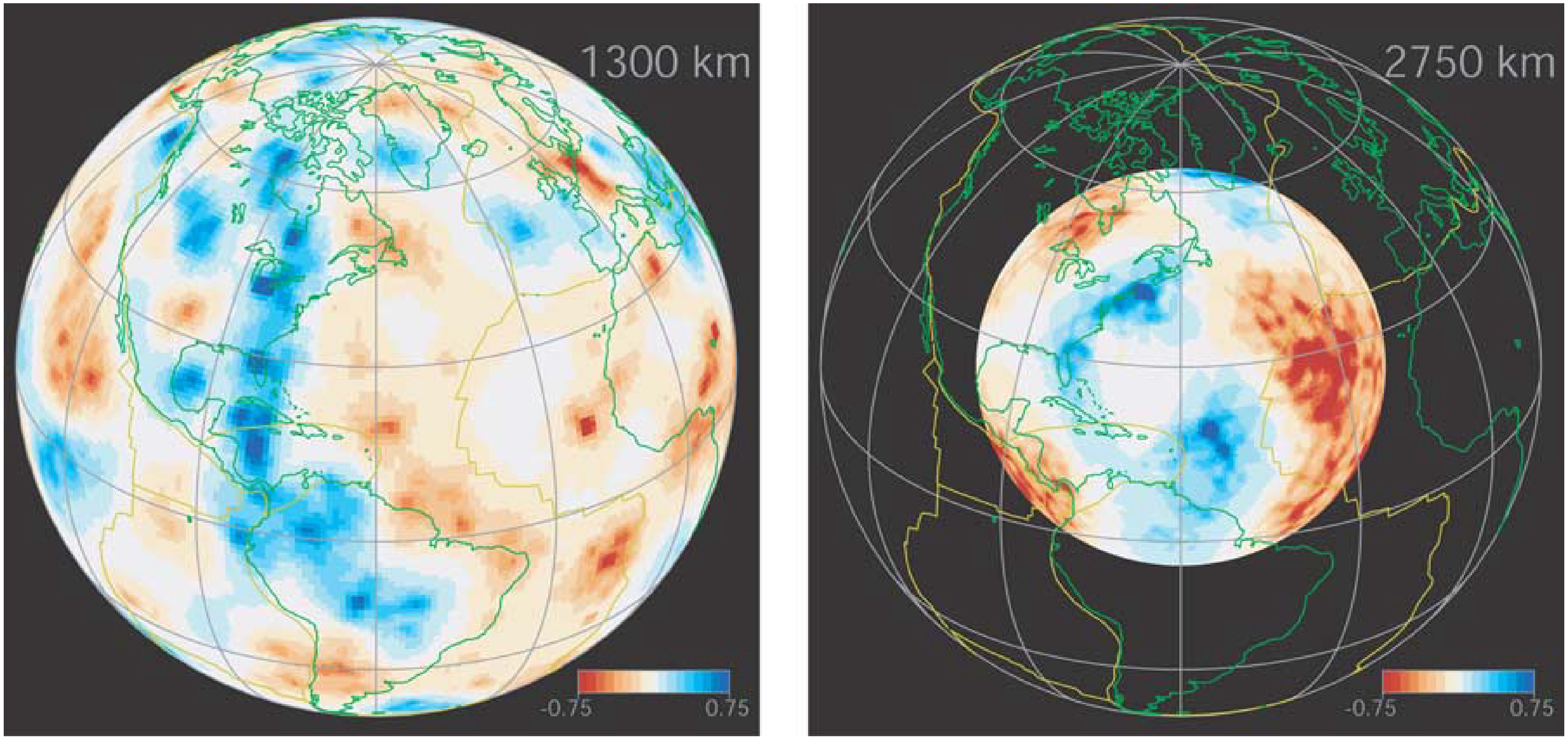
Positron emission tomography (PET) + CT



Single-Photon Emission Computed Tomography (SPECT)

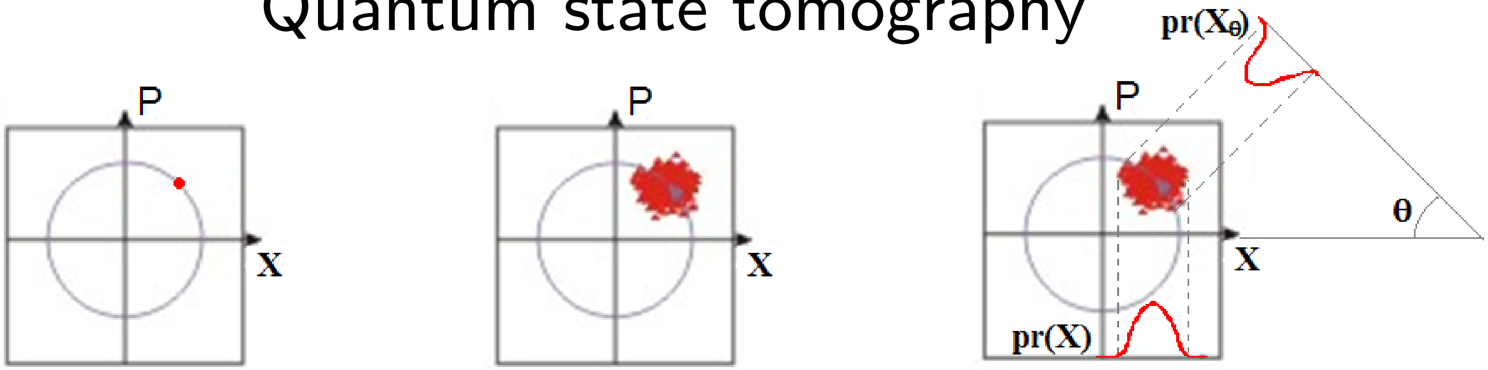
Examples of tomographic imaging

Seismic tomography



source: Sambridge et al. G3 Vol.4 Nr.3 (2003)

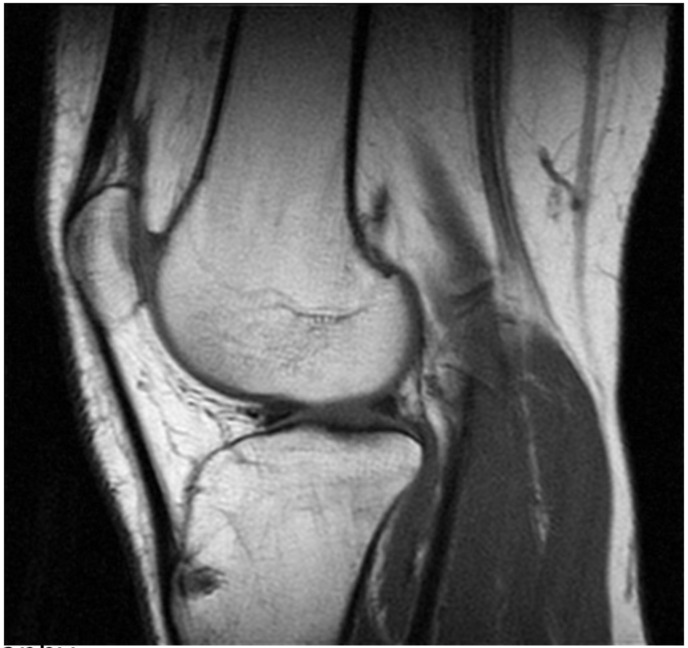
Quantum state tomography

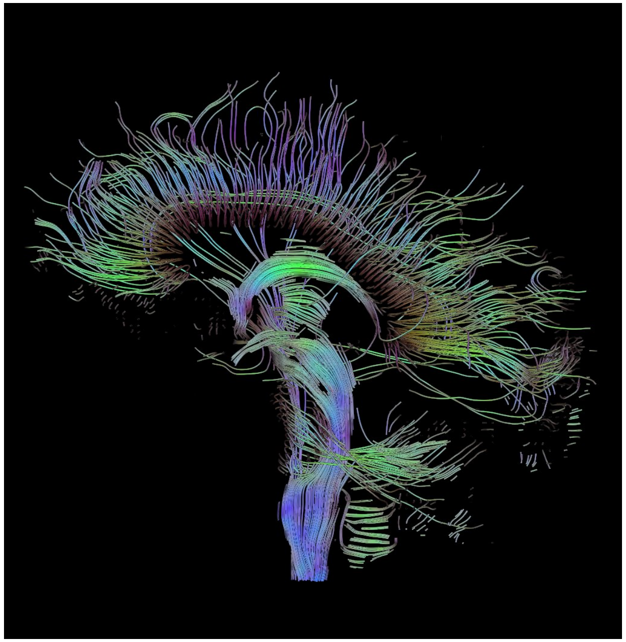


Examples of tomographic imaging

Ultrasonography/tomography (US/UST)

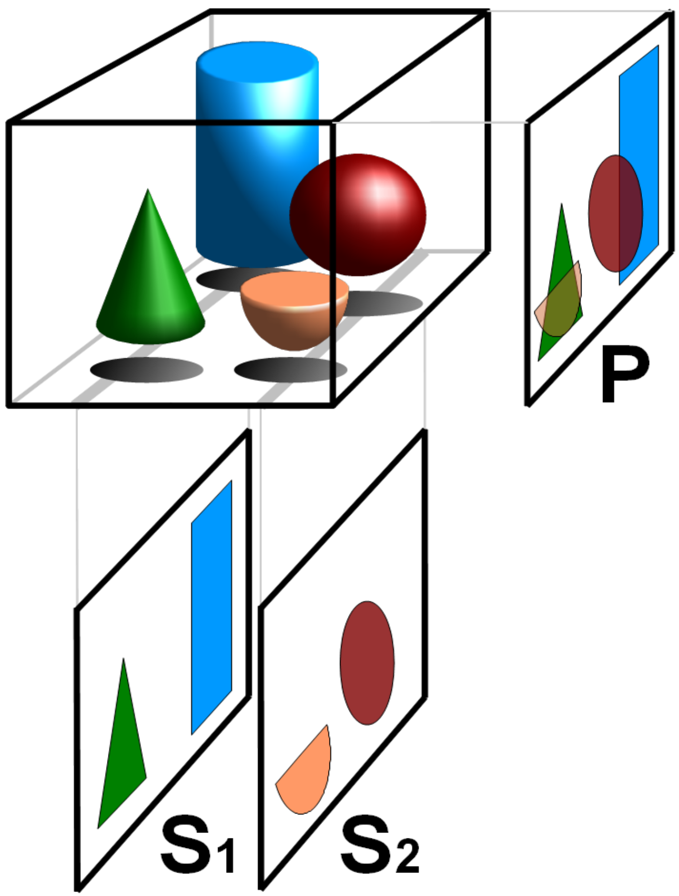
Magnetic resonance imaging/tomography (MRI/MRT)



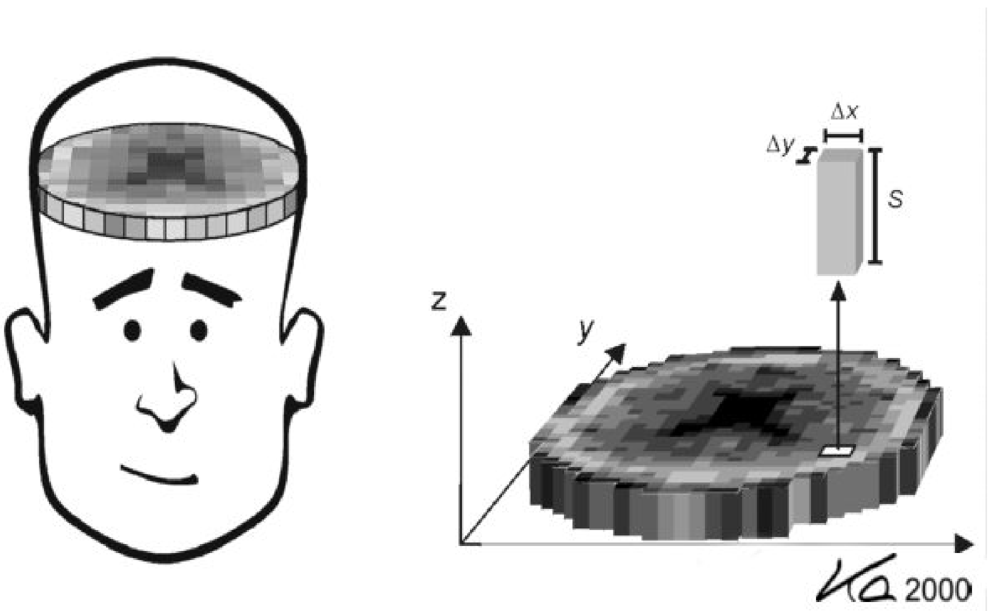


Reconstructions from projections

Reconstruction of volume from projections

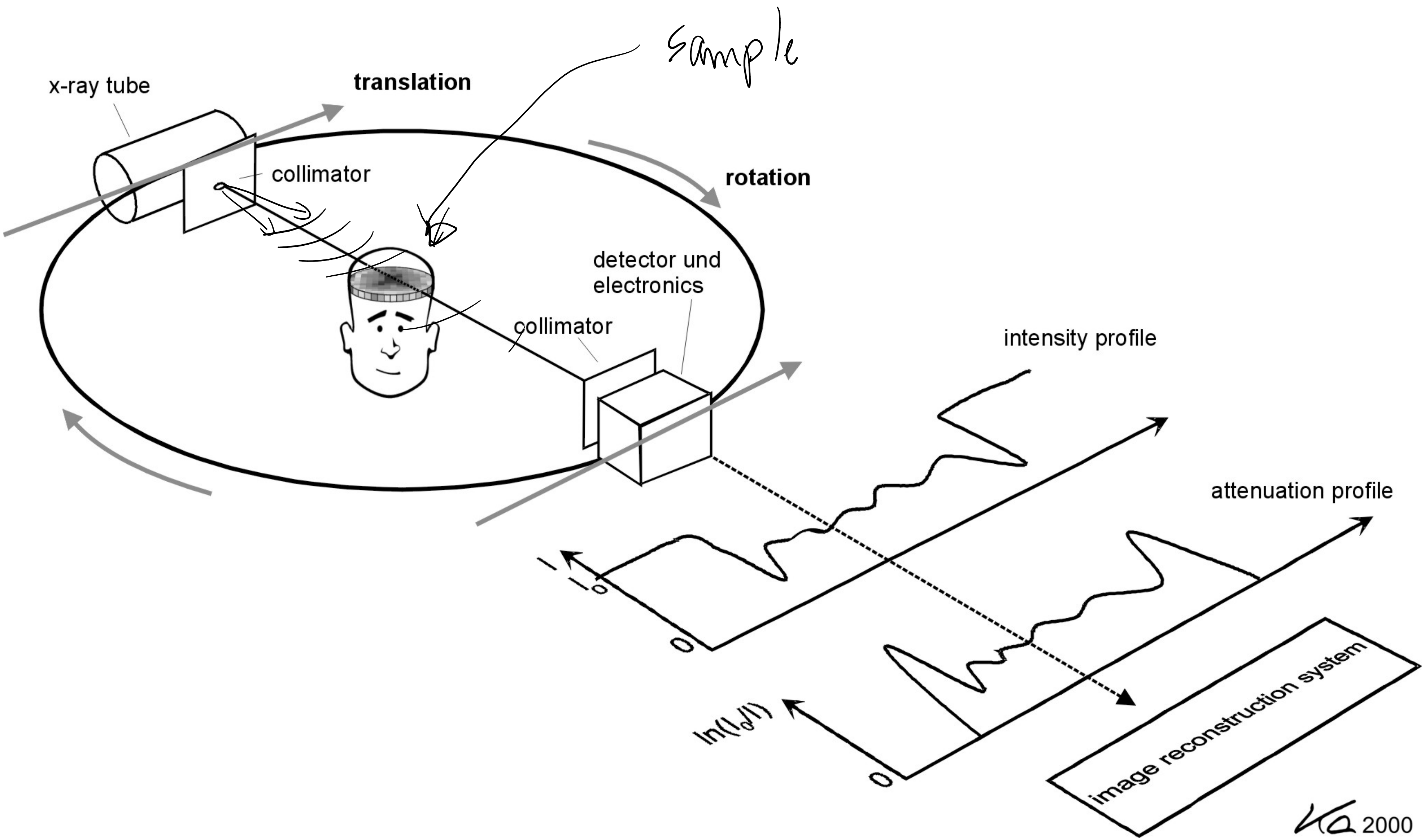


Digitization into voxels



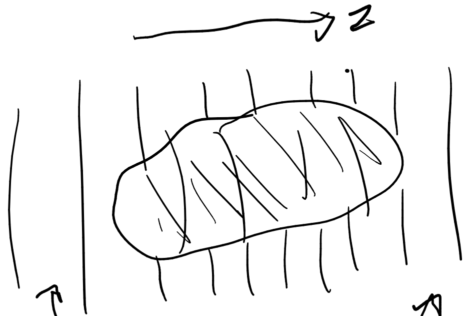
source: W. Kalender, Publicis, 3rd ed. 2011

Principles of X-ray CT



source: W. Kalender, Publicis, 3rd ed. 2011

Beer-Lambert law



X-rays: index of refraction very close to 1

$$n(\vec{r}) = 1 - \delta(\vec{r}) + i\beta(\vec{r})$$

$\vec{r}$: function of space! $\delta(\vec{r})$: real part, $\beta(\vec{r})$: imaginary part

$$\psi = \psi_0 \exp\left(ik\int (n-1)\,dz\right) \leftarrow \text{projection approximation}$$

$$= \psi_0 \exp\left(ik\int \delta\, dz - k\int \beta\, dz\right)$$

$$I = |\psi|^2 = |\psi_0|^2 \underbrace{\exp\left(-2k\int \beta\, dz\right)}_{\text{attenuation}}$$

$2k\beta = \mu$: linear attenuation coefficient

$$I = I_0 e^{-\int \mu\, dz} \leftarrow \text{Beer-Lambert law}$$

$$\int \mu\, dz = -\ln\left(I/I_0\right)$$

Radon transform

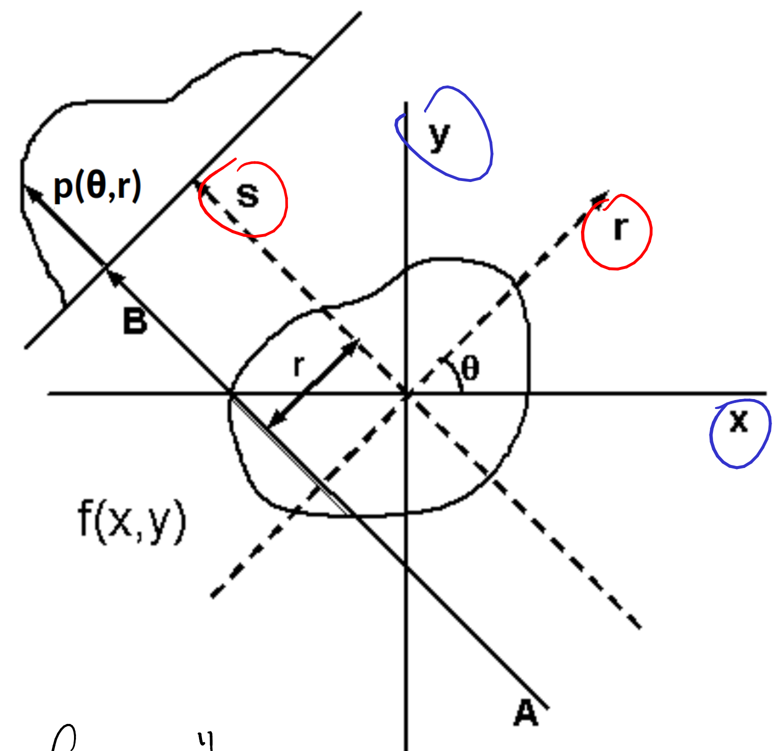
$\int dz$ becomes $\int ds$

Rotated coordinate system

$$\begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Radon transform

$$p(\theta, r) = \int ds\, f(x = r\cos\theta - s\sin\theta,\ y = s\cos\theta + r\sin\theta)$$



Radon transform is linear

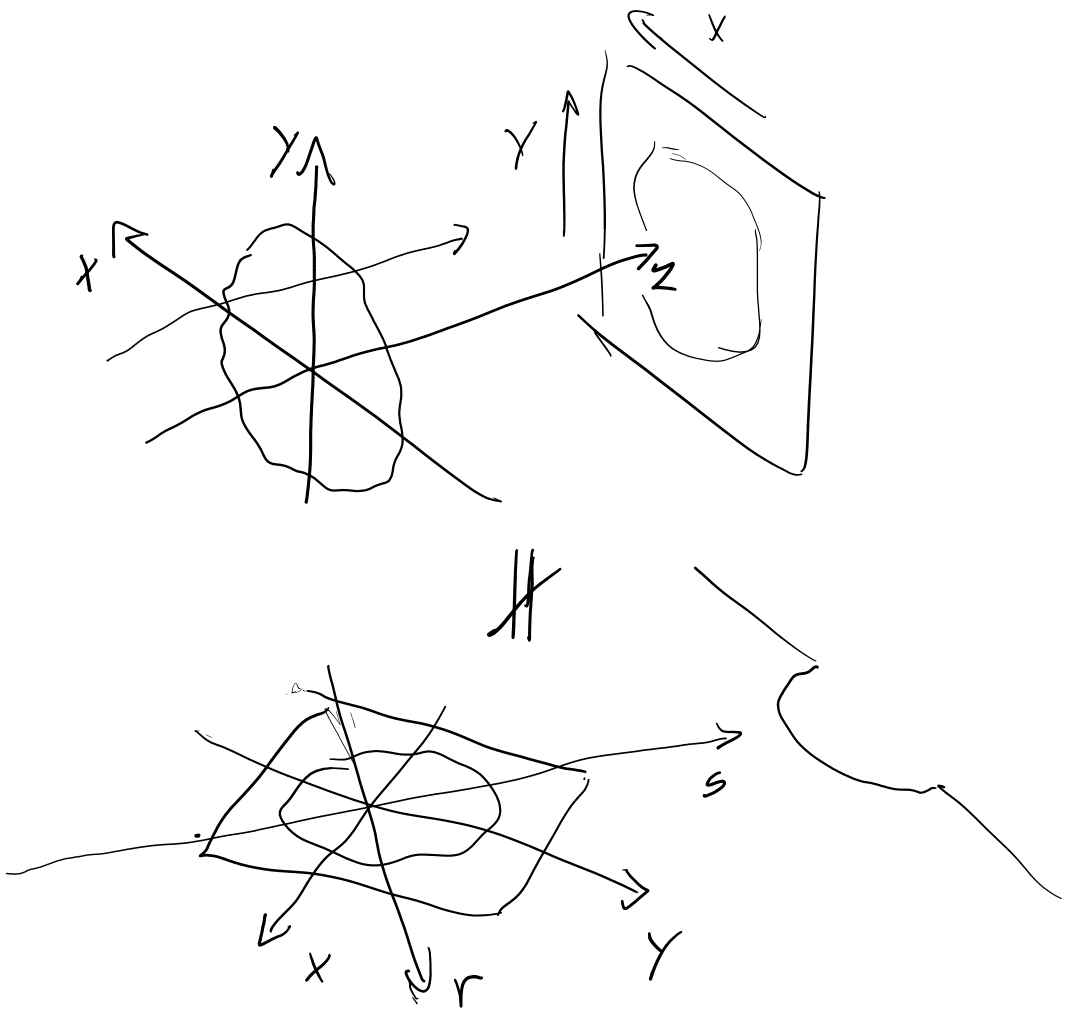
problem: recover f from p
"inverse Radon transform"

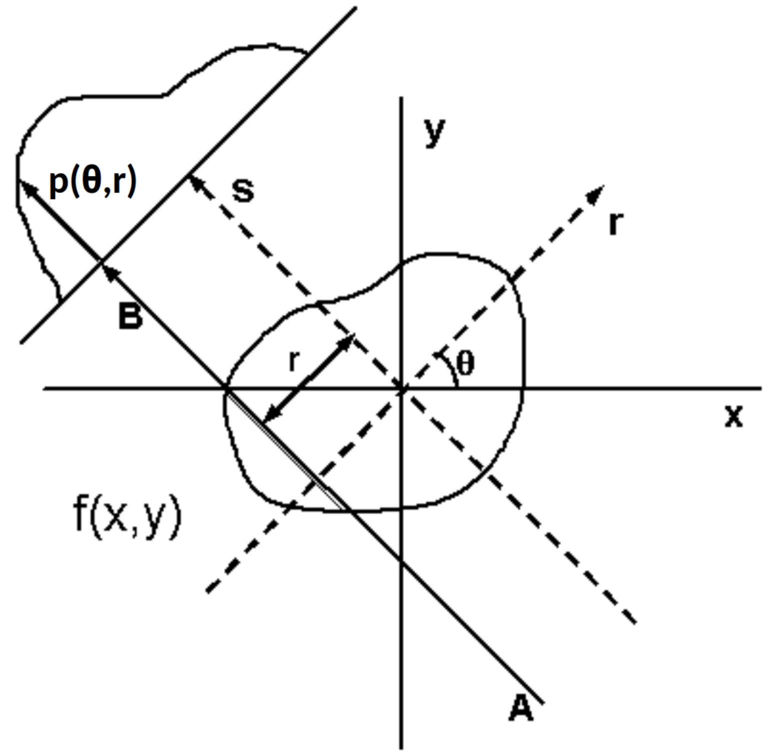
Warning: $(\theta, r) \neq$ polar coordinates (r can be negative)

Radon transform

Rotated coordinate system

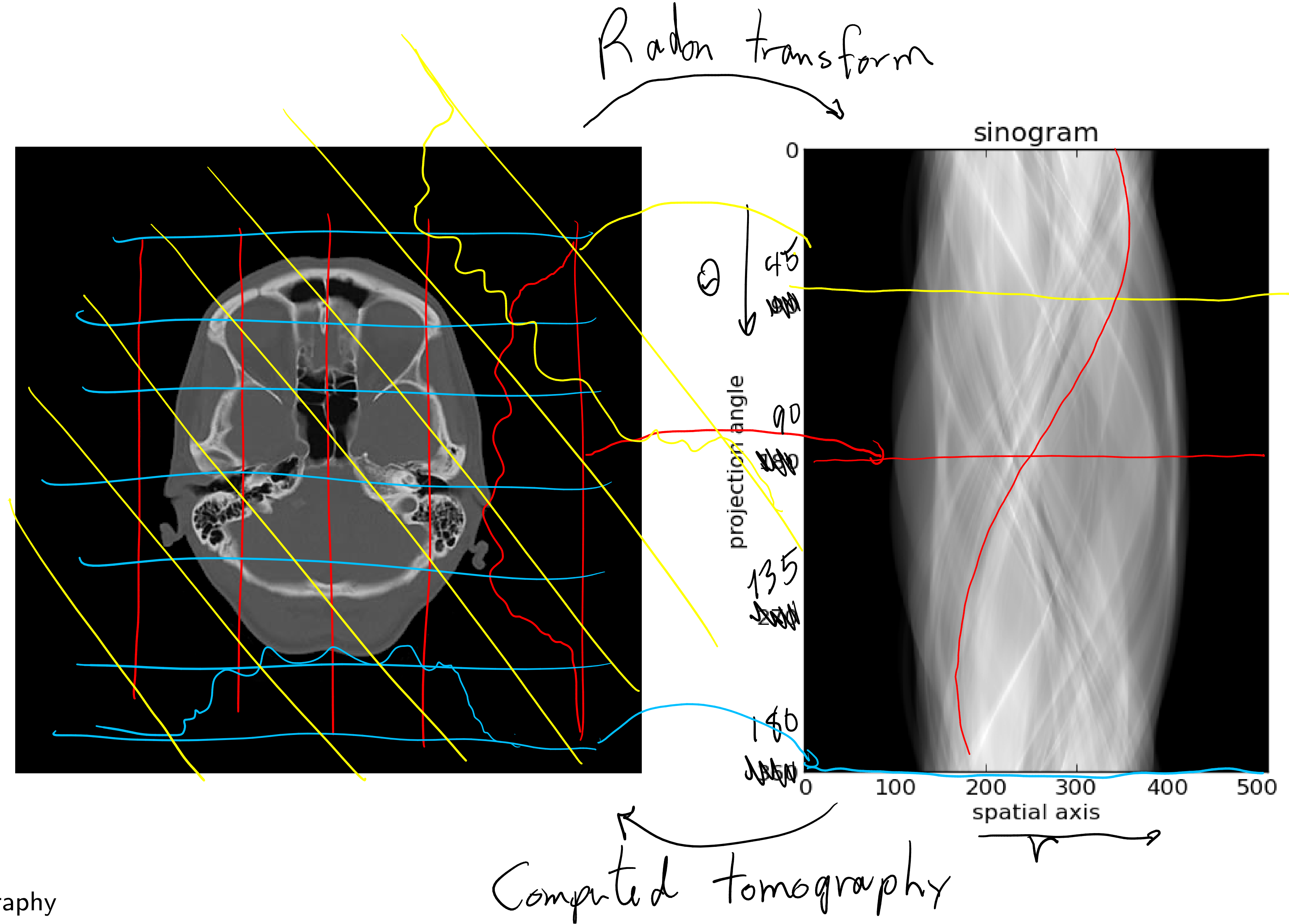
Radon transform



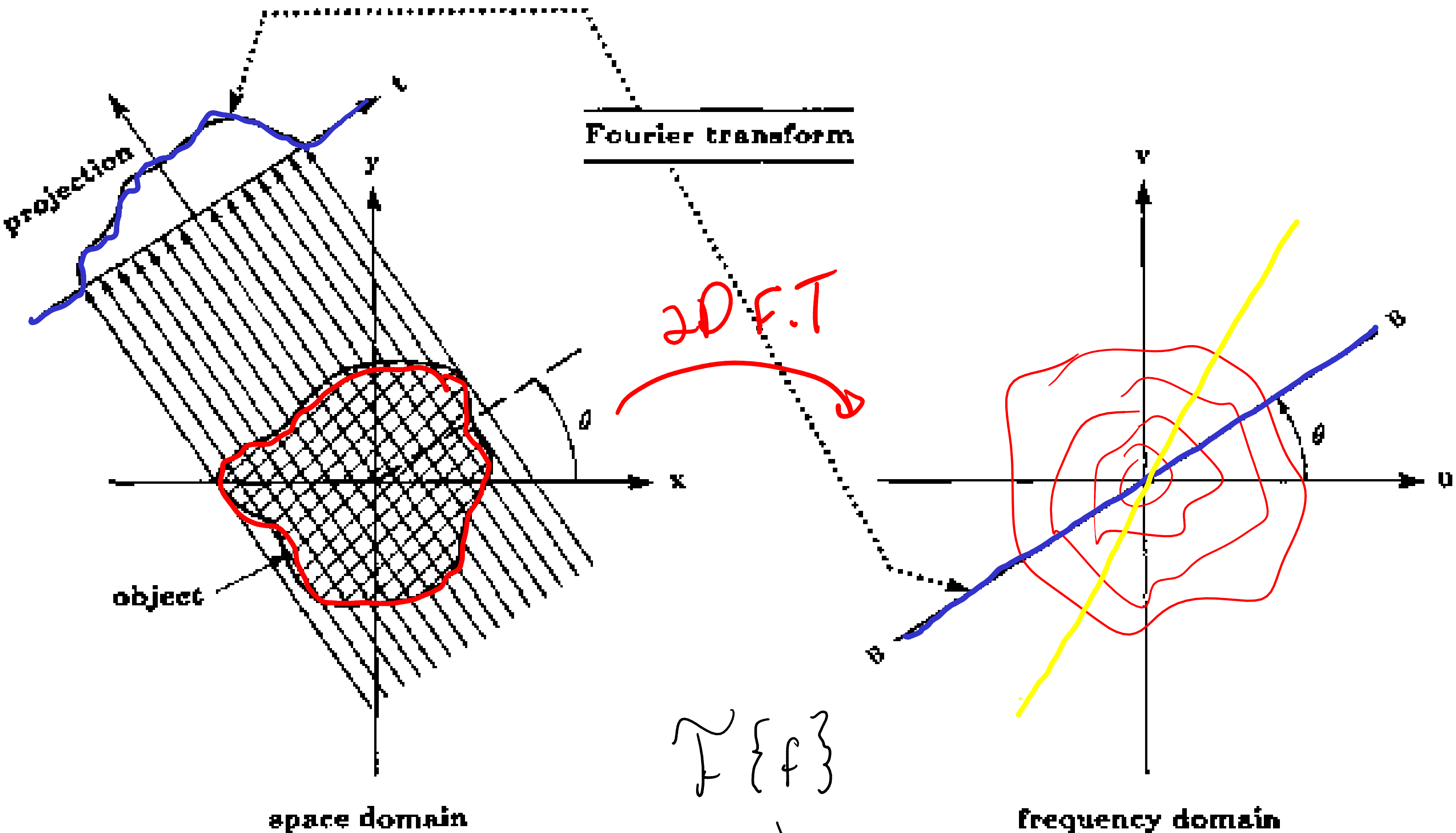


Sinogram

Representation of projection measured by
a single detector line as a function of angle



The Fourier slice theorem



$$\mathcal{F}_{r \to w}\{p(\theta, r)\} = F(u = w\cos\theta,\ v = w\sin\theta)$$

The Fourier slice theorem

$$p(\theta, r) = \int f(x = r\cos\theta - s\sin\theta,\ y = s\cos\theta + r\sin\theta)\, ds$$

$$= \int ds \int F(u,v)\, e^{i 2\pi u(r\cos\theta - s\sin\theta)}\, e^{i 2\pi v(s\cos\theta + r\sin\theta)}\, du\, dv$$

$$= \int ds \int F(u,v)\, e^{i 2\pi s(v\cos\theta - u\sin\theta)}\, e^{i 2\pi r(u\cos\theta + v\sin\theta)}\, du\, dv$$

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \Rightarrow \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix}$$

$$= \int ds \int F(u'\cos\theta - v'\sin\theta,\ u'\sin\theta + v'\cos\theta)\, e^{i 2\pi s v'}\, e^{i 2\pi r u'}\, du'\, dv'$$

$$= \int F(\ldots)\, e^{i 2\pi r u'} \underbrace{\left(\int e^{i s v'}\, ds\right)}_{\delta(v')} du'\, dv'$$

The Fourier slice theorem

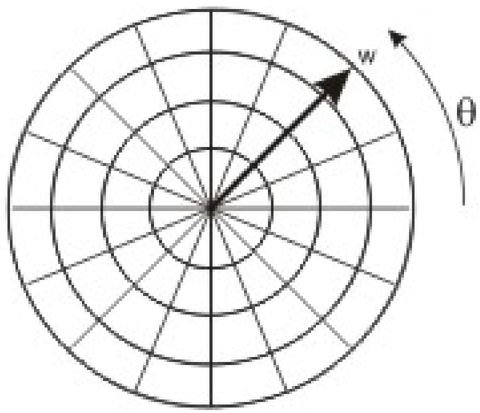
$$p(\theta, r) = \int F(u = u' \cos\theta, v = u' \sin\theta) e^{i 2\pi u' r} du'$$

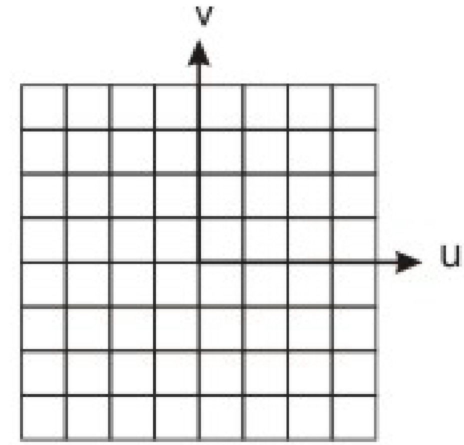
$\downarrow$

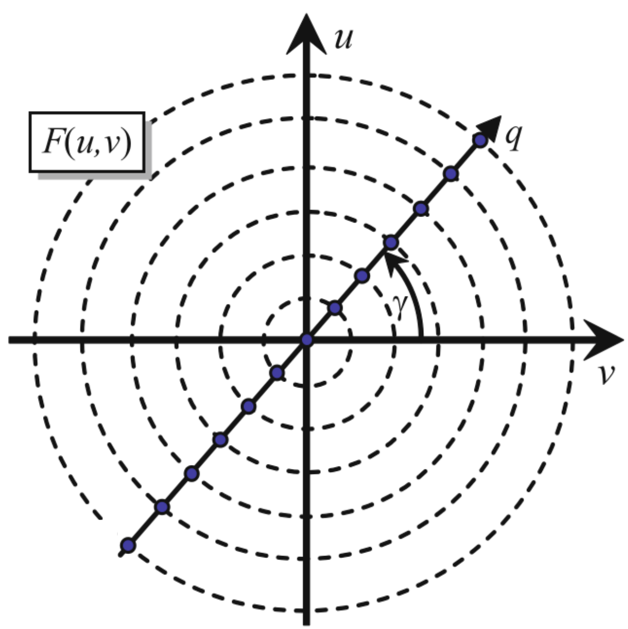
$$\int p(\theta, r) e^{-2\pi i w r} dr = F(u = w\cos\theta, v = w\sin\theta)$$

Frequency space sampling

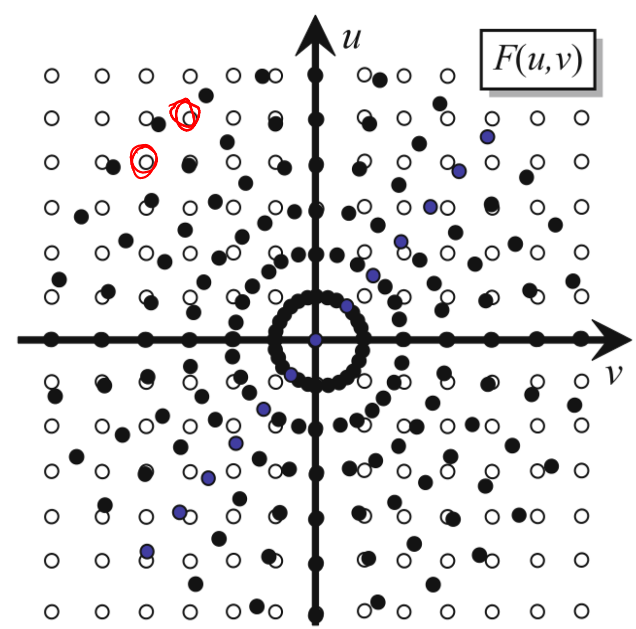
Change of sampling grid from polar to rectangular requires interpolation







a



b

"regridding"

Filtered back-projection

$$f(x,y) = \mathcal{F}^{-1}\{F(u,v)\}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)\, e^{2\pi i(ux+vy)}\, du\, dv$$

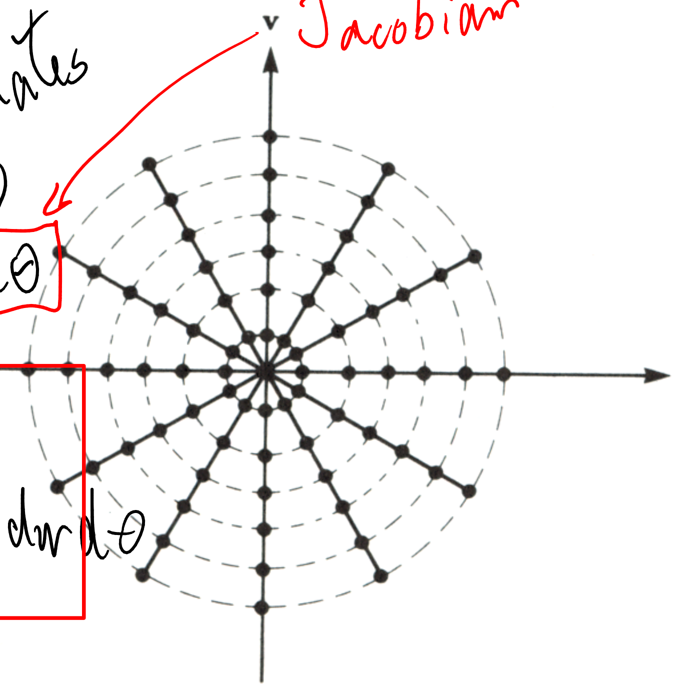
polar coordinates (w,θ)

$$= \int_{0}^{2\pi}\int_{0}^{\infty} F(w\cos\theta, w\sin\theta)\, e^{2\pi i(xw\cos\theta + yw\sin\theta)}\, w\, dw\, d\theta$$

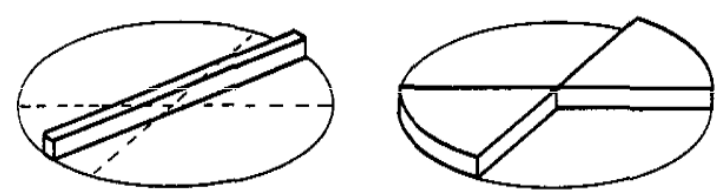
Jacobian

$$= \int_{0}^{\pi}\int_{-\infty}^{\infty} F(w\cos\theta, w\sin\theta)\, e^{2\pi i w(\cos\theta x + \sin\theta y)}\, |w|\, dw\, d\theta$$

filtered sinogram

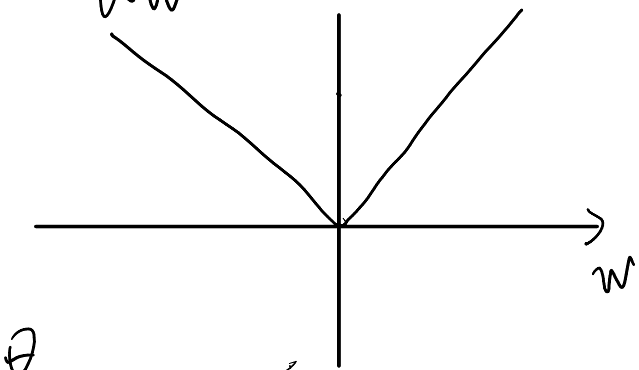


frequency domain

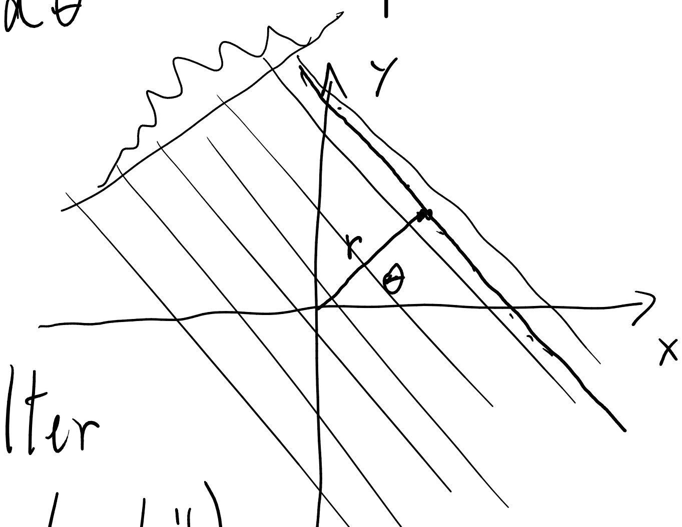


Filtered back-projection

$$p'(\theta, r) = \int_{-\infty}^{\infty} \mathcal{F}\{p(\theta, r)\} \, |w| \, e^{2\pi i w r} \, dw$$



$$f(x,y) = \int_0^{\pi} \underbrace{p'(\theta, x\cos\theta + y\sin\theta)}_{\text{back-projection}} \, d\theta$$

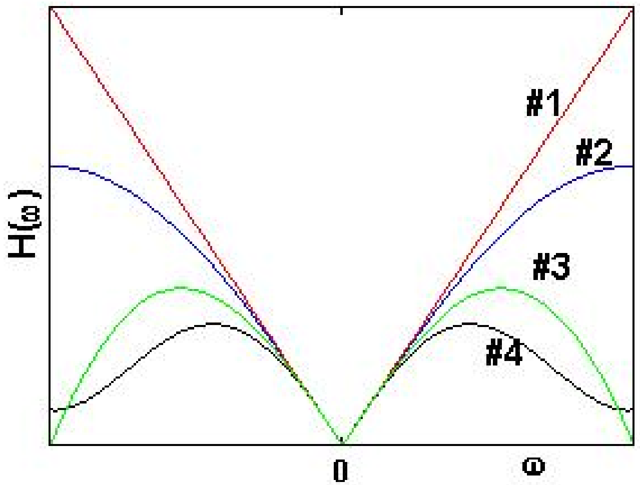


Recipe: 1) filter sinogram (ramp filter "Ram-Lack")

2) back project for each angle.

Filtered back-projection

- Filter can be tuned to achieve image enhancement
- Trade-off between noise and sharpness

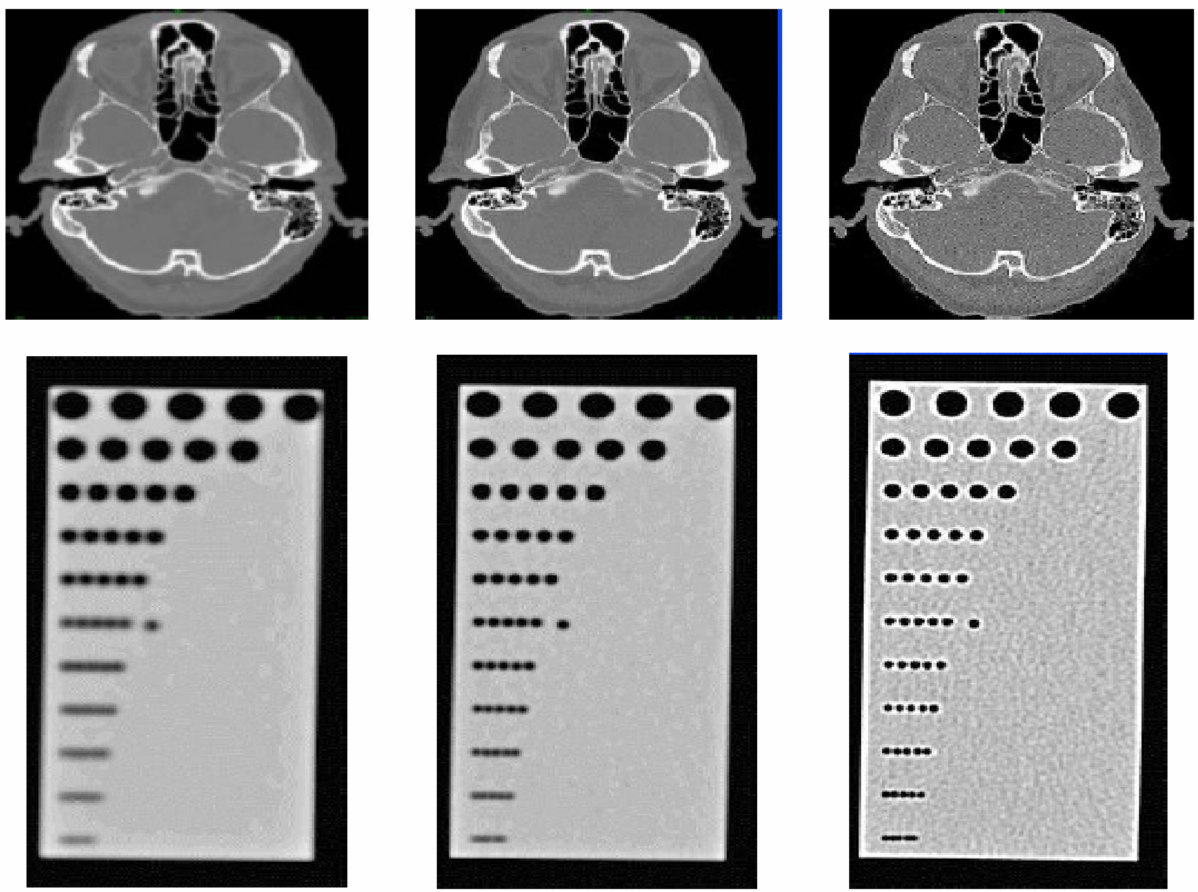


#1 ram-lak (ramp)

#2 Shepp-Logan

#3 cosine

#4 Hamming



smoothing standard edge enhanced

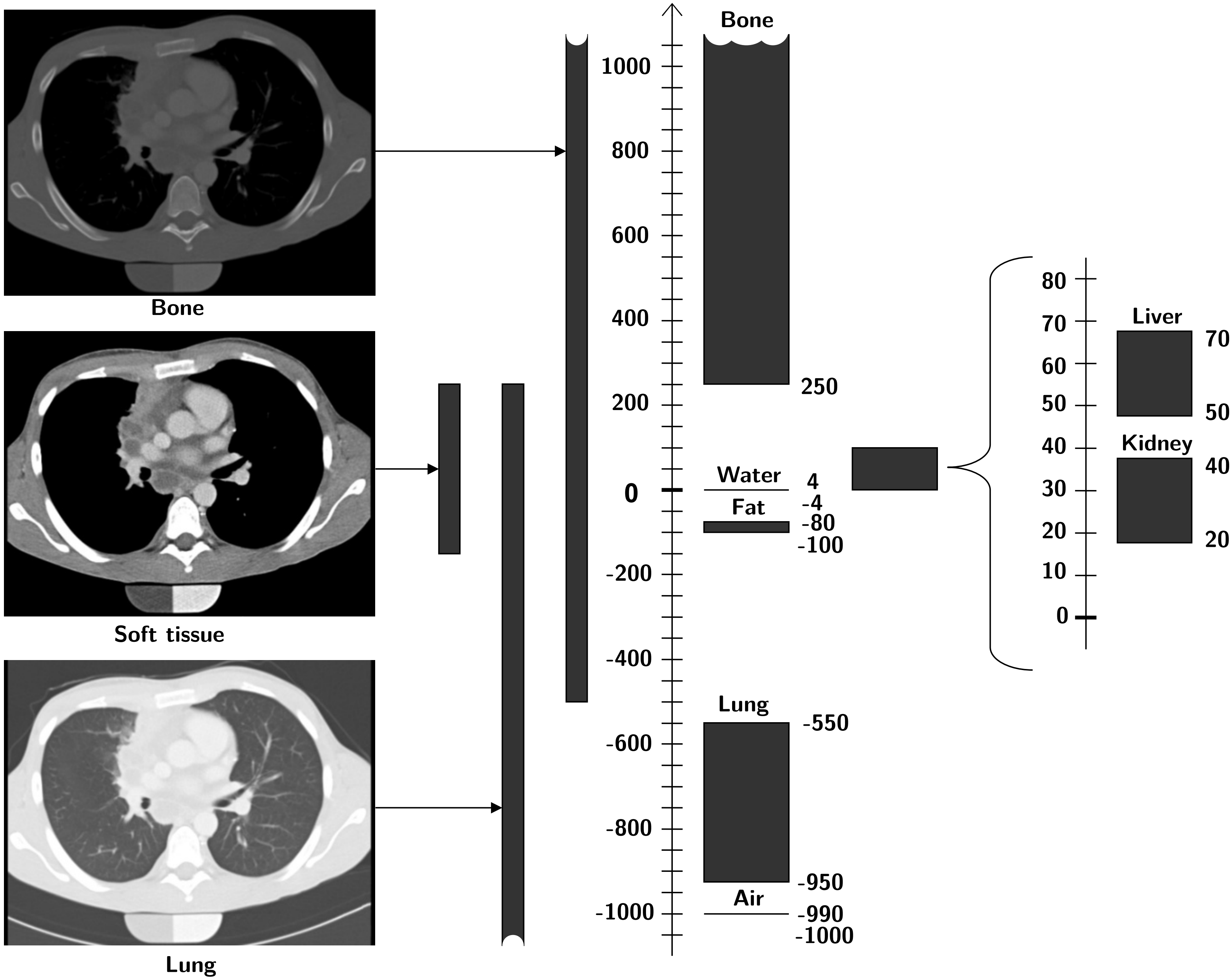
Medical CT

- Mathematical methods developed by Allan M. Cormack in the early 60s.
- First clinically useful CT instrument developed by Godfrey Hounsfield in the early 70s.
- Cormack & Hounsfield were awarded the Nobel prize in 1979 "for the development of computer assisted tomography".

Photo from the Nobel Foundation archive.
Allan M. Cormack

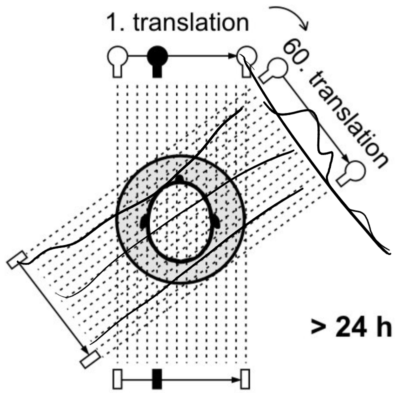
Photo from the Nobel Foundation archive.
Godfrey N. Hounsfield

Hounsfield units



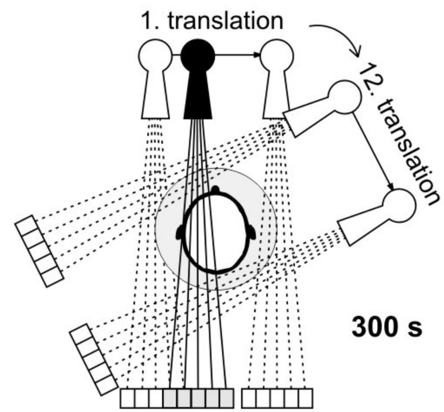
Geometries

pencil beam (1970)

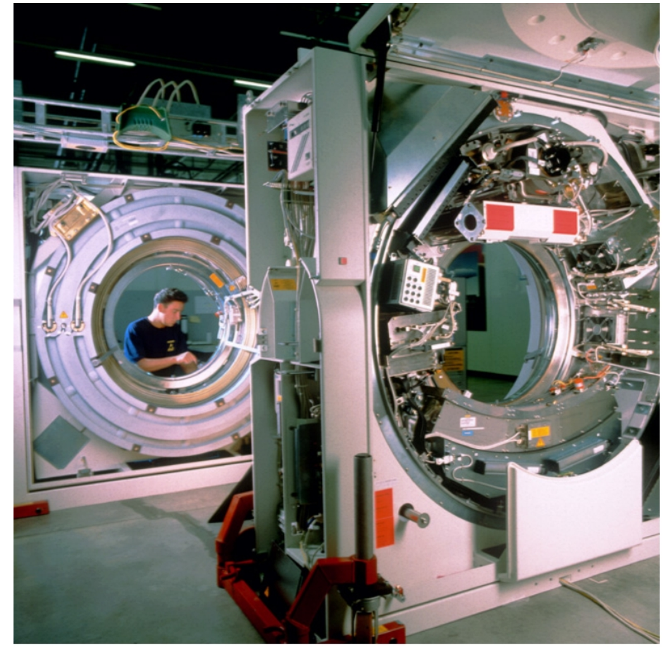


1st generation: translation / rotation

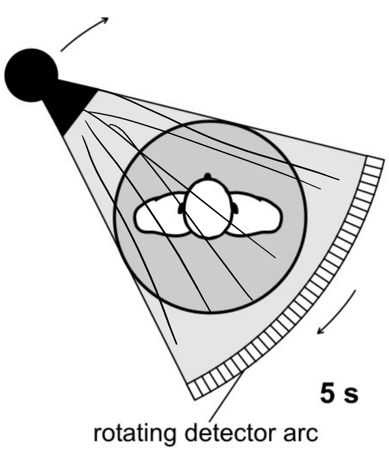
partial fan beam (1972)



2nd generation: translation / rotation

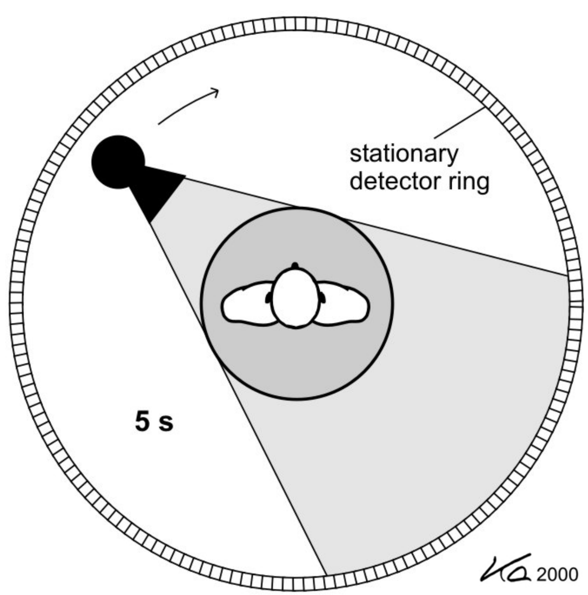


fan beam (1976)

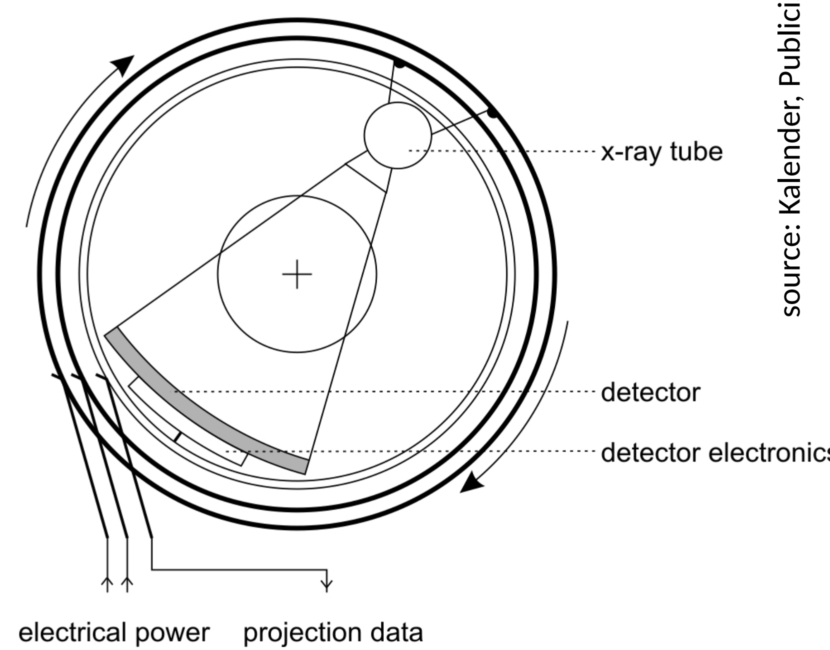


3rd generation: continuous rotation

fan beam (1978)

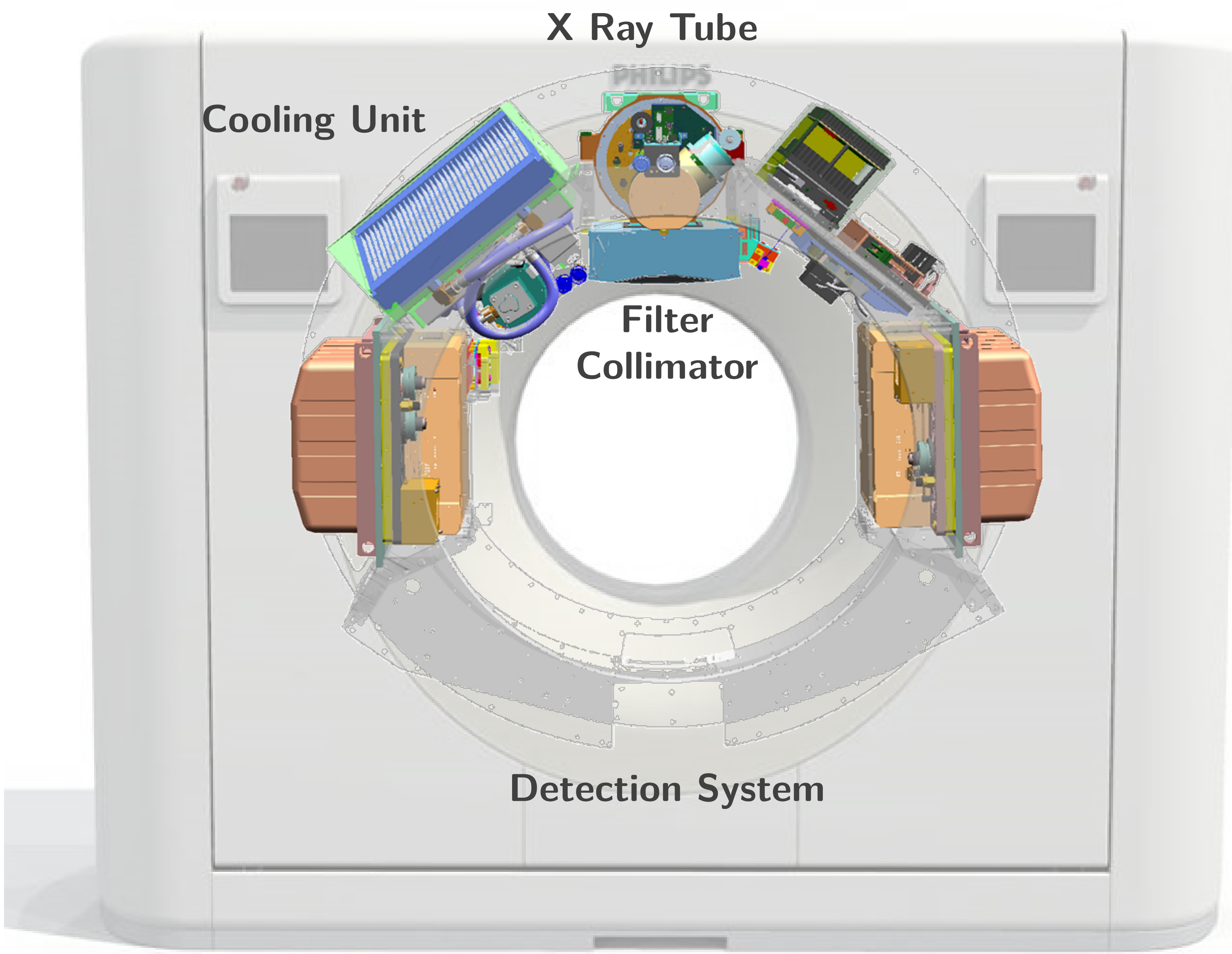


4th generation: continuous rotation



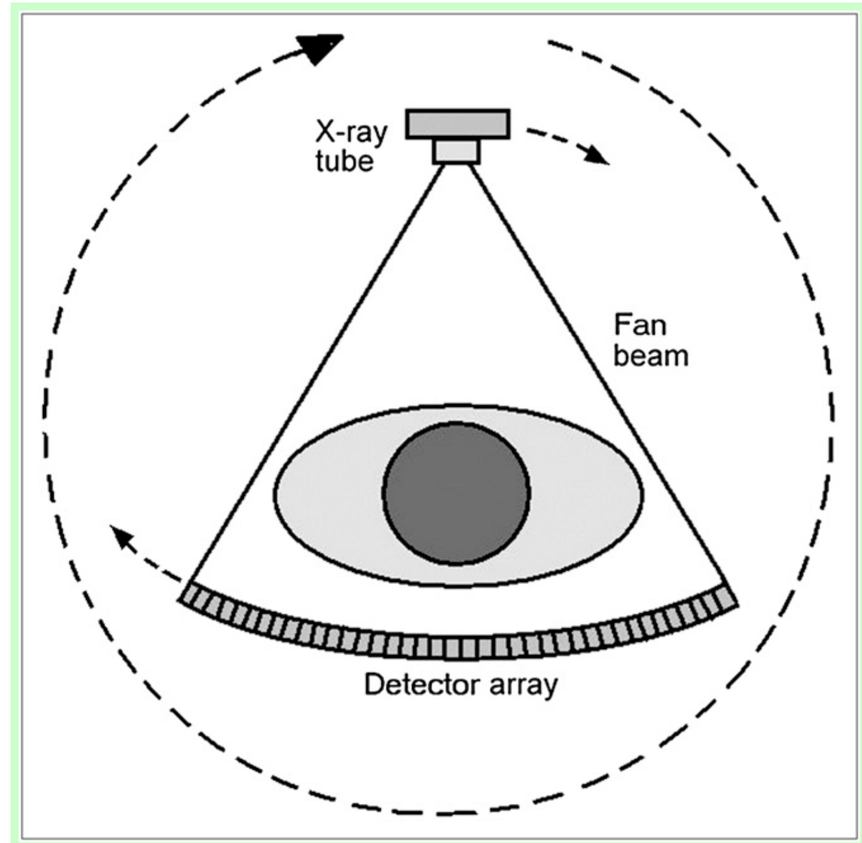
source: Kalender, Publicis, 3rd ed. 2011

CT systems


X Ray Tube
PHILIPS
Cooling Unit
Filter
Collimator
Detection System

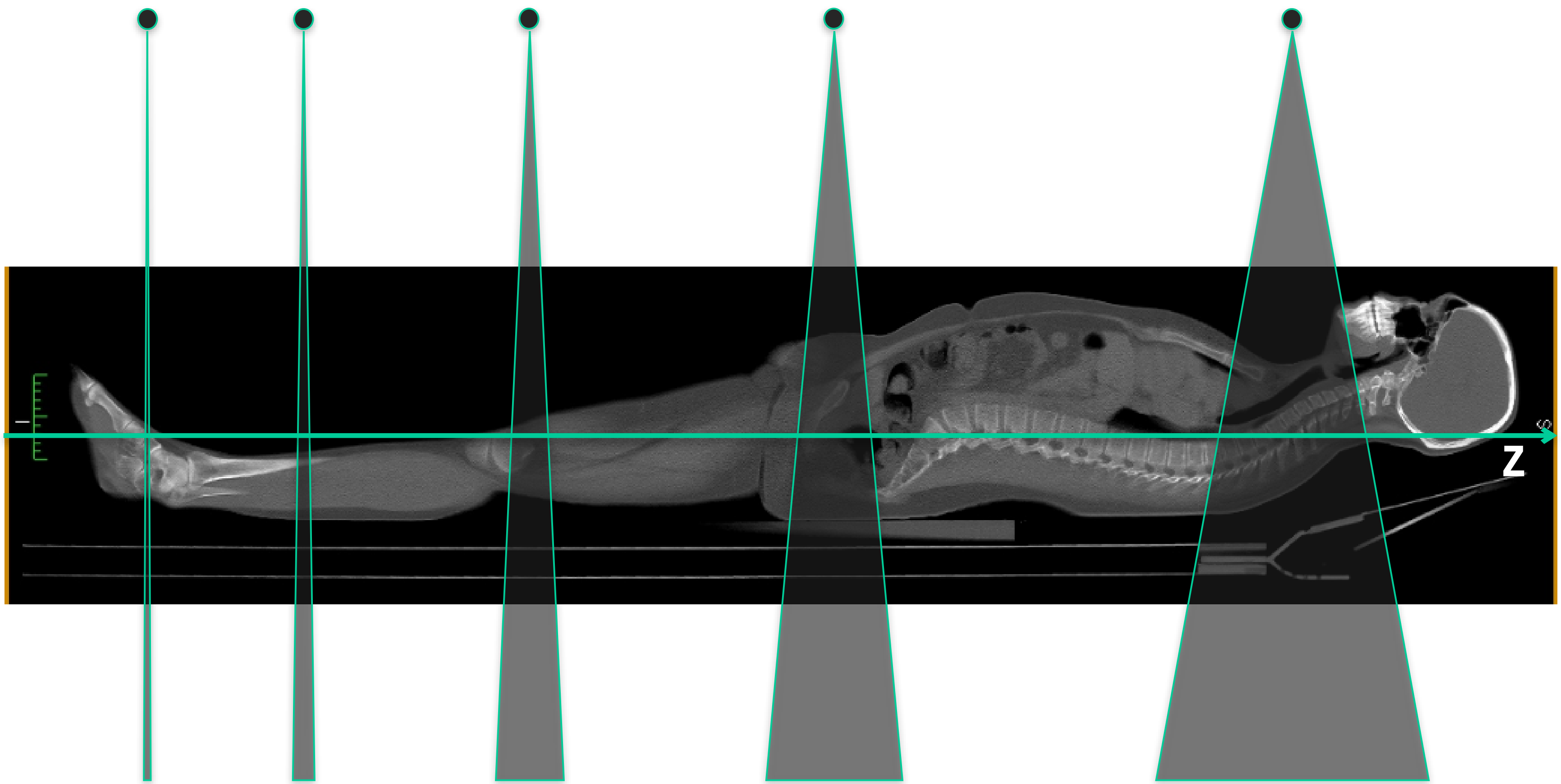
CT systems

- 3^{rd} generation scanners:
  - 1 X-ray source
  - 1D detector
  - *Fan beam* geometry
  - Total scan time less than 5 second.



CT systems

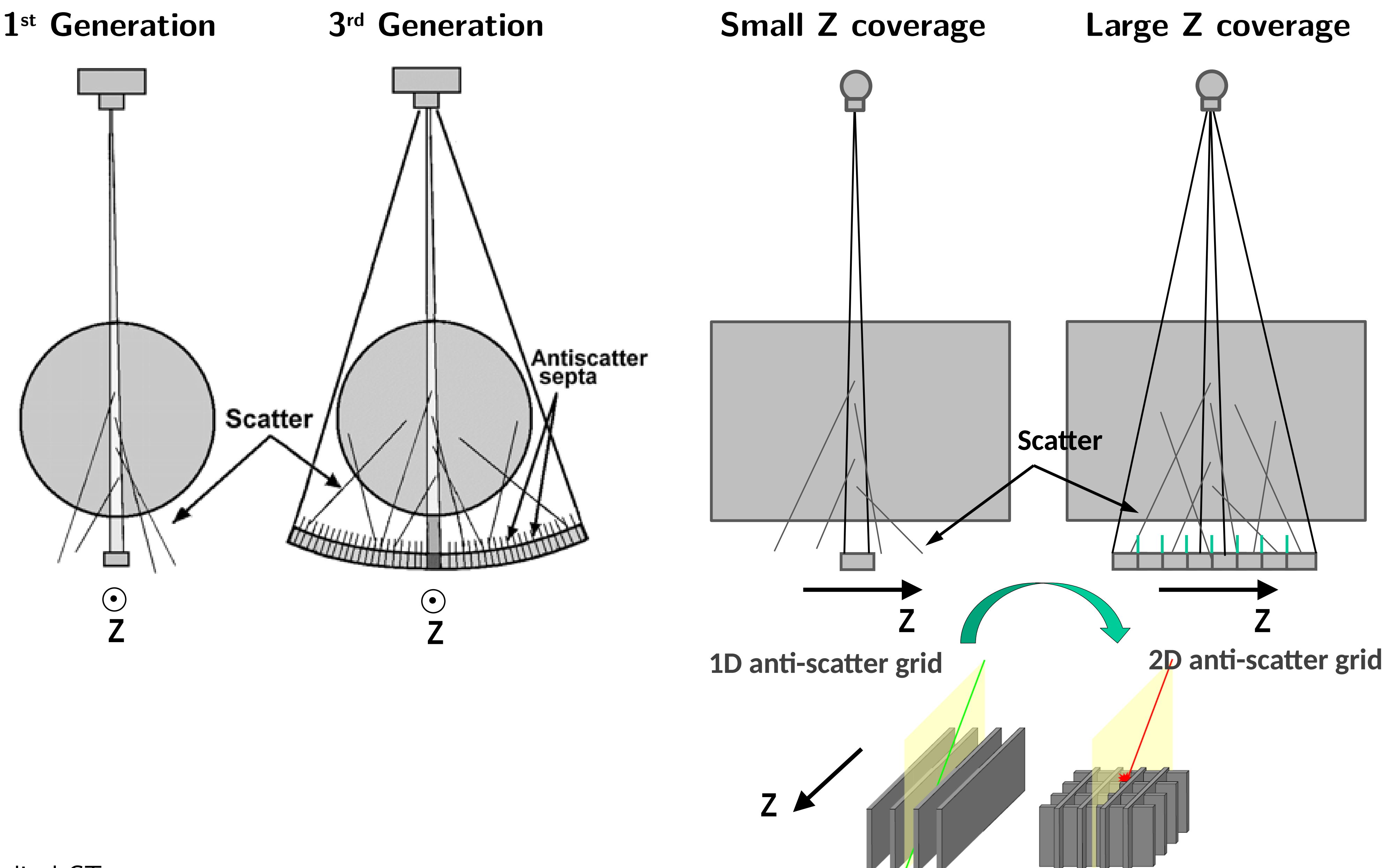
- Recent scanners: 2D detectors (cone beam)



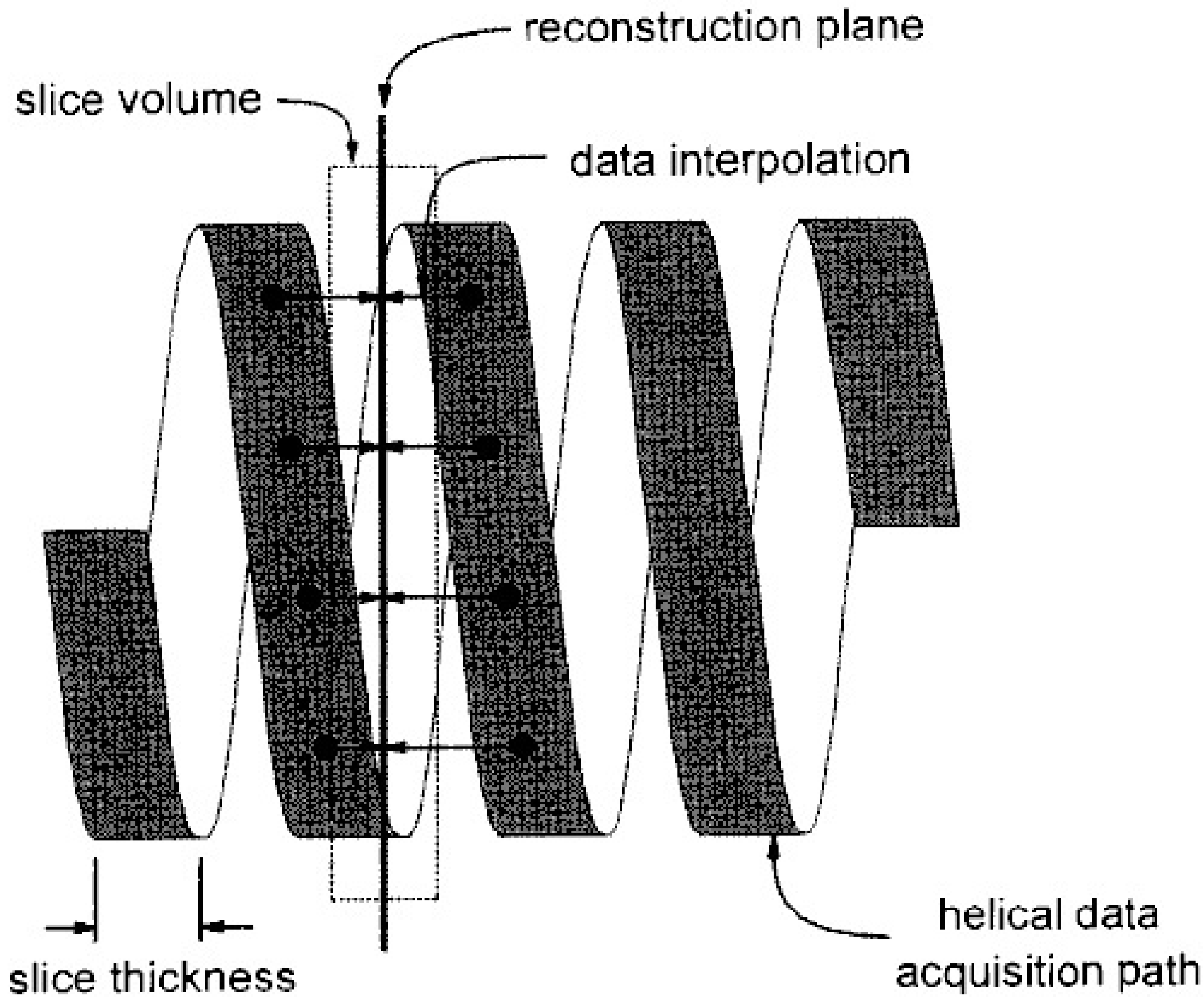
NxT	1x5mm	4x1mm	16x0.75mm	64x0.6mm	320x0.5mm
t_{rot}	0.75s	0.5s	0.42s	0.33s	0.27s
year	1995	1998	2001	2004	2008

CT systems

Anti-scatter

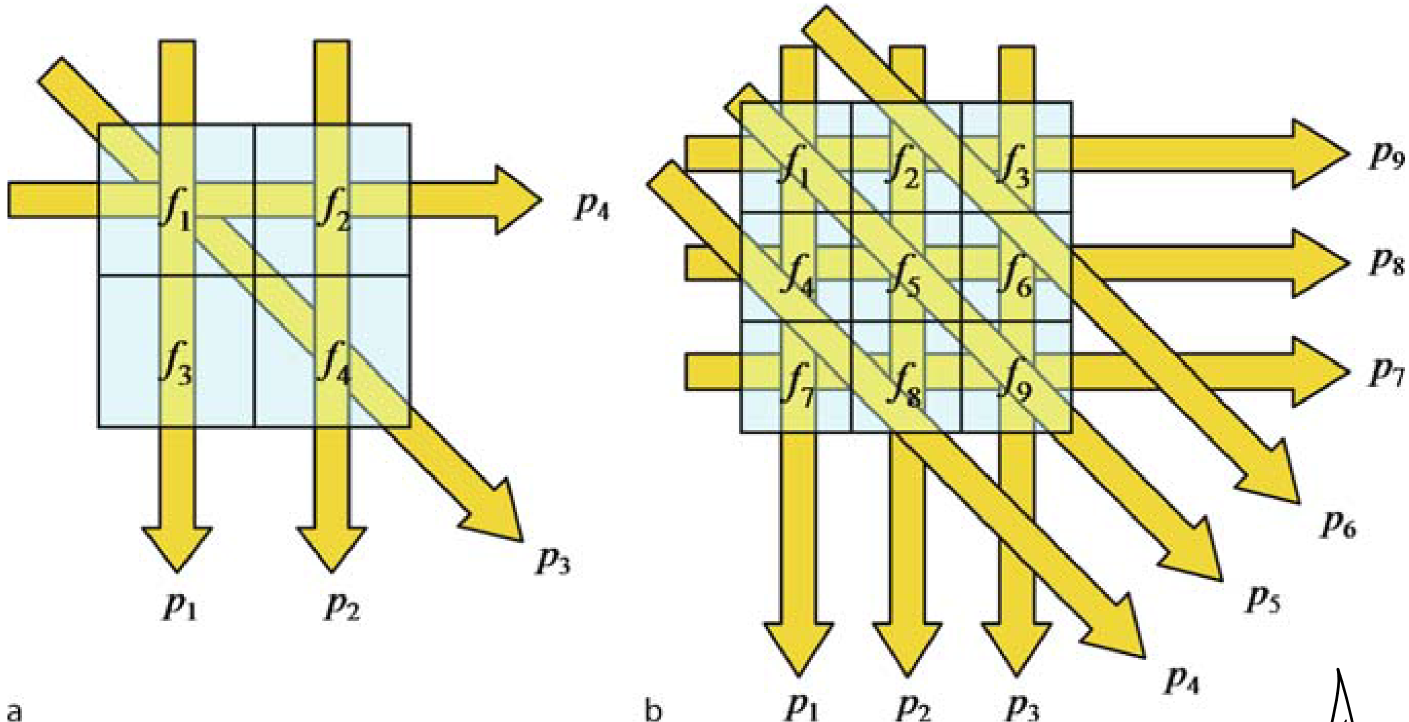


CT systems
Helical CT



Algebraic formulation

Tomography can be formulated as a set of linear equations



$Ax = b$

$$f_1 + f_2 = p_4$$
$$f_1 + f_3 = p_1$$
$$f_2 + f_4 = p_2$$
$$f_1 + f_4 = p_3$$

system matrix

$$\longrightarrow \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{pmatrix}$$

source: Buzug, Springer, 1st ed. 2008

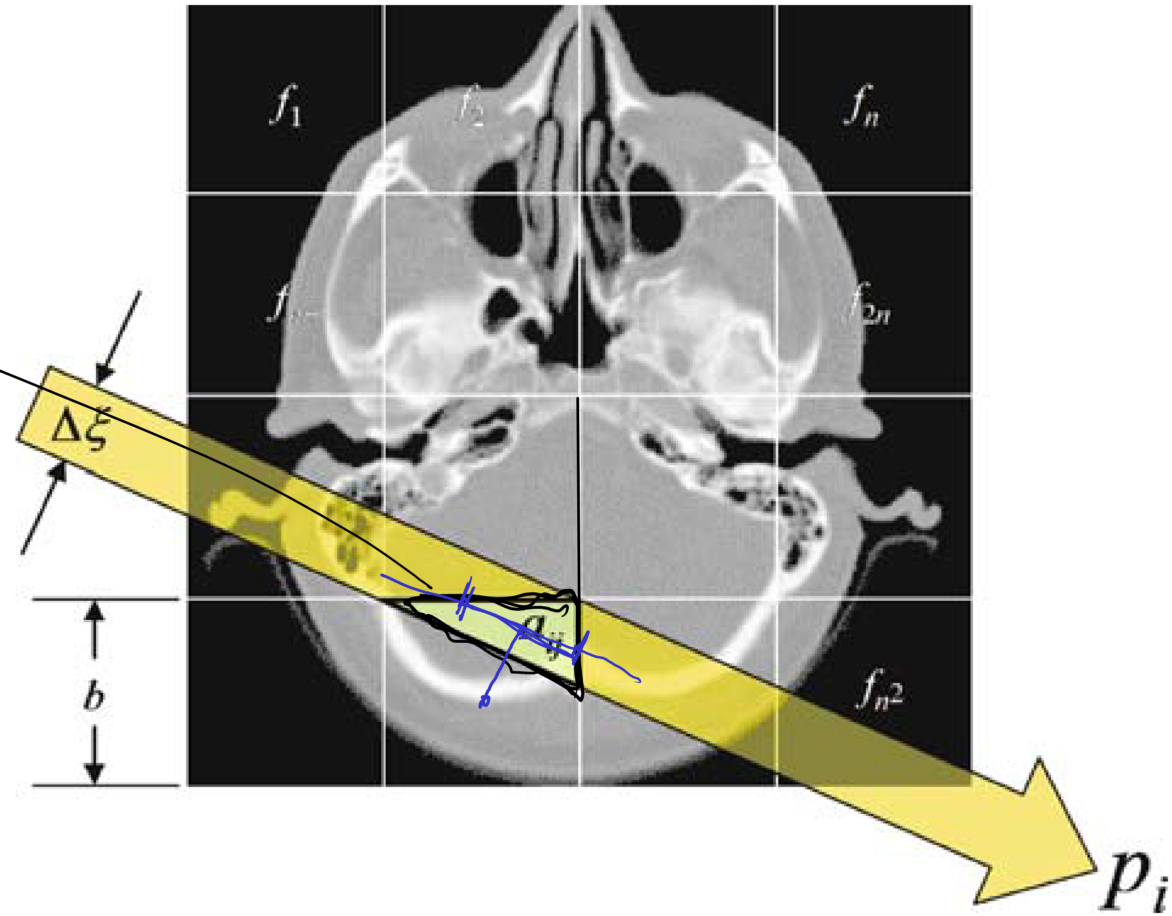
Weighting coefficients

Weighting measures:

- Logic

  0 or 1

- Area
- Path length
- Distance to pixel center



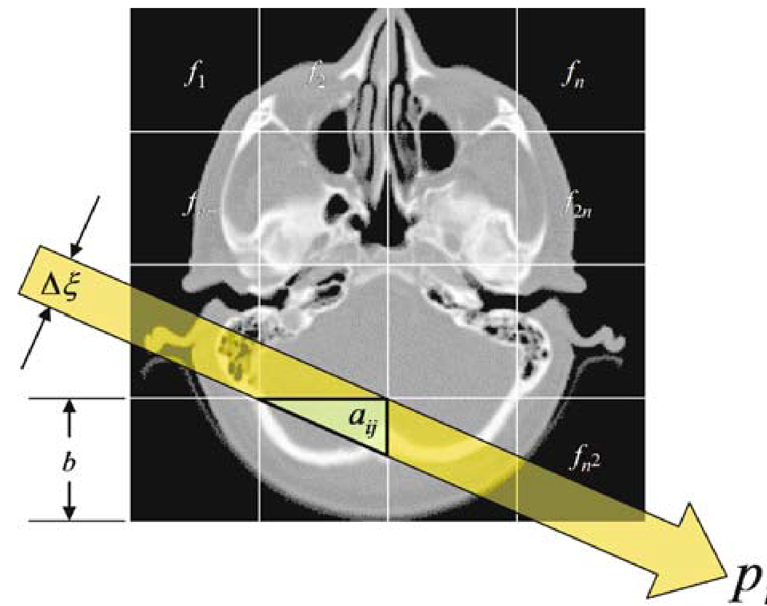
Differences in calculation effort, smoothness, noise sensitivity, ...

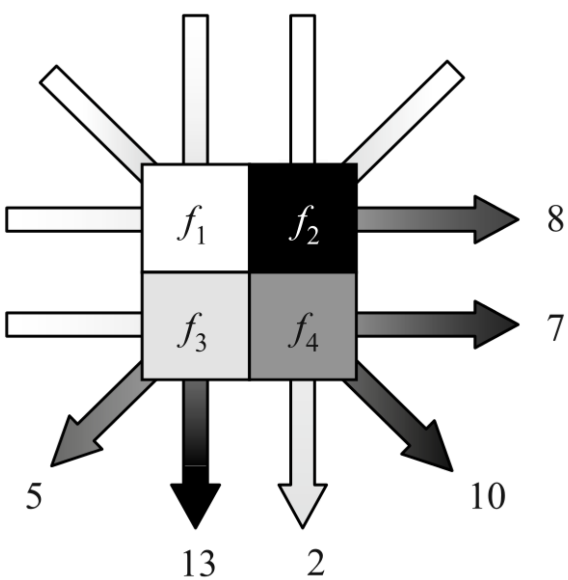
source: Buzug, Springer, 1st ed. 2008

System Matrix

In general:

- very sparse
- made of entries between 0 and 1
- not square



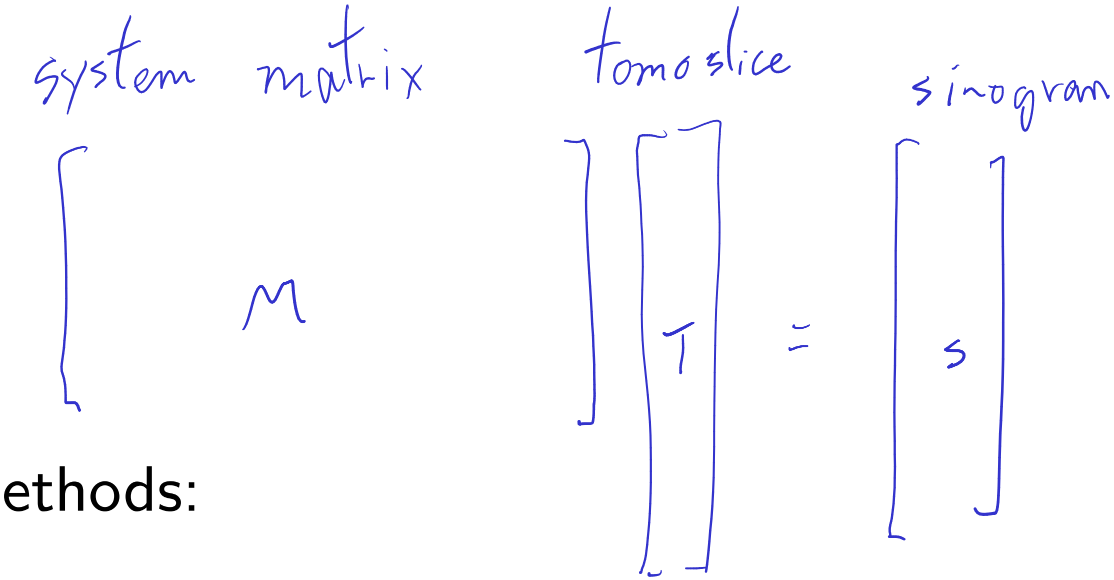


$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = \begin{pmatrix} 5 \\ 13 \\ 2 \\ 10 \\ 7 \\ 8 \end{pmatrix}$$

source: Buzug, Springer, 1st ed. 2008

Matrix (pseudo)-inversion

Tomographic reconstruction = linear system inversion



$T \approx 10^6$ entries

$S \approx 10^6$ entries

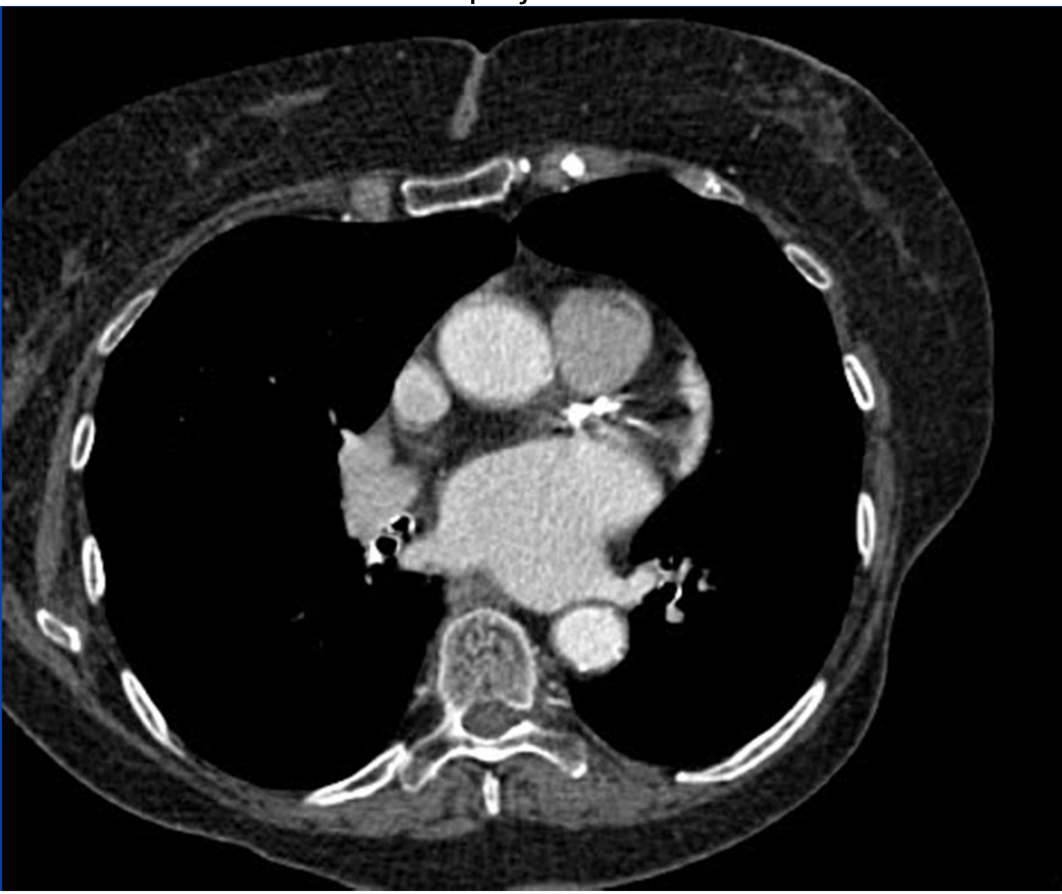
$M : 10^6 \times 10^6$ matrix

Iterative methods:

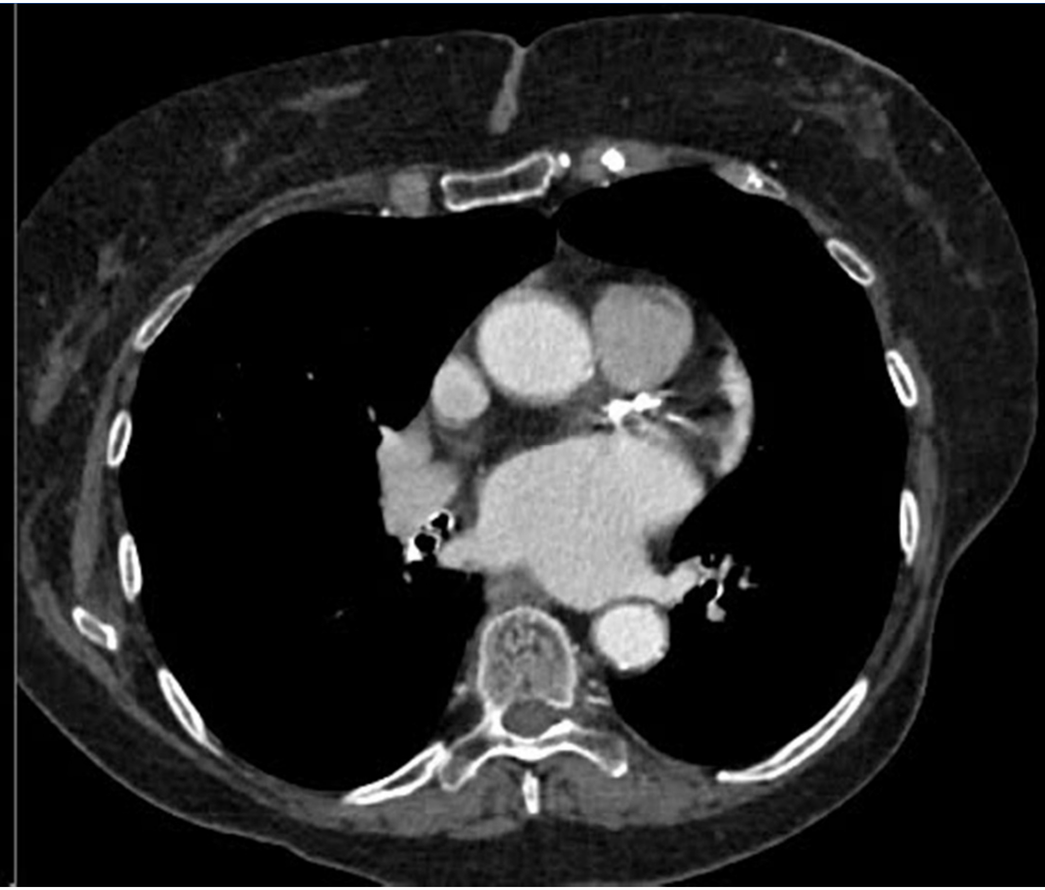
- ART Algebraic reconstruction technique
- SART Simultaneous algebraic reconstruction technique
- SIRT Simultaneous iterative reconstruction technique
- MART Multiplicative algebraic reconstruction technique
- MLEM Maximum likelihood expectation maximization
- OSEM Ordered subset expectation maximization
- ... and many, many more

FBP vs algebraic methods

Filtered backprojection 100% dose

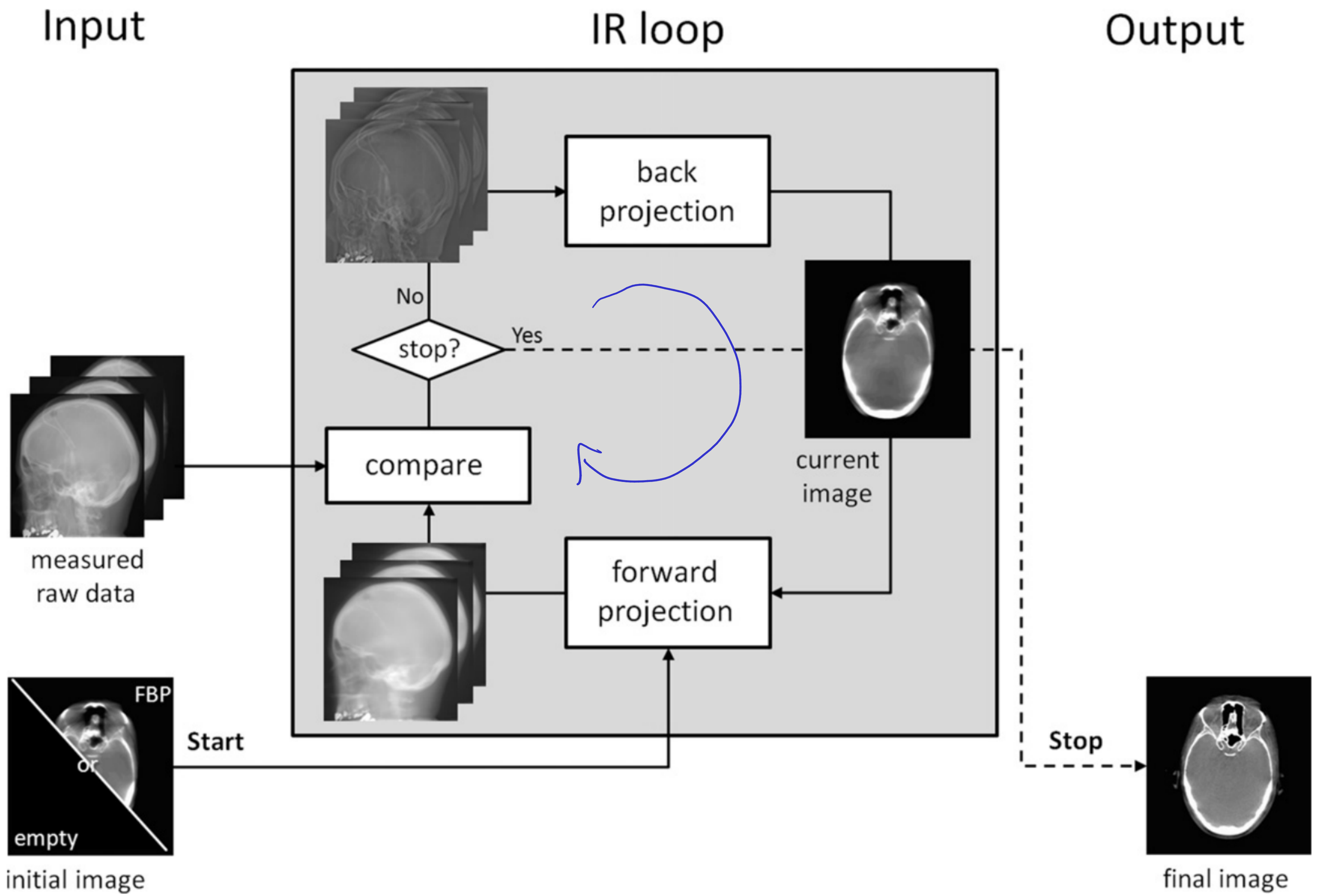


iterative 40% dose



source: Kachelries, http://www.dkfz.de/en/medphysrad/workinggroups/ct/ct_conference_contributions/BasicsOfCTImageReconstruction_Part2.pdf

Iterative methods



Iterative methods

Abbreviation	Meaning
ART	Algebraic reconstruction technique
SART	Simultaneous ART
SIRT	Simultaneous iterative reconstruction technique
OS-SIRT	Ordered subset SIRT
MART	Multiplicative algebraic reconstruction technique
ML-EM	Maximum likelihood expectation-maximization
OS-EM	Ordered subset expectation-maximization
OSC	Ordered subset convex algorithm
ICD	Iterative coordinate descent
OS-ICD	Ordered subset ICD
MBIR	Model-based iterative reconstruction

Gordon et al. 1970

Anderson & Kak, 1984

Gilbert 1972

Gordon et al. 1970

Lange & Carson 1984

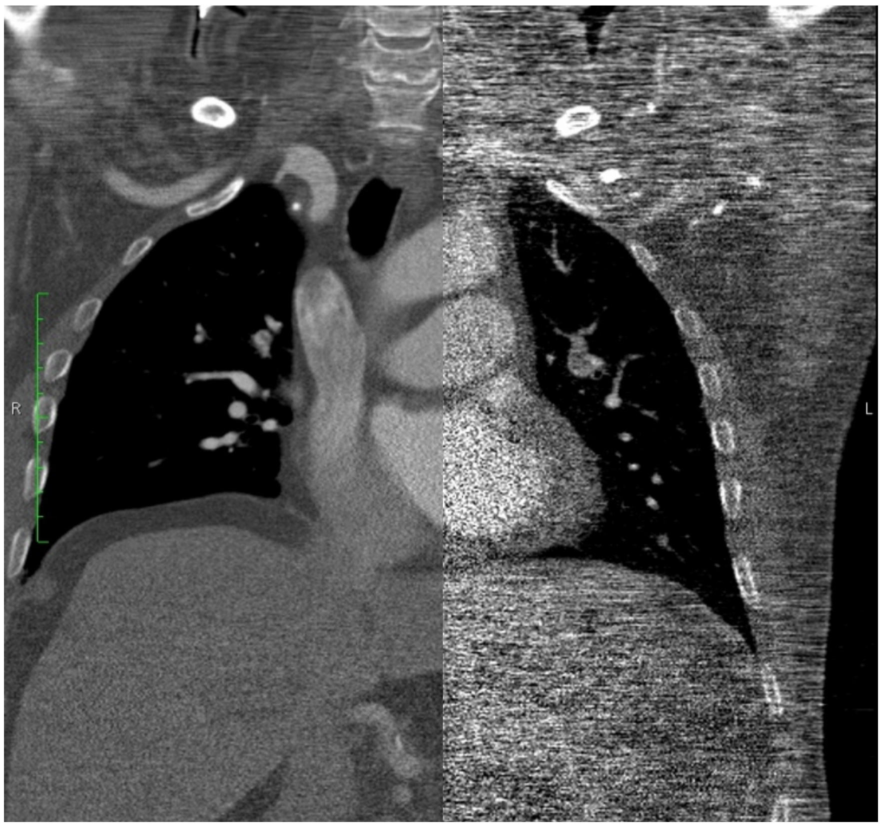
Manglos et al 1995

Kamphuis & Beekman 1998
Erdogan & Fessler 1999

...

Image quality

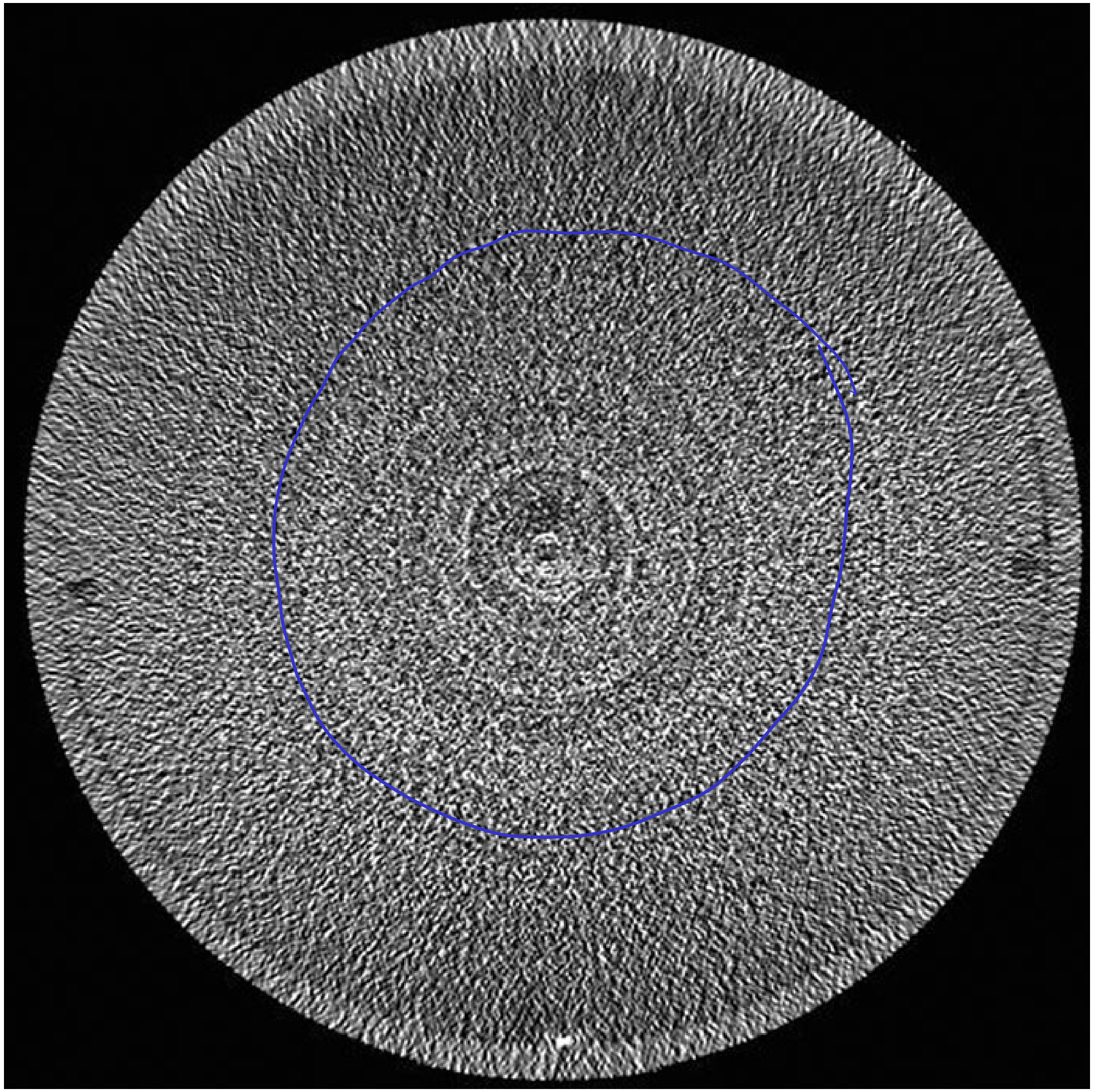
Signal to noise

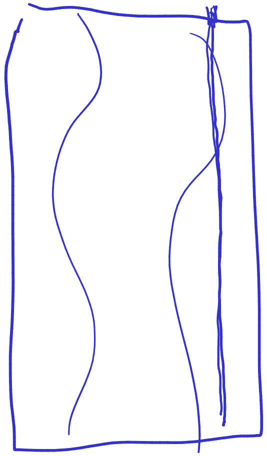


Artifacts

Artifacts

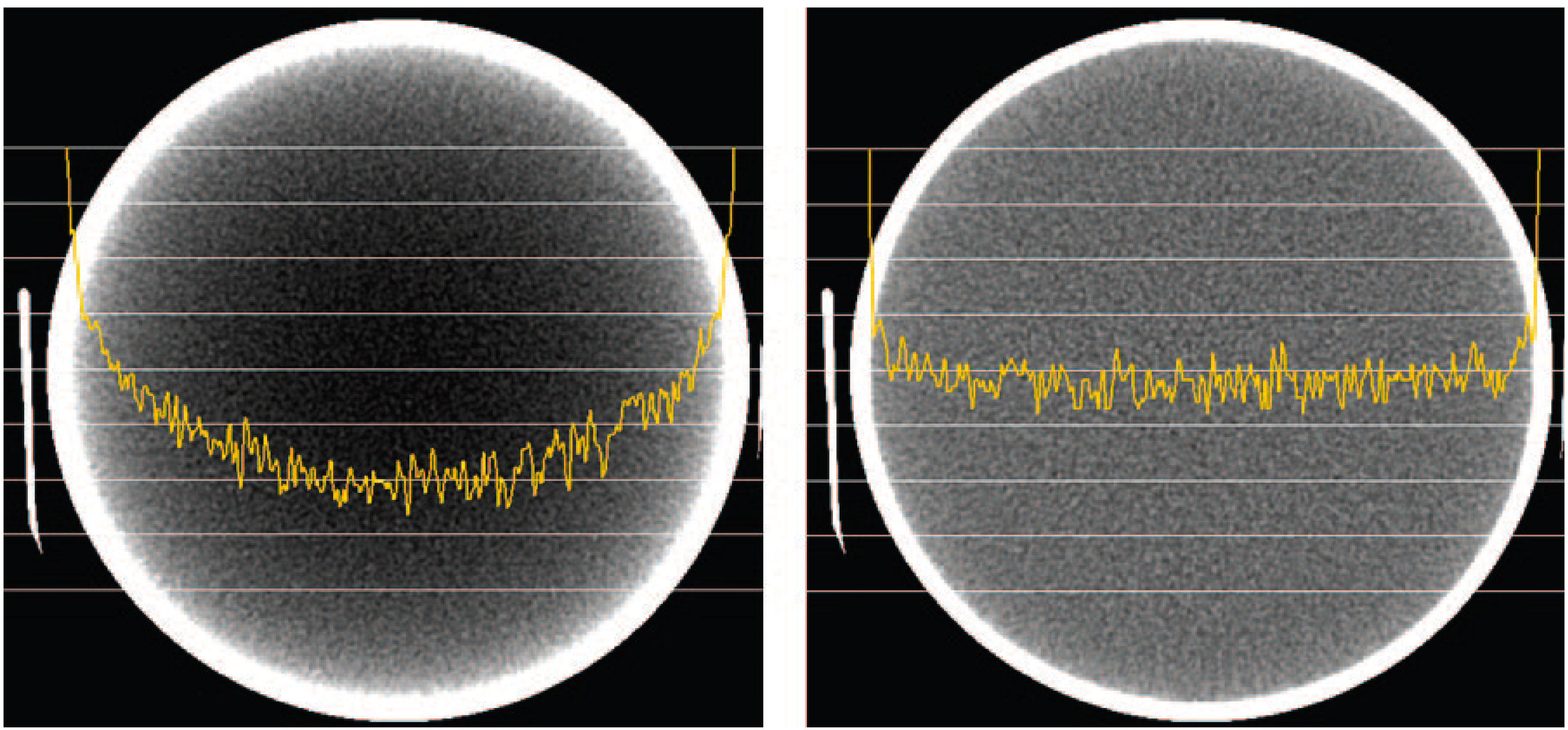
Ring artifacts: caused by damaged or miscalibrated detector pixels.





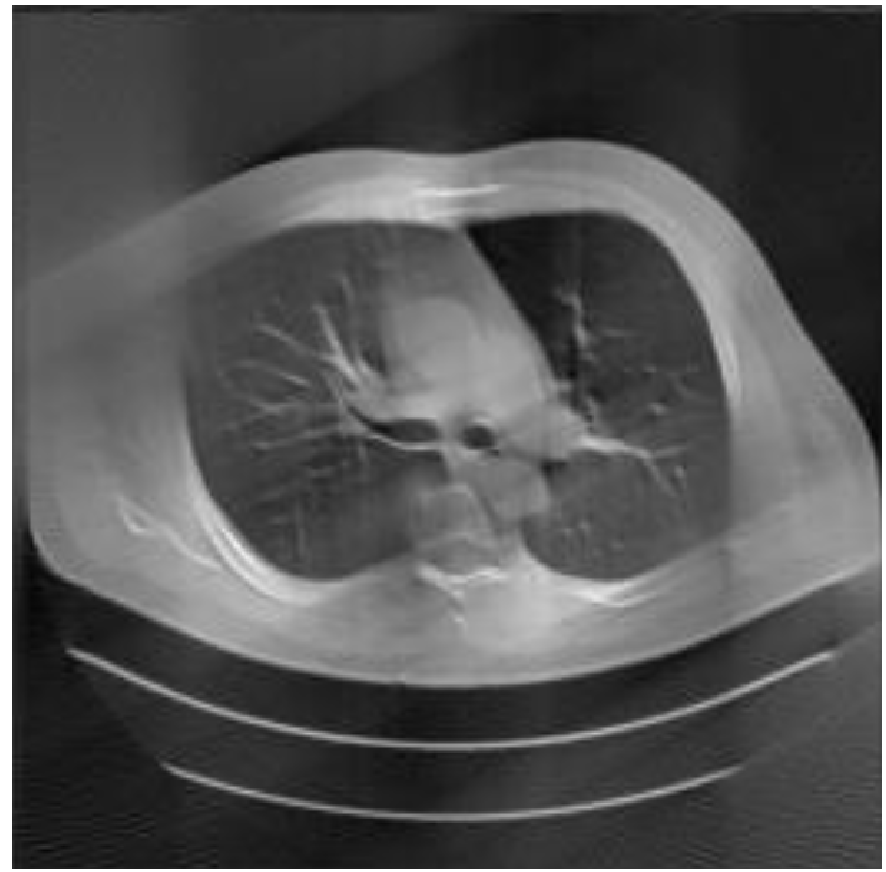
Artifacts

Beam hardening: deviation from the exponential law caused by the attenuation of a broad spectrum (thicker objects seem more transmissive than they should)



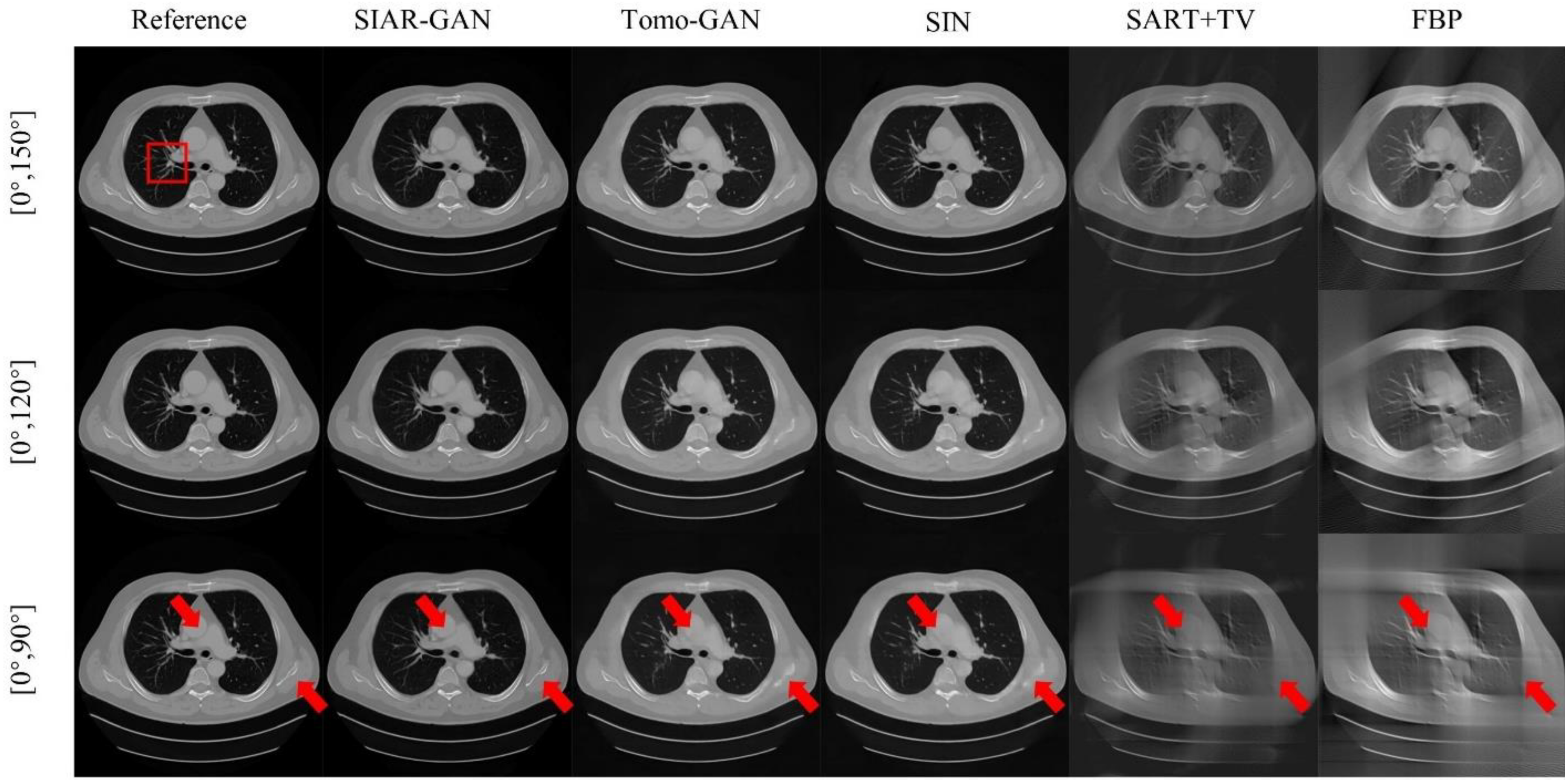
Artifacts

"Missing wedge": caused by an incomplete sinogram (also called limited angle tomography)



Artifacts

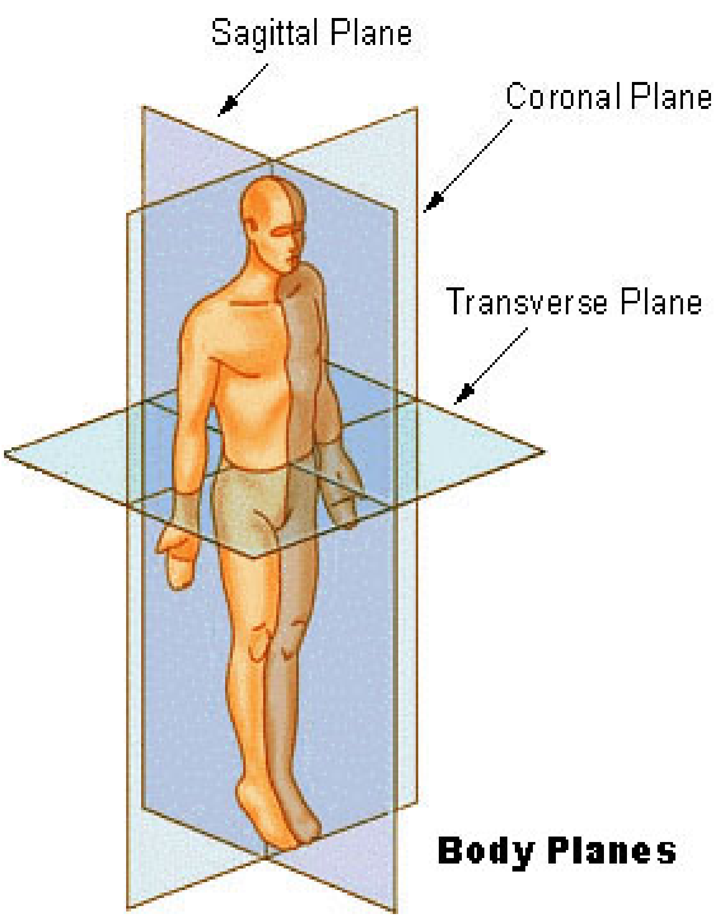
“Missing wedge”: promising results using machine learning

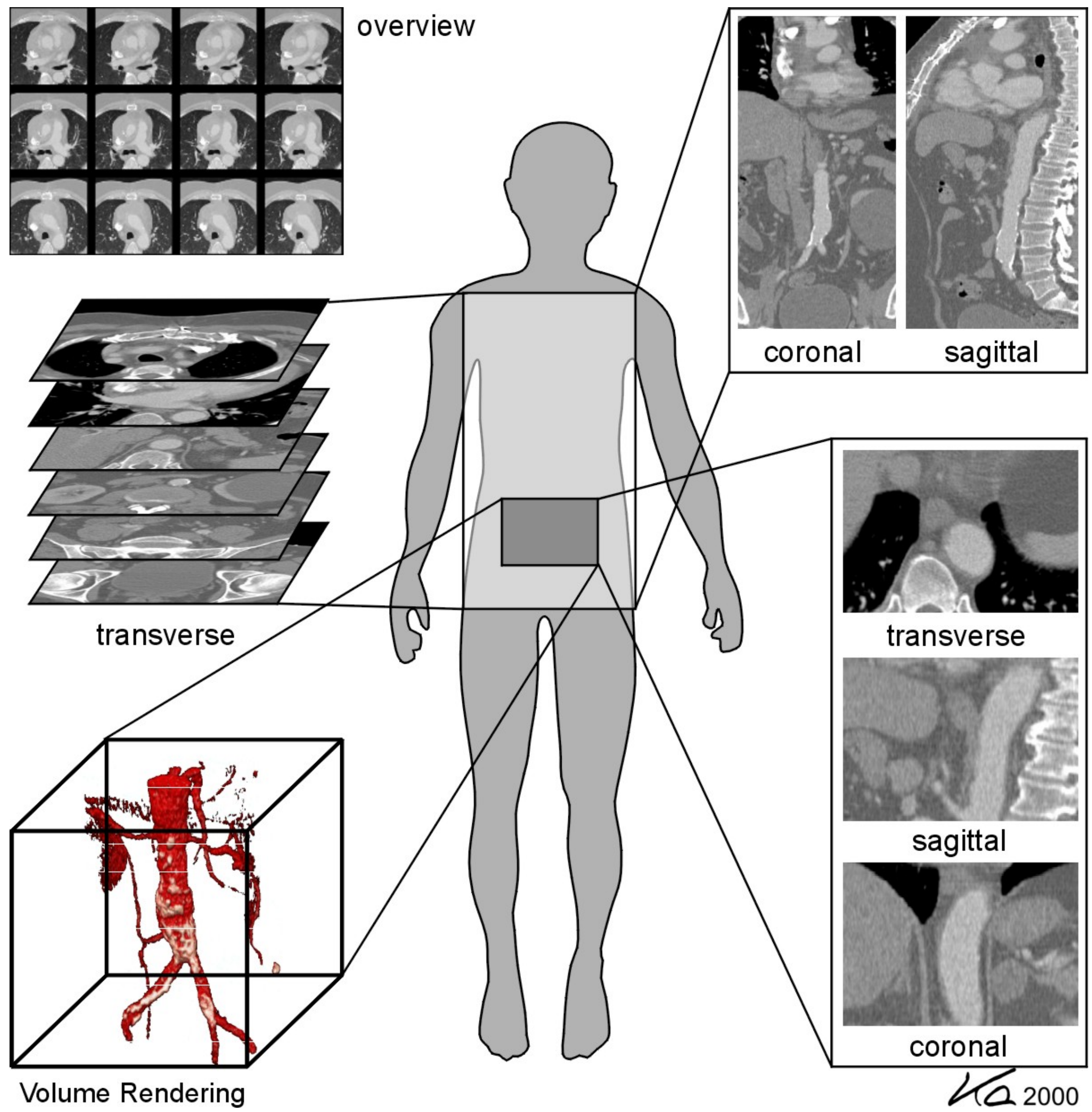


Artifacts

Photon starvation: strongly absorbing features discard useful signal for the reconstruction of nearby areas

Tomographic Display

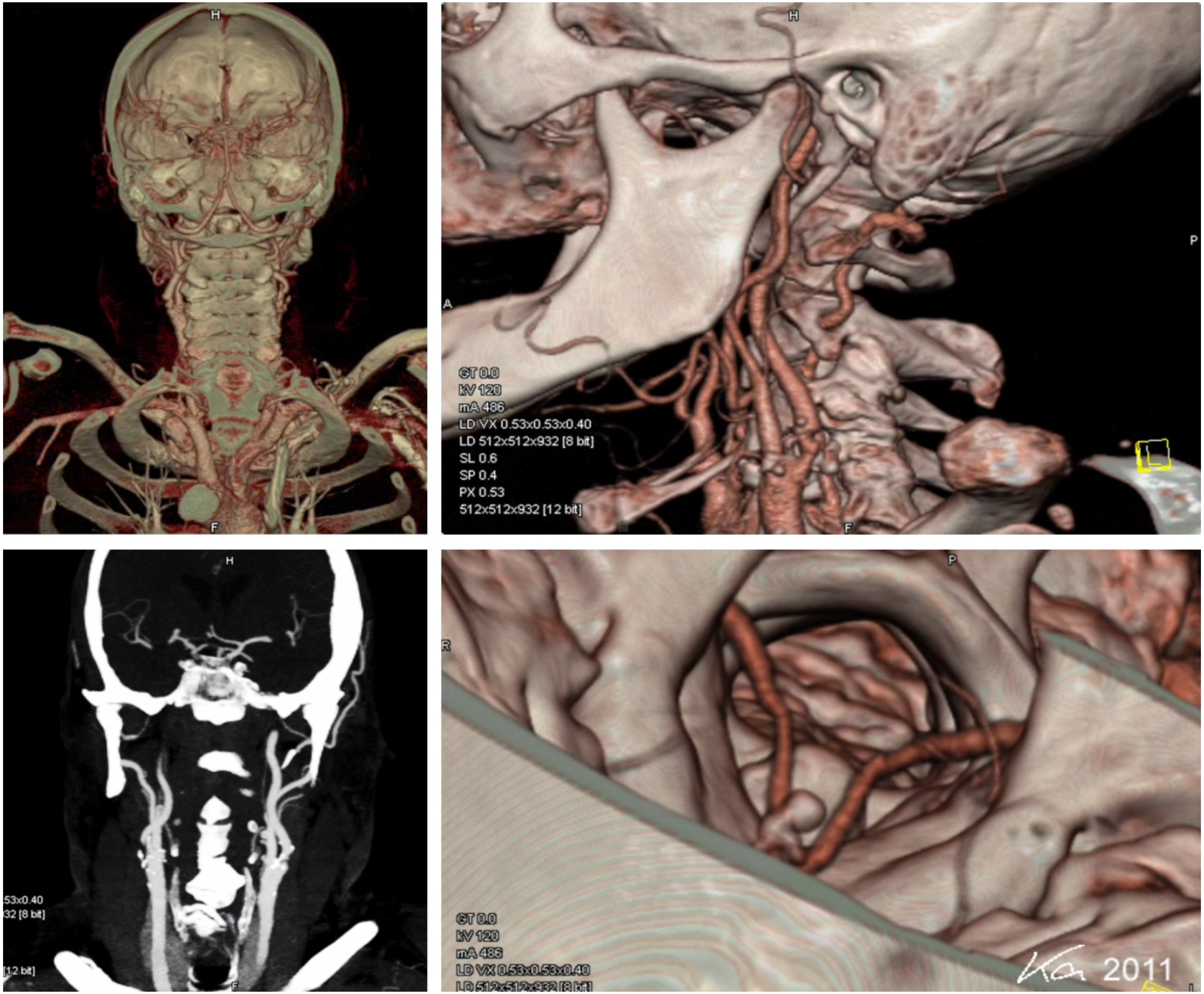




source: http://wikipedia.org

source: W. Kalender, Publicis, 3rd ed. 2011

Volume rendering display



source: Kalender, Publicis, 3rd ed. 2011

Summary

- Computed tomography: reconstruction from projections
- Analytic approach:
  - Projections and tomographic slices are related by the Fourier slice theorem
  - Standard algorithm uses filtered back-projection
- Algebraic approach:
  - Tomography as a system of linear equations
  - Iterative methods are used for large matrix inversions
  - More powerful but computationally more costly
- Imperfect data leads to artifacts