Nov 27

Theor $L^p(X, d\mu)$ $1 \le p \le \infty$ is a B-space

Theor $L^p(X, d\mu)$ $2 \le p < \infty$ $(1 < p < \infty)$

are reflexive

Pf We have Clarkson inequality

$$\left\| \frac{f+g}{2} \right\|_{L^p}^p + \left\| \frac{f-g}{2} \right\|_{L^p}^p \le \frac{1}{2} \left(\|f\|_{L^p}^p + \|g\|_{L^p}^p \right)$$

$2 \le p < \infty$

$f, g \in \overline{D_{L^p}(0,1)}$

$\|f - g\|_{L^p} \ge \varepsilon > 0$

$$\left\| \frac{f+g}{2} \right\|_{L^p}^p \le 1 - \frac{\varepsilon^p}{2^p}$$

$$\left\| \frac{f+g}{2} \right\|_{L^p} \le \left(1 - \frac{\varepsilon^p}{2^p} \right)^{\frac{1}{p}} =$$

$$= 1 - \underbrace{\left(1 - \left(1 - \frac{\varepsilon^p}{2^p} \right)^{\frac{1}{p}} \right)}_{\delta > 0}$$

$$\left\| \frac{f+g}{2} \right\|_{L^p}^p + \left\| \frac{f-g}{2} \right\|_{L^p}^p \le \frac{1}{2} \left(\|f\|_{L^p}^p + \|g\|_{L^p}^p \right)$$

$2 \le p < \infty$

$$\left| \frac{a+b}{2} \right|^p + \left| \frac{a-b}{2} \right|^p \le \frac{1}{2} \left(|a|^p + |b|^p \right)$$

$2 \le p < \infty$

$$\alpha^p + \beta^p \le (\alpha^2 + \beta^2)^{\frac{p}{2}} \qquad 2 \le p < \infty$$

$\alpha = \left| \frac{a+b}{2} \right| \qquad \beta = \left| \frac{a-b}{2} \right|$

$$\left| \frac{a+b}{2} \right|^p + \left| \frac{a-b}{2} \right|^p \le \left(\left| \frac{a+b}{2} \right|^2 + \left| \frac{a-b}{2} \right|^2 \right)^{\frac{p}{2}}$$

$$= \left(\frac{a^2}{2} + \frac{b^2}{2} \right)^{\frac{p}{2}} \le \frac{1}{2} (a^2)^{\frac{p}{2}} + \frac{1}{2} (b^2)^{\frac{p}{2}}$$

$t \to t^{\frac{p}{2}} \qquad \frac{p}{2} \ge 1 \qquad = \frac{a^p}{2} + \frac{1}{2} b^p$

$a = \alpha^2 \qquad b = \beta^2 \qquad q = \frac{p}{2}$

$a^q + b^q \le (a+b)^q \qquad q \ge 1$

$$\left(\frac{a}{a+b} \right)^q + \left(\frac{b}{a+b} \right)^q \le 1$$

?

$$\le \frac{a}{a+b} + \frac{b}{a+b} = 1$$

Thm L^p $1<p<2$ is reflexive.

Pf $1<p<+\infty$ $\qquad \frac{1}{p}+\frac{1}{p'}=1$

$$f\in L^p \longrightarrow Tf\in (L^{p'})'$$

$$\langle Tf, g\rangle_{(L^{p'})'\times L^{p'}} = \int f g$$

Hölder inequality

$$\left|\langle Tf, g\rangle_{(L^{p'})'\times L^{p'}}\right| \le \left|\int f g\right| \le |f|_{L^p}\, |g|_{L^{p'}}$$

$$\le |f|_{L^p}$$

$$\Rightarrow \|Tf\|_{(L^{p'})'} \le |f|_{L^p}$$

$$f\in L^p \qquad g=|f|^{p-2}f \in L^{p'}$$

$$\int g|_{L^{p'}} = |\,|g|\,|_{L^{p'}} = |\,|f|^{p-1}|_{L^{p'}} =$$

$$= |\,|f|^{p-1}|_{L^{\frac{p}{p-1}}}$$

$$\left|\,|f|^{\alpha}\right|_{L^q} = |f|^{\alpha}_{L^{q\alpha}}$$

$$= |f|^{p-1}_{L^p} < +\infty$$

$$\langle Tf, g\rangle_{(L^{p'})'\times L^{p'}} = \int f g = \int f\, |f|^{p-2} f$$

$$= \int |f|^p\, dx = |f|^p_{L^p} \le |Tf|_{(L^{p'})'}\, |f|^{p-1}_{L^p}$$

$$|f|_{L^p} \le |Tf|_{(L^{p'})'} \underset{=}{\le} |f|_{L^p}$$

$T: L^p \longrightarrow (L^{p'})'$ is an isometry

$R(T) = \cancel{T}\, L^p$ is closed in $(L^{p'})'$

$L^{p'}$ reflexive $\Rightarrow (L^{p'})'$ is $\Rightarrow R(T)$ is reflexive

$\Rightarrow L^p$ $1<p\le 2$ reflexive

$$L^p \subset (L^{p'})' \qquad 1<p<2$$

Thm (Riesz representation theorem)

$1 < p < \infty$ and $\phi \in (L^p)'$.

Then $\exists\, u \in L^{p'}$ s.t.

$$\phi(f) = \int_X u f \, d\mu \quad \forall f \in L^p$$

$\left(\exists\, u \in L^{p'} \text{ s.t. } \phi = Tu \right)$

Pf $\quad T : L^{p'} \longrightarrow (L^p)'$

$TL^{p'}$ is closed inside $(L^p)'$

If $TL^{p'} \subsetneq (L^p)'$

then $\exists\, h'' \in (L^p)''$ s.t. $\quad \langle Tu, h'' \rangle_{(L^p)' \times (L^p)''} = 0 \quad \forall u$
$h'' \neq 0$, $h'' = Jh$

$h'' = Jh \quad h \in L^p \qquad \left(\langle Tu, h \rangle_{(L^p)' \times L^p} = \right.$

$$= \int u h = 0 \quad \forall u \in L^{p'}$$

Here $h \in L^p$
$h \neq 0$

$$u = |h|^{p-2} h \qquad \int u h = \int |h|^p = \|h\|_{L^p}^p \neq 0$$

$TL^{p'} = L^p$

Theorem $\phi \in (L^1(X))'$ X σ-finite

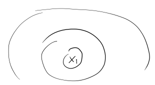
Then $\exists$ $u \in L^\infty(X)$ s.t. $\phi(f) = \int f u \, d\mu$ $\forall f \in L^1(X)$

Pf $X = \bigcup_{n=1}^{N} X_n$ $N \in \mathbb{N} \cup \{\infty\}$ $\mu(X_n) < +\infty$

$X_1 \subset X_2 \subset X_3 \subset \cdots$

Claim $\exists$ $w \in L^2(X)$ s.t. $\forall n$ $\exists$ $C_n > 0$ s.t. $w(x) \geq C_n$ in X_n

Pf



$w = \sum_{n=2}^{\infty} c_n 1_{X_n \setminus X_{n-1}} + c_1 1_{X_1}$

$c_1, c_2, \in \mathbb{R}_+$

$c_1 \mu(X_1) + \sum_{n=2}^{\infty} c_n \mu(X_n \setminus X_{n-1}) < +\infty$

$c_1 \mu(X_1) = 1$ $c_n \mu(X_n \setminus X_{n-1}) = \frac{1}{2^n}$

$w \in L^1(X)$

$w \in L^2(X)$

$\int |w|^2 = \int c_1^2 1_{X_1} + \sum_{n=2}^{\infty} \int c_n^2 1_{X_n \setminus X_{n-1}} < \infty$

$\phi \in (L^1(X))'$

$f \in L^2(X) \longrightarrow \langle \phi, fw \rangle$ $(L^1(X))' \times L^1(X)$

This is a bounded functional in $L^2(X)$

$|\langle \phi, fw \rangle| \leq |\phi|_{(L^1(X))'} |f|_{L^2} |w|_{L^2}$

$\exists$ $g \in L^2(X)$ st

$\langle \phi, fw \rangle = \int f g$ $\forall f \in L^2(X)$

$= \int f w \frac{g}{w}$

$\langle \phi, h \rangle = \int h u$ $\forall h \in L^1$

We want to show that $u = \frac{g}{w}$ is the function

$u \in L^\infty$ $|u|_{L^\infty} \leq |\phi|_{(L^1)'}$

Let $C > |\phi|_{(L^1)'}$ and

$A_\pm = \{x : \pm u(x) > C\}$

we show $|A_+| = 0$. $|A_+| > 0$

$\exists n$ st $|A_+ \cap X_n| > 0$

$C \int_{A_+ \cap X_n} w \leq \int_{A_+ \cap X_n} u w = \int_{A_+ \cap X_n} g$ ($u = \frac{g}{w}$)

$= \int 1_{A_+ \cap X_n} g = \langle \phi, 1_{A_+ \cap X_n} w \rangle_{(L^1)' \times L^1}$

$\leq |\phi|_{(L^1)'} |1_{A_+ \cap X_n} w|_{L^1}$ $\sim \int_{A_+ \cap X_n} w$

$C < |\phi|_{(L^1)'} < C$.

$|A_+| = 0$ $|u|_{L^\infty} \leq |\phi|_{(L^1)'}$

$\geq$

$f \in L^1 \cap L^2$

$\langle \phi, 1_{X_n} f \rangle = \langle \phi, (1_{X_n} \frac{f}{w}) w \rangle = \int 1_{X_n} \frac{f}{w} g$

$= \int 1_{X_n} f u$

$\langle \phi, f \rangle$

$\langle \phi, 1_{X_n} f \rangle = \int 1_{X_n} f u \to \int f u$ $\forall f \in L^1 \cap L^2$ $n \to +\infty$

$1_{X_n} f \to f$

$1_{X_n} f \xrightarrow{n \to +\infty} f$ in L^1

$|(1_{X_n} f)(x)| \leq |f(x)|$

$\langle \phi, f_n \rangle = \int f_n u$ ($\in L^1 \cap L^2$) $\forall f \in \underbrace{L^1 \cap L^2}_{\text{is dense in } L^1}$

$\langle \phi, f \rangle = \int f u$

$|\langle \phi, f \rangle| \leq |\int f u| \leq |f|_{L^1} |u|_{L^\infty}$ $\forall f \in L^1$

$|\phi|_{(L^1)'} = |u|_{L^\infty}$

Furthermore we prove $\forall \phi \in (L^1)'$ $\exists u \in L^\infty$

$\langle \phi, f \rangle = \int f u$