28th of November

Theorem (convolution) Let $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$ $\quad p, q \in [1, +\infty]$

set

$$f * g(x) = \int_{\mathbb{R}^d} f(x-y)\, g(y)\, dy$$

Then (Young's convolution inequality)

$$|f * g|_{L^r(\mathbb{R}^d)} \leq |f|_{L^p(\mathbb{R}^d)}\, |g|_{L^q(\mathbb{R}^d)}$$

for $\quad \frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$

Proof

$$L^p(\mathbb{R}^d) \times L^q(\mathbb{R}^d) \to L^r(\mathbb{R}^d)$$

$T : X \to Y$

$$\sup \left\{ \|Tx\|_Y : \|x\|_X \leq 1,\ x \in D \text{ where } D \text{ is dense in } X \right\}$$

We will ~~assume~~ that $C_c^0(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$ $\forall\ 1 \leq p < +\infty$.

$$\int \underbrace{\left(\int_{\mathbb{R}^d} f(x-y)\, g(y)\, dy \right)}_{L^r(\mathbb{R}^d)} h(x)\, dx \qquad h \in L^{r'}(\mathbb{R}^d)$$

$$I(f,g,h) = \int f(x-y)\, g(y)\, h(x)\, dx\, dy$$

It is enough to move $\quad I(f,g,h) \leq 1 \quad$ if

$$|f|_{L^p(\mathbb{R}^d)} = |g|_{L^q(\mathbb{R}^d)} = |h|_{L^{r'}(\mathbb{R}^d)} = 1$$

$f \geq 0$, $g \geq 0$, $h \geq 0$ $\qquad \frac{1}{r'} = 1 - \frac{1}{r} \qquad \frac{1}{r} = 1 - \frac{1}{r'}$

if $p < +\infty$ $\quad f \in C_c^0(\mathbb{R}^d)$ $\qquad \frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$

$q < +\infty$ $\quad g \in C_c^0(\mathbb{R}^d)$ $\qquad 2 = \frac{1}{r'} + \frac{1}{p} + \frac{1}{q}$

$r' < +\infty$ $\quad h \in C_c^0(\mathbb{R}^d)$

$$\left(2 - \frac{1}{p} - \frac{1}{q}\right) r' = 1$$

$$\left(2 - \frac{1}{p} - \frac{1}{r'}\right) q = 1 \qquad \left(2 - \frac{1}{r'} - \frac{1}{q}\right) p = 1$$

$$\begin{cases} \left(1 - \frac{1}{p}\right) r' + \left(1 - \frac{1}{q}\right) r' = 1 \\ \left(1 - \frac{1}{p}\right) q + \left(1 - \frac{1}{r'}\right) q = 1 \\ \left(1 - \frac{1}{r'}\right) p + \left(1 - \frac{1}{q}\right) p = 1 \end{cases} \qquad \frac{1}{r} p + \frac{1}{q'} p = 1$$

$$I(f,g,h) = \int f(y)\, g(x-y)\, h(x)\, dx\, dy =$$

$$= \int \left(f^p(y)\, g^q(x-y)\right)^{1-\frac{1}{r'}} \left(f^p(y)\, h^{r'}(x)\right)^{1-\frac{1}{q}} \left(g^q(x-y)\, h^{r'}(x)\right)^{1-\frac{1}{p}} dx\, dy$$

$$\frac{1}{r} + \frac{1}{p'} + \frac{1}{q'} = 1$$

$$\leq \left| \left(f^p(y)\, g^q(x-y)\right)^{\frac{1}{r}} \right|_{L^r(\mathbb{R}^d \times \mathbb{R}^d)} \left| \left(f^p(y)\, h^{r'}(x)\right)^{\frac{1}{q'}} \right|_{L^{q'}(\mathbb{R}^d \times \mathbb{R}^d)}$$

$$\left| \left(g^q(x-y)\, h^{r'}(x)\right)^{\frac{1}{p'}} \right|_{L^{p'}(\mathbb{R}^d \times \mathbb{R}^d)}$$

$$= \left| f^p(y)\, g^q(x-y) \right|_{L^1(\mathbb{R}^d \times \mathbb{R}^d)}^{\frac{1}{r}} \left| f^p(y)\, h^{r'}(x) \right|_{L^1(\mathbb{R}^d \times \mathbb{R}^d)}^{\frac{1}{q'}}$$

$$= \left| g^q(x-y)\, h^{r'}(x) \right|_{L^1(\mathbb{R}^d \times \mathbb{R}^d)}^{\frac{1}{p'}}$$

$$= \left(\int_{\mathbb{R}^d} dy\, f^p(y) \int_{\mathbb{R}^d} g^q(x-y)\, dx \right)^{\frac{1}{r}} \left(\int_{\mathbb{R}^d} f^p(y)\, dy \int_{\mathbb{R}^d} h^{r'}(x)\, dx \right)^{\frac{1}{q'}}$$

$$\cdot \left(\int_{\mathbb{R}^d} dx\, h^{r'}(x) \int_{\mathbb{R}^d} g^q(x-y)\, dy \right)^{\frac{1}{p'}}$$

$$= |f|_{L^p}^{\frac{p}{r}}\ |g|_{L^q}^{\frac{q}{r}}\ |f|_{L^q}^{\frac{p}{q'}}\ |h|_{L^{r'}}^{\frac{r'}{q'}}$$

$$|h|_{L^{r'}}^{\frac{r'}{p'}}\ |g|_{L^q}^{\frac{q}{p'}} \qquad p\left(\frac{1}{p} + \frac{1}{q'}\right) = 1$$

$$= 1$$

$p < +\infty$, $\quad q < +\infty$ $\quad r' < +\infty$

$$I : \overbrace{C_c^0(\mathbb{R}^d)}^{L^p} \times \overbrace{C_c^0(\mathbb{R}^d)}^{L^q} \times \overbrace{C_c^0(\mathbb{R}^d)}^{L^{r'}} \longrightarrow \mathbb{R}$$

$$|I(f,g,h)| \leq |f|_{L^p}\, |g|_{L^q}\, |h|_{L^{r'}}$$

Proof $f \in C_c^k(\mathbb{R}^d)$ $\quad g \in L^1_{loc}(\mathbb{R}^d)$

Then $f * g \in C^k(\mathbb{R}^d)$

$$\partial_x^\alpha (f * g) = (\partial_x^\alpha f) * g$$

$\forall \ |\alpha| \leq k$

$$f * g = g * f$$

<u>Theorem</u> $\rho \in L^1(\mathbb{R}^d)$ $\int_{\mathbb{R}^d} \rho(x)\,dx = 1$

$$\rho_\varepsilon(x) = \varepsilon^{-d} \rho\left(\frac{x}{\varepsilon}\right) \quad \forall \varepsilon > 0$$

For any $f \in L^p(\mathbb{R}^d)$ with $1 \le p < +\infty$

we have $\boxed{\lim_{\varepsilon \to 0^+} \rho_\varepsilon * f = f}$ in $L^p(\mathbb{R}^d)$

$\Downarrow$

In particular $\boxed{\lim_{t \to 0^+} e^{t\Delta} f = f \text{ in } L^p(\mathbb{R}^d)}$

$$e^{t\Delta} f(x) = \frac{1}{(4\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4t}} f(y)\,dy$$

$\rho_{\sqrt{t}}$ $\rho(x) = (4\pi)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4}}$

$$\begin{cases} (\partial_t - \Delta) u = 0 & t > 0 \\ u(0) = f \in L^\infty(\mathbb{R}) \end{cases}$$

$p < +\infty \qquad f \in C_c^0(\mathbb{R}^d) \subseteq L^p(\mathbb{R}^d)$

$\rho_\varepsilon * f(x) - f(x) =$

$= \int_{\mathbb{R}^d} f(x-y)\, \varepsilon^{-d} \rho\left(\frac{y}{\varepsilon}\right) dy - \int_{\mathbb{R}} f(x) \rho(z)\, dz$

$z = \frac{y}{\varepsilon}$

$= \int_{\mathbb{R}^d} \big(f(x - \varepsilon z) - f(x) \big) \rho(z)\, dz$

$\| \rho_\varepsilon * f - f \|_{L^p(\mathbb{R}^d)} = \left\| \int_{\mathbb{R}^d} \big(f(\cdot - \varepsilon z) - f \big) \rho(z)\, dz \right\|_{L^p(\mathbb{R}^d)}$

$\leq \int_{\mathbb{R}^d} \underbrace{\| f(\cdot - \varepsilon z) - f \|_{L^p(\mathbb{R}^d)}}_{\Delta(\varepsilon z)} |\rho(z)|\, dz$

$= \int_{\mathbb{R}^d} \underline{\Delta(\varepsilon z) |\rho(z)|}\, dz$

$\lim_{y \to 0} \Delta(y) = 0 \qquad \Delta(y) = \| f(\cdot - y) - f \|_{L^p(\mathbb{R}^d)}$

$\big| \big(f(x-y) - f(x) \big) \big|^p =$

$= \big| 1_K(x) \big(f(x-y) - f(x) \big) \big|^p \leq \underline{C\, 1_K(x)}$

$\lim_{y \to 0} \| f(\cdot - y) - f \|^p_{L^p(\mathbb{R}^d)} = \lim_{y \to 0} \int_{\mathbb{R}^d} | f(x-y) - f(x) |^p dx$

$= \int_{\mathbb{R}} \underbrace{\lim_{y \to 0} | f(x-y) - f(x) |^p}_{0} dx \qquad |\Delta(y)| \leq 2 \|f\|_{L^p(\mathbb{R}^d)}$

$| \Delta(\varepsilon z) \rho(z) | \leq 2 \|f\|_{L^p} \rho(z)$

$$\| \rho_\varepsilon * f - f \|_{L^p(\mathbb{R}^d)} = \left\| \int_{\mathbb{R}^d} (f(\cdot - \varepsilon z) - f) \rho(z) \, dz \right\|_{L^p(\mathbb{R}^d)}$$

$$\le \int_{\mathbb{R}^d} \underbrace{\| f(\cdot - \varepsilon z) - f \|_{L^p(\mathbb{R}^d)}}_{\Delta(\varepsilon z)} |\rho(z)| \, dz$$

$$= \int_{\mathbb{R}^d} \Delta(\varepsilon z) |\rho(z)| \, dz$$

$$0 \le \underbrace{\lim_{\varepsilon \to 0} \| \rho_\varepsilon * f - f \|_{L^p(\mathbb{R}^d)}}_{= 0} \le \lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} \Delta(\varepsilon z) |\rho(z)| \, dz$$

$$\int_{\mathbb{R}^d} \underbrace{\lim_{\varepsilon \to 0} \Delta(\varepsilon z)}_{0} |\rho(z)| \, dz = 0$$

If $f \in C_c^0(\mathbb{R}^d)$. If $f \in L^p(\mathbb{R}^d) \setminus C_c^0(\mathbb{R}^d)$

we pick $\tilde{f} \in C_c^0(\mathbb{R}^d)$ s.t. $\| f - \tilde{f} \|_{L^p(\mathbb{R}^d)} \le \delta$
with $\delta > 0$.

$$\| \rho_\varepsilon * f - f \|_{L^p} = \| \rho_\varepsilon * (f - \tilde{f}) + \rho_\varepsilon * \tilde{f} - \tilde{f} + \tilde{f} - f \|_{L^p}$$

$$\le \| \rho_\varepsilon * (f - \tilde{f}) \|_{L^p} + \underbrace{\| \rho_\varepsilon * \tilde{f} - \tilde{f} \|_{L^p}}_{\xrightarrow{\varepsilon \to 0} 0} + \underbrace{\| \tilde{f} - f \|_{L^p}}_{< \delta}$$

$$\le \underbrace{\| \rho_\varepsilon \|_{L^1}}_{\| \rho \|_{L^1}} \| f - \tilde{f} \|_{L^p}$$

$$\frac{1}{p} + 1 = \cancel{1} + \frac{1}{p} \qquad r = p \qquad q = 1$$

$$\| f * g \|_{L^r} \le \| f \|_{L^p} \| g \|_{L^q} \qquad \frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$$

$$\limsup_{\varepsilon \to 0} \| \rho_\varepsilon * f - f \| \le 2\delta \qquad \forall \delta > 0$$

$k \in L^q(\mathbb{R}^d)$

$\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$

$f \rightarrow Tf = k * f$

$L^p(\mathbb{R}^d) \longrightarrow L^r(\mathbb{R}^d)$

Hörmander

$r \geq p$

$\forall y \in \mathbb{R}^d$

$\tau_y f(x) = f(x-y)$

$\tau_y T = T \tau_y$

Theorem (K R F)

Let $\mathcal{F} \subseteq L^p(\mathbb{R}^d)$ $\quad p<+\infty$

be a bounded subset of $L^p(\mathbb{R}^d)$ s.t.

(a) $\forall \varepsilon>0 \ \exists \ \delta(\varepsilon)>0$ st $|h|<\delta(\varepsilon) \Longrightarrow |\tau_h f - f|_{L^p(\mathbb{R})}<\varepsilon \quad \forall f \in \mathcal{F}$.

Then $\forall \ \Omega \subset \mathbb{R}^d$ bounded $\mathcal{F}|_{\Omega}$ is relatively

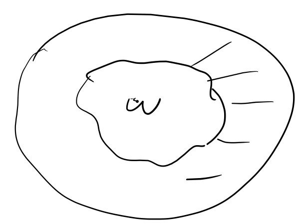
compact in $L^p(\Omega)$.

Pf We need ~~need~~ to prove that

$\forall \varepsilon>0 \quad \mathcal{F}|_{\Omega}$ is in a finite union of balls of radius ε in $L^p(\Omega)$.

(1) $\exists \ \omega \subset\subset \Omega$ st

$|f|_{L^p(\Omega \setminus \omega)} \le \frac{\varepsilon}{3} \quad \forall f \in \mathcal{F}$



Ω

$T(a,b) = \{ f \in C^1(\mathbb{R}^d) : |f|_{L^\infty(\mathbb{R}^d)} \le a, \ |\nabla f|_{L^\infty(\mathbb{R}^d)} \le b \}$

$T(a,b)|_{\omega}$ is relatively compact $C^0(\omega)$

$\rho \in C_c^\infty(D_{\mathbb{R}^d}(0,1), [0,+\infty))$ $\qquad \rho_n(x) = n^d \rho(nx)$

$D_{\mathbb{R}^d}(0, \frac{1}{n})$

$n > \frac{1}{\delta(\frac{\varepsilon}{4})}$ $\qquad \frac{1}{n} < \delta(\frac{\varepsilon}{4})$

$$|\rho_n * f - f|_{L^p(\mathbb{R}^d)} = \Big| \int_{\mathbb{R}^d} \rho_n(y) \big(\overbrace{f(x-y)}^{\tau_y f(x)} - f(x)\big) dy \Big|_{L^p(\mathbb{R}^d_x)} \le$$

$$\le \int_{|y|<\delta(\frac{\varepsilon}{4})} \rho_n(y) \underbrace{|\tau_y f - f|_{L^p(\mathbb{R}^d)}}_{\frac{\varepsilon}{4}} dy \le \int_{\mathbb{R}^d} \rho_n(y)\, dy \ \frac{\varepsilon}{4} = \frac{\varepsilon}{4}$$

$|\rho_n * f(x)| \le a_n \qquad \forall f \in \mathcal{F}$

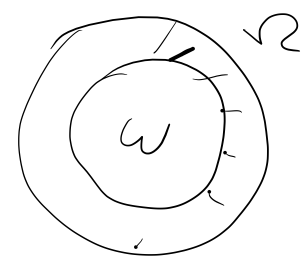
$|\rho_n * f(x)| = \left|\int_{\mathbb{R}^d} \rho_n(x-y)\, f(y)\, dy\right| \le$

$\le \int_{\mathbb{R}^d} \rho_n(x-y)\, |f(y)|\, dy \le |\rho_n|_{L^{p'}} |f|_{L^p} \qquad \forall f \in \mathcal{F}$

$\le C\, |\rho_n|_{L^{p'}}$

$|\nabla(\rho_n * f)(x)| \le b_n \qquad \forall f \in \mathcal{F}$

$\{\rho_n * f\}_{f \in \mathcal{F}} \subseteq T(a_n, b_n)$



There exists a finite family of

$T(a_n, b_n)\big|_\omega \subseteq \bigcup_{j=1}^{N} D_{C^0(\omega)}\left(u_j, \frac{\varepsilon}{C_1}\right)$

$C^0(\omega) \subseteq L^p(\omega)$

$T(a_n, b_n)\big|_\omega \subseteq \bigcup_{j=1}^{N} D_{L^p(\omega)}\left(u_j, \frac{\varepsilon}{3}\right)$

$\mathcal{F}\big|_\Omega \subseteq \bigcup_{j=1}^{N} D_{L^p(\Omega)}\left(u_j, \varepsilon\right)$