VETRI DI SPIN

"The history of spin glass may be the best example I know of the dictum that a real scientific mistery is worth pursuing to the ends of the Earth for its own sake, independently of any obvious practical importance or intellectual glamour."

— P.W. Anderson (1988)

Nobel Prize in Physics 2021

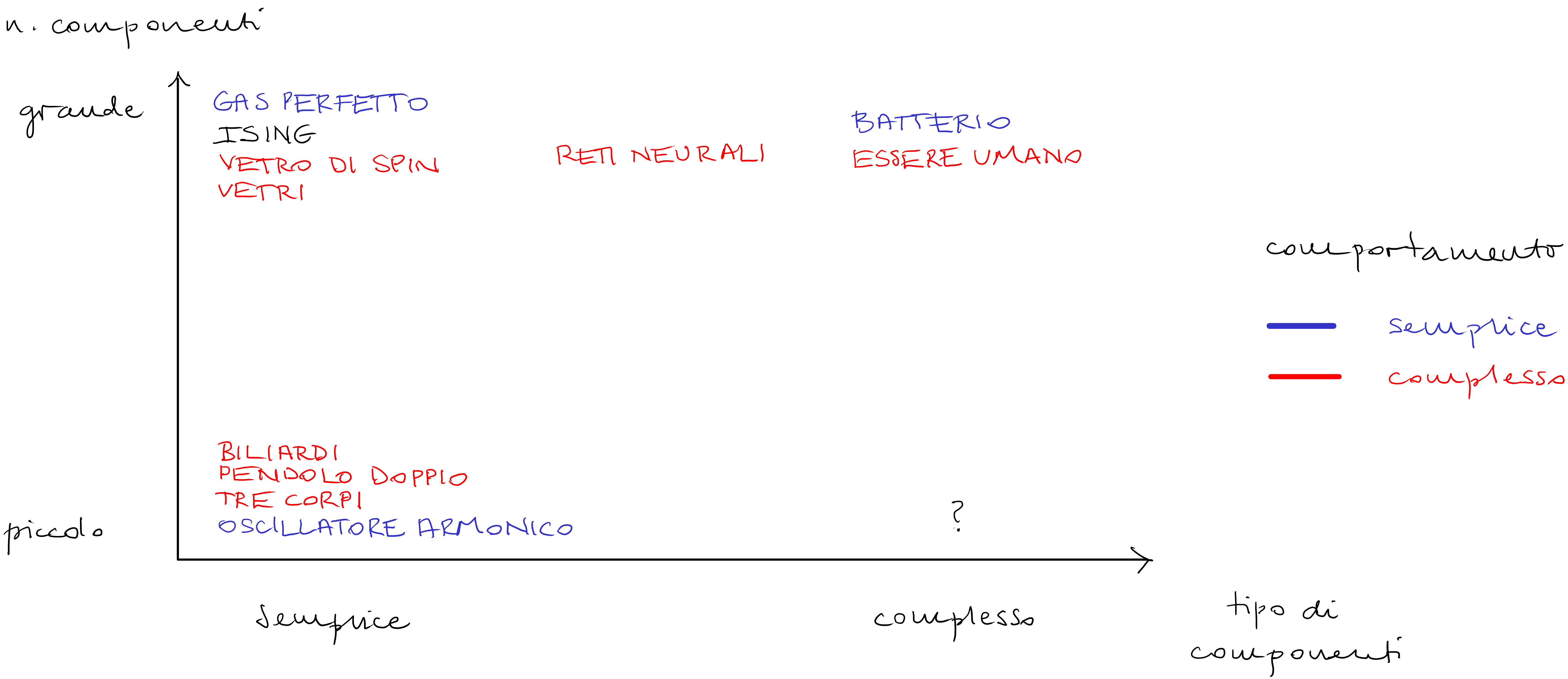
"for groundbreaking contributions to our understanding of complex physical systems"

Syukuro Manabe and Klaus Hasselmann

"for the physical modelling of Earth's climate, quantifying variability and reliably predicting global warming"

Giorgio Parisi

"for the discovery of the interplay of disorder and fluctuations in physical systems from atomic to planetary scales"



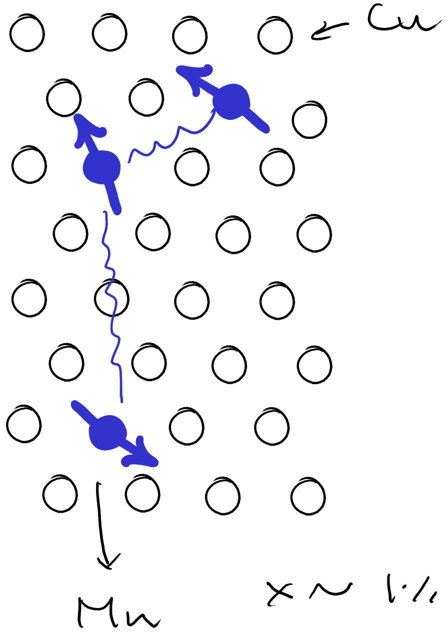
n. componenti
grande
piccolo
GAS PERFETTO
ISING
VETRO DI SPIN
VETRI
RETI NEURALI
BATTERIO
ESSERE UMANO
BILIARDI
PENDOLO DOPPIO
TRE CORPI
OSCILLATORE ARMONICO
?
semplice
complesso
tipo di componenti
comportamento
semplice
complesso

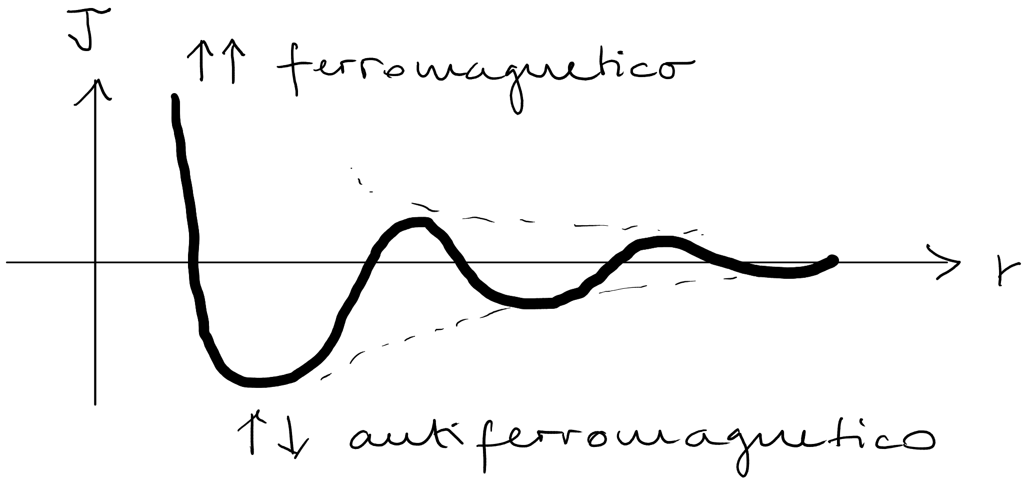
INTRODUZIONE

Vetri di spin vs. vetri strutturali

Fenomenologia

Miscela : metallo nobile (Au, Ag, Pt, Cu) → $(1-x)$ } concentrazione
elementi con magnetismo (Fe, Mn) → x





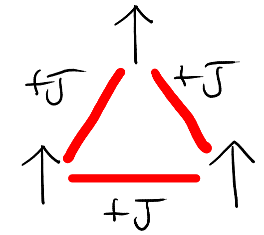
RKKY = Ruderman - Kittel - Kasuya - Yosida

$$J(r) \sim \frac{\cos(2kr)}{(kr)^3}$$

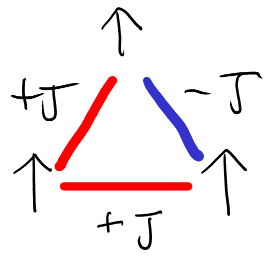
1. disordine gelato

② frustrazione $\quad -J_{ij}\sigma_i\sigma_j \quad +J \; \uparrow\uparrow \quad -J \; \uparrow\downarrow$

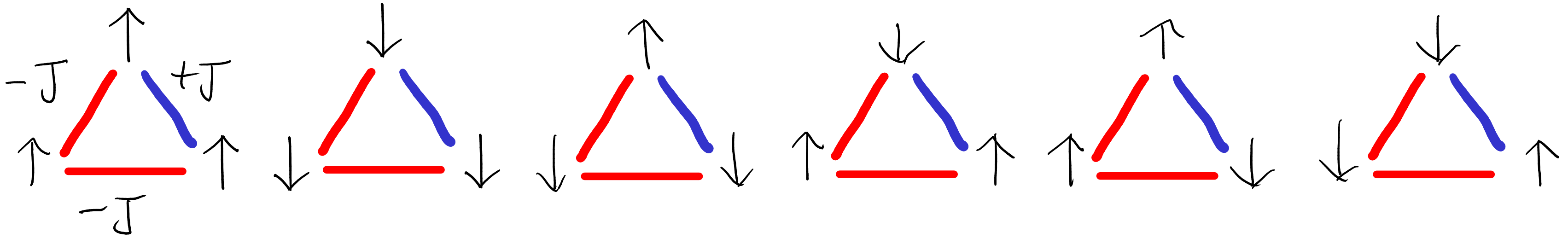
→ impossibilità di rendere favorevoli tutti gli accoppiamenti



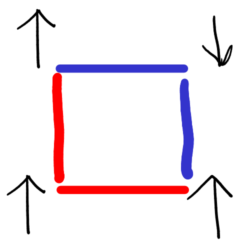
$U = -3J$



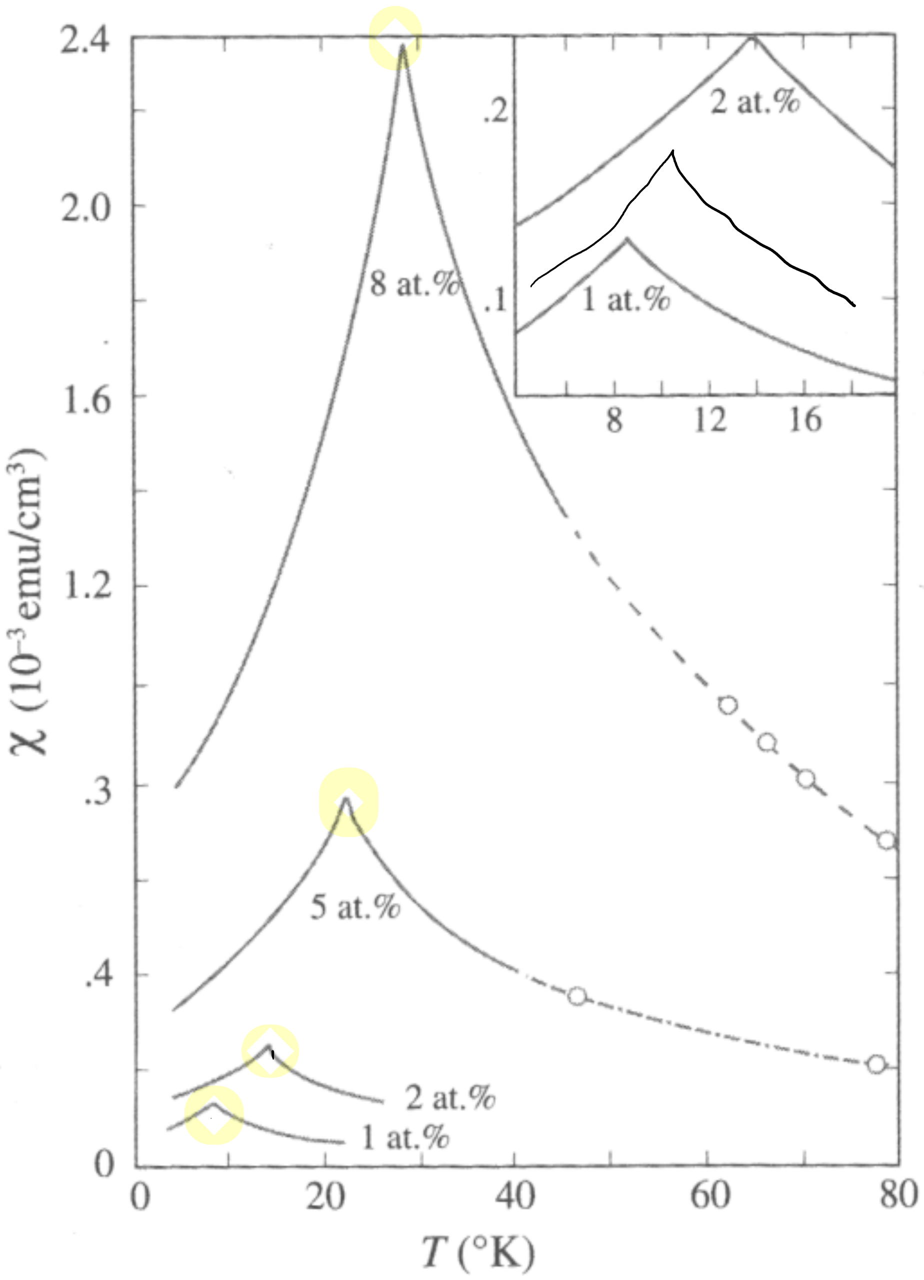
$U = -2J$

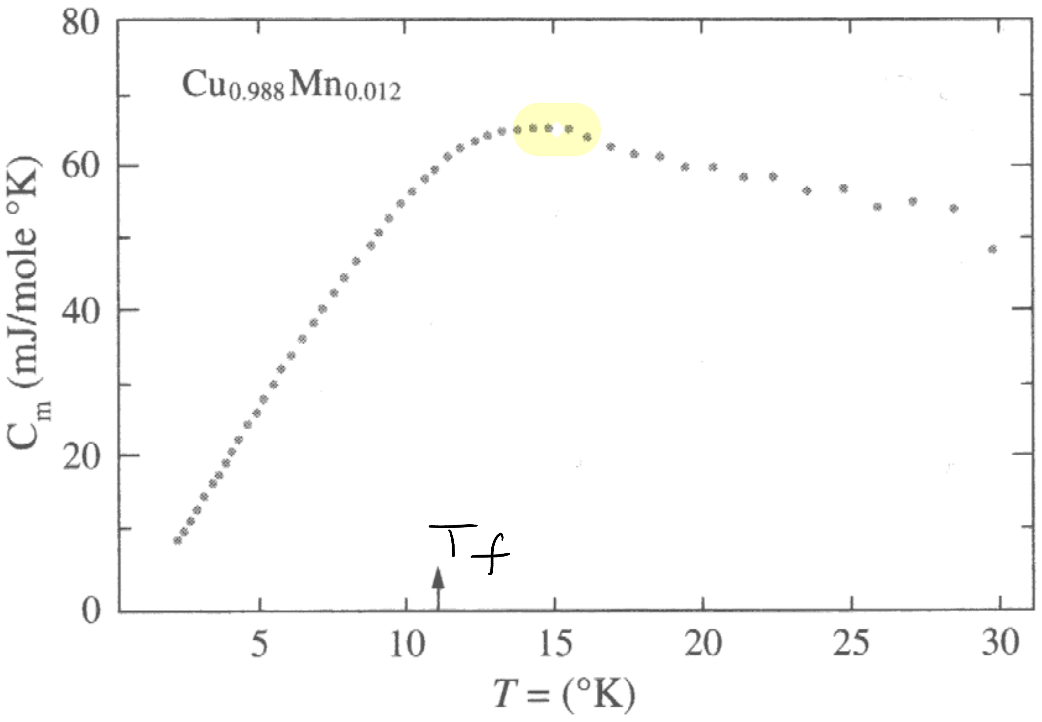


degenerazione = 6



$Cu_{1-x}Mn_x \qquad x \sim 1\%$





$$C_V = \frac{\partial U}{\partial T}$$

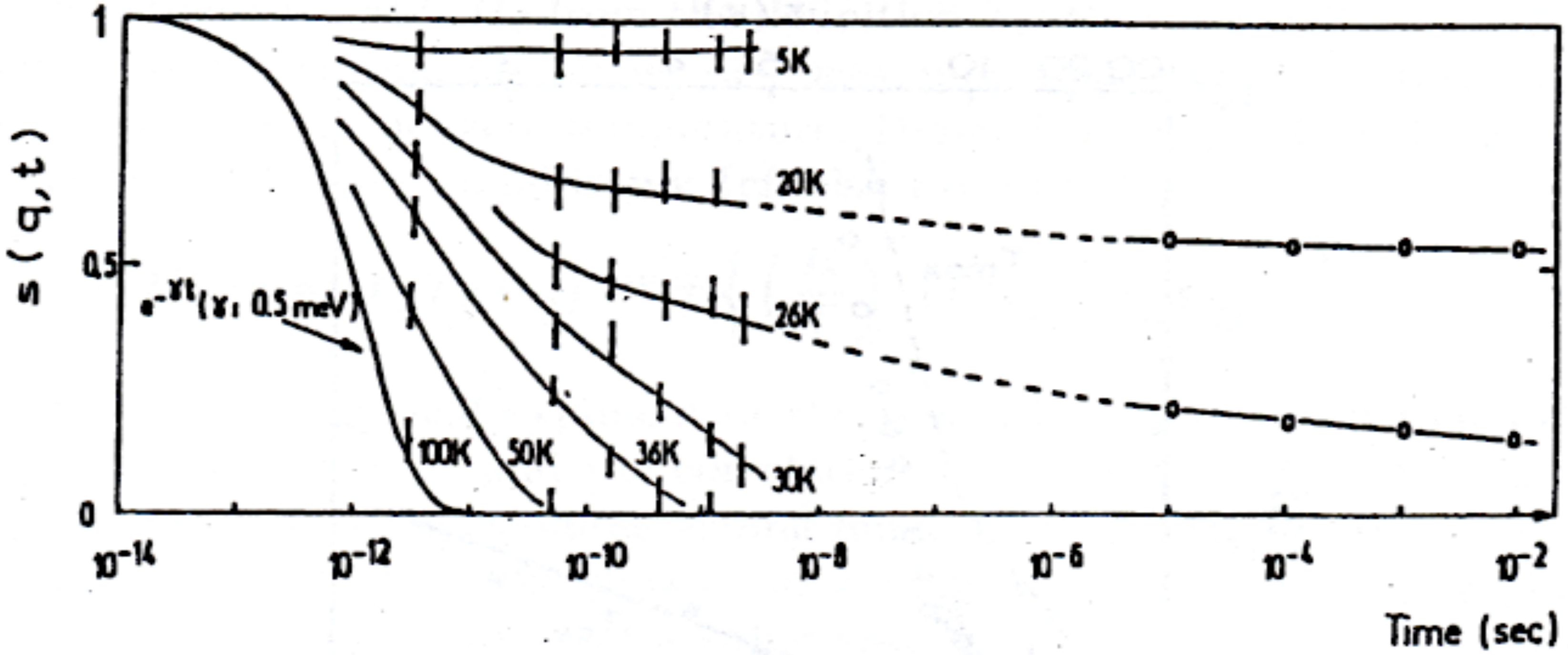
suscettibilità magnetica : $\chi = \frac{\partial M}{\partial h}\Big|_{h=0}$

$$\delta \tilde{A} = \chi \, \phi$$

$M = \chi \, h \leftarrow$ campo magnetico

↑

magnetizzazione



rilassamento degli spin

$\sim F(k,t)$

Modelli teorici

dofs: spins $\{\sigma_i\}$ $i = 1, \dots, N$

interazioni: J_{ij} → disordine gelato

Hamiltoniana

$$H = - \sum_{i=1}^{N} \sum_{j>i}^{N} J_{ij} \sigma_i \sigma_j - h \sum_{i=1}^{N} \sigma_i$$ $h = 0$ nel seguito

- Spins : m componenti $|\sigma_i| = 1$

$m=1$ Ising $\sigma_i = \pm 1$ ↑↓

$m=2$ XY $\vec{\sigma}_i \to \theta_i$

$m=3$ Heisenberg $\to \theta_i, \varphi_i$

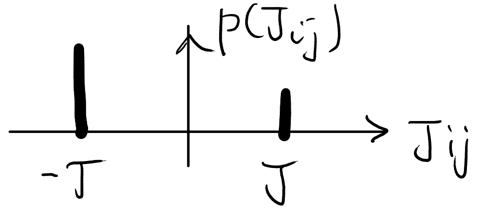
- range

"fully connected" $\sum_i \sum_{j>i}$

"short-range" $\sum_{\langle ij \rangle}$ [Edwards - Anderson]

- <u>accoppiamenti</u> J_{ij} variabili aleatorie gelate

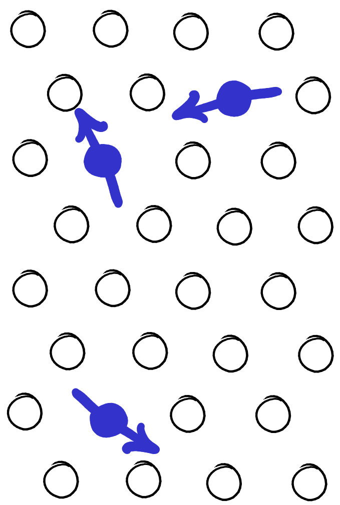
±J model $p(J_{ij}) = x\,\delta(J_{ij}-J) + (1-x)\,\delta(J_{ij}+J)$

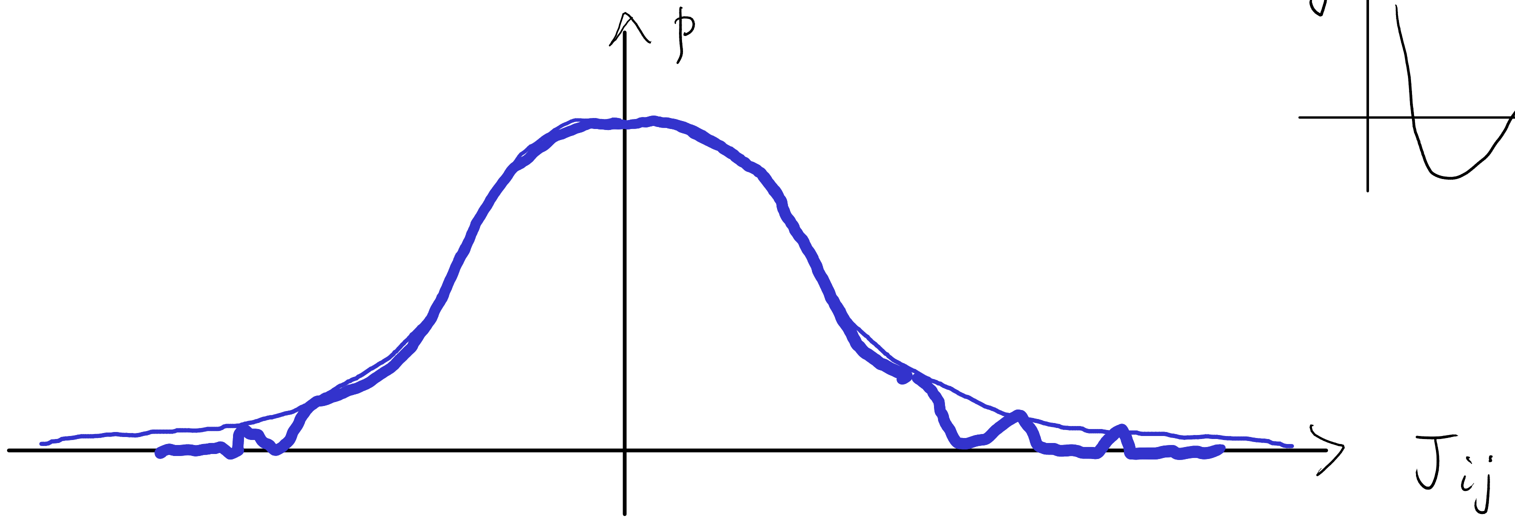


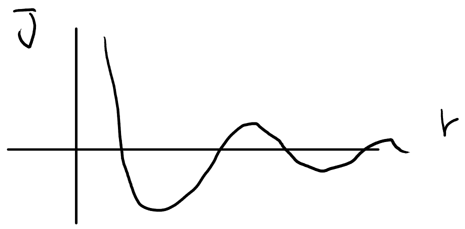
gaussiana

$$p(J_{ij}) = \frac{1}{\sqrt{2\pi\Delta J^2}} \exp\left[-\frac{(J_{ij}-J_0)^2}{2\Delta J^2}\right]$$ short range

$$p(J_{ij}) = \frac{1}{\sqrt{2\pi\Delta J^2/N}} \exp\left[-\frac{(J_{ij}-J_0)^2}{2\Delta J^2/N}\right]$$ fully connected







Media sul disordine

$A(\sigma^N; J) \qquad \sigma^N = \{\sigma_1 \dots, \sigma_N\} \qquad H[\sigma^N; J]$

$$\langle A \rangle = \mathrm{Tr}\left[e^{-\beta H[\sigma^N; J]} A(\sigma^N; J) \right] / \mathrm{Tr}\left[e^{-\beta H} \right]$$

$$= \int d\sigma^N \, e^{-\beta H[\sigma^N; J]} A(\sigma^N; J) / Z(N; J)$$

media termica $\langle \dots \rangle$

$$\overline{A} = \int dJ \, p(J) \, A(N; J)$$

media sul disordine $\overline{\dots}$

Osservabile è self-averaging se

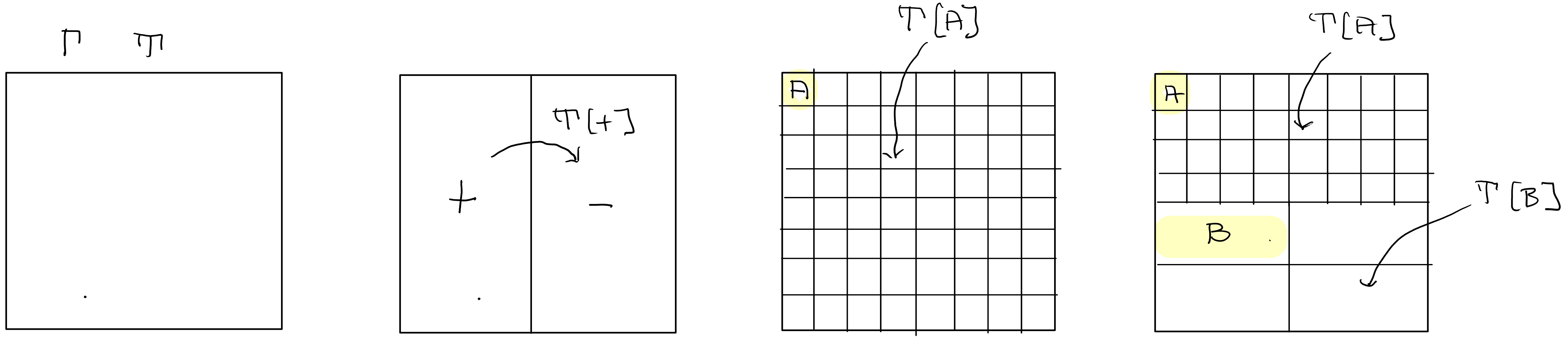
$$A = \lim_{N\to\infty} A(N; J) = \lim_{N\to\infty} \int dJ \, p(J) \, A(N; J)$$

Es: energia libera

$$F(N; J) = -k_B T \ln(Z(N; J)) \qquad f(N; J) = \frac{F(N; J)}{N}$$

$$-\beta f = \lim_{N\to\infty} \frac{1}{N} \int dJ \, p(J) \ln(Z(N; J)) = \lim_{N\to\infty} \frac{1}{N} \overline{\ln(Z(N; J))}$$

Rottura di simmetria $\Rightarrow$ rottura di ergodicità $\quad N \to \infty$



$T > T_c$ | $T < T_c$ ferromagnetismo | $T < T_M$ cristallizzazione | $T < T_c$ "replica symmetry breaking"

Stato : ensemble ristretto ("stato puro")

$$\langle \ldots \rangle = \sum_\alpha w_\alpha \langle \ldots \rangle_\alpha \qquad Z_\alpha = \int_\alpha d\Gamma \, e^{-\beta H}$$

$$\langle A \rangle = \frac{1}{Z} \int d\Gamma \, e^{-\beta H[\Gamma]} A(\Gamma) = \frac{1}{Z} \sum_\alpha \int_\alpha d\Gamma \, e^{-\beta H} A = \sum_\alpha \underbrace{\frac{Z_\alpha}{Z}}_{w_\alpha} \underbrace{\frac{1}{Z_\alpha} \int_\alpha d\Gamma \, e^{-\beta H} A}_{\langle A \rangle_\alpha}$$

Es. : Ising $\quad T < T_c \quad \langle \sigma_i \rangle = \frac{1}{2} \underbrace{\langle \sigma_i \rangle_+} + \frac{1}{2} \underbrace{\langle \sigma_i \rangle_-} = 0$

Parametro d'ordine

Magnetizzazione

$$m = \frac{1}{N} \sum_{i=1}^{N} \langle \sigma_i \rangle \qquad \langle \sigma_i \rangle_t = m$$

Disordine gelato $\quad p(J_{ij}) \quad$: $\quad m = \frac{1}{N} \sum_{i=1}^{N} \overline{\langle \sigma_i \rangle} = 0$

Parametro d'ordine di Edwards - Anderson

$$q_{EA} = \frac{1}{N} \sum_{i=1}^{N} \overline{\langle \sigma_i \rangle^2}$$

Overlap

- Configurazioni $\sigma^N = \{\sigma_i\}$ $i = 1, \dots, N$

σ^N, σ'^N $\qquad q_{\sigma\sigma'} = \frac{1}{N}\sum_{i=1}^{N} \sigma_i \sigma'_i$ $\begin{cases} 1 \\ \approx 0 \text{ scorrelate} \\ -1 \end{cases}$ $\qquad q_{\sigma\sigma} = 1$

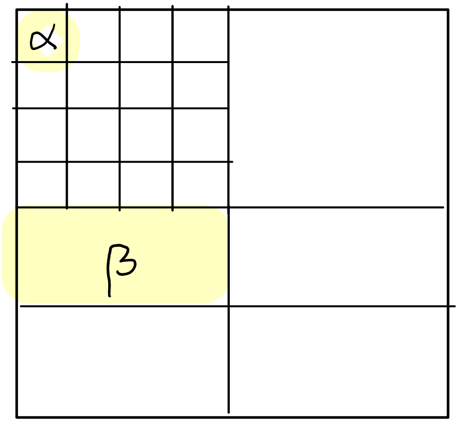
- Stati

α, β

$$q_{\alpha\beta} = \frac{1}{N}\sum_{i=1}^{N} \langle \sigma_i \rangle_\alpha \langle \sigma_i \rangle_\beta$$

$$= \frac{1}{N}\sum_{i=1}^{N} \frac{1}{Z_\alpha}\int_\alpha d\sigma^N e^{-\beta H[\sigma^N]} \sigma_i \frac{1}{Z_\beta}\int_\beta d\sigma'^N e^{-\beta H[\sigma'^N]} \sigma'_i$$

$$= \int_\alpha d\sigma^N \int_\beta d\sigma'^N \underbrace{\frac{1}{Z_\alpha} e^{-\beta H[\sigma^N]} \frac{1}{Z_\beta} e^{-\beta H[\sigma'^N]}}_{\text{peso statistico}} \underbrace{\frac{1}{N}\sum_{i=1}^{N} \sigma_i \sigma'_i}_{q_{\sigma\sigma'}}$$



$q_{\alpha\alpha}$ self-overlap

$q_{\alpha\alpha} = \frac{1}{N}\sum_{i=1}^{N} \langle \sigma_i \rangle_\alpha^2 \qquad \overline{q_{\alpha\alpha}} = q_{EA} \qquad \mathbb{Z}_2$

Distribuzione degli overlap

2 copie del sistema per J_{ij} fissato $\longrightarrow$ repliche $\rightarrow$ σ^N, σ'^N

$$p(q) = \frac{1}{Z^2} \int d\sigma \int d\sigma' \, e^{-\beta H[\sigma; J]} e^{-\beta H[\sigma'; J]} \, \delta(q - q_{\sigma\sigma'}) = \dots =$$

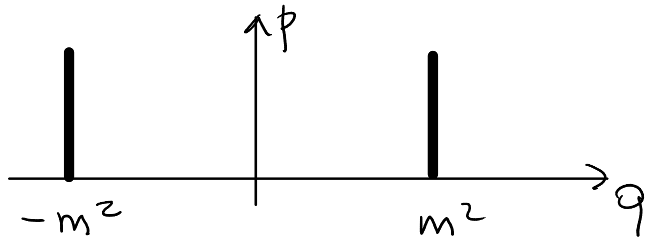
$$p(q) = \sum_\alpha \sum_\beta w_\alpha w_\beta \, \delta(q - q_{\alpha\beta}) \qquad [\text{def.}]$$

Es.: Ising $\quad w_+ = w_- = \frac{1}{2} \quad T < T_c$

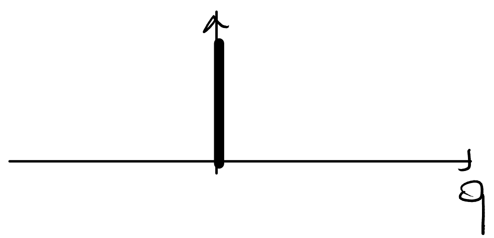
$$q_{++} = \frac{1}{N} \sum_{i=1}^{N} \langle \sigma_i \rangle_+^2 = m^2 = q_{--}$$

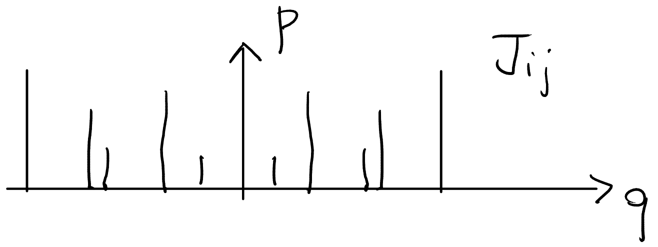
$$q_{+-} = q_{-+} = -m^2$$

$T < T_c$



$T > T_c$





$\rightarrow \overline{p}(q)$

$T < T_c$

vetro di spin

Metodo delle repliche

$$F = -K_B T\, \overline{\log Z} = -K_B T \int dJ\, p(J) \log Z(N;J)$$

$$Z^n = \exp(n \log Z) \approx 1 + n \log Z + O(n^2)$$

$$\log Z = \lim_{n\to 0} \frac{Z^n - 1}{n}$$

$$\boxed{\overline{\log Z} = \lim_{n\to 0} \frac{\overline{Z^n} - 1}{n}}$$

"replica trick"

$$Z^n = \underbrace{Z \dots Z}_{\times n}$$

n intero + continuazione analitica $n \to 0$

$$Z^n = \int d\sigma_1^N \dots \int d\sigma_n^N\, e^{-\beta \sum_{a=1}^{n} H[\sigma_a^N; J]}$$

$a, b \to$ repliche

$$\overline{Z^n} = \underbrace{\int d\sigma_1^N \dots \int d\sigma_n^N}_{Tr_\sigma}\, \underbrace{\overline{e^{-\beta \sum_{a=1}^{n} H[\sigma_a^N; J]}}}_{e^{-\beta H_{eff}}} = Tr_\sigma \left[e^{-\beta H_{eff}} \right]$$

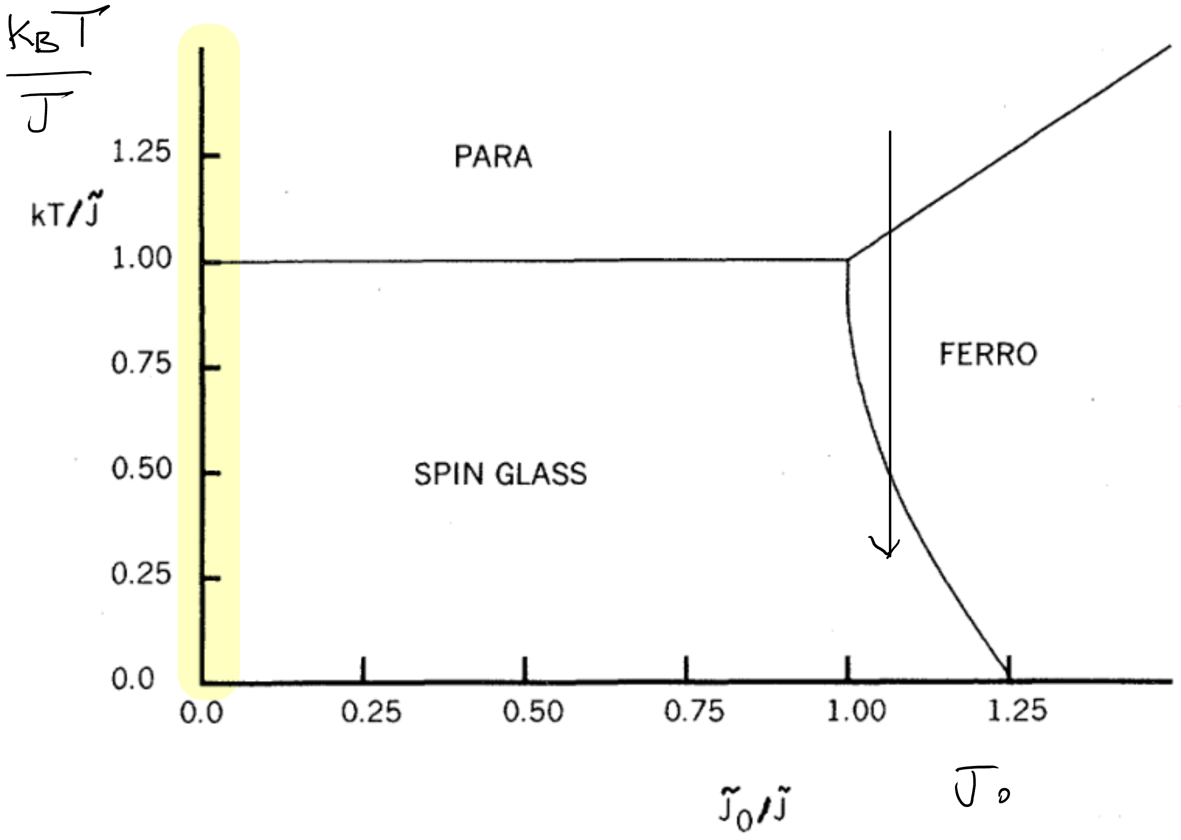
$$-\beta f = \lim_{N\to\infty} \lim_{n\to 0} \frac{\overline{Z^n} - 1}{nN}$$

① $N \gg 1$ $\lim_{n\to 0}$

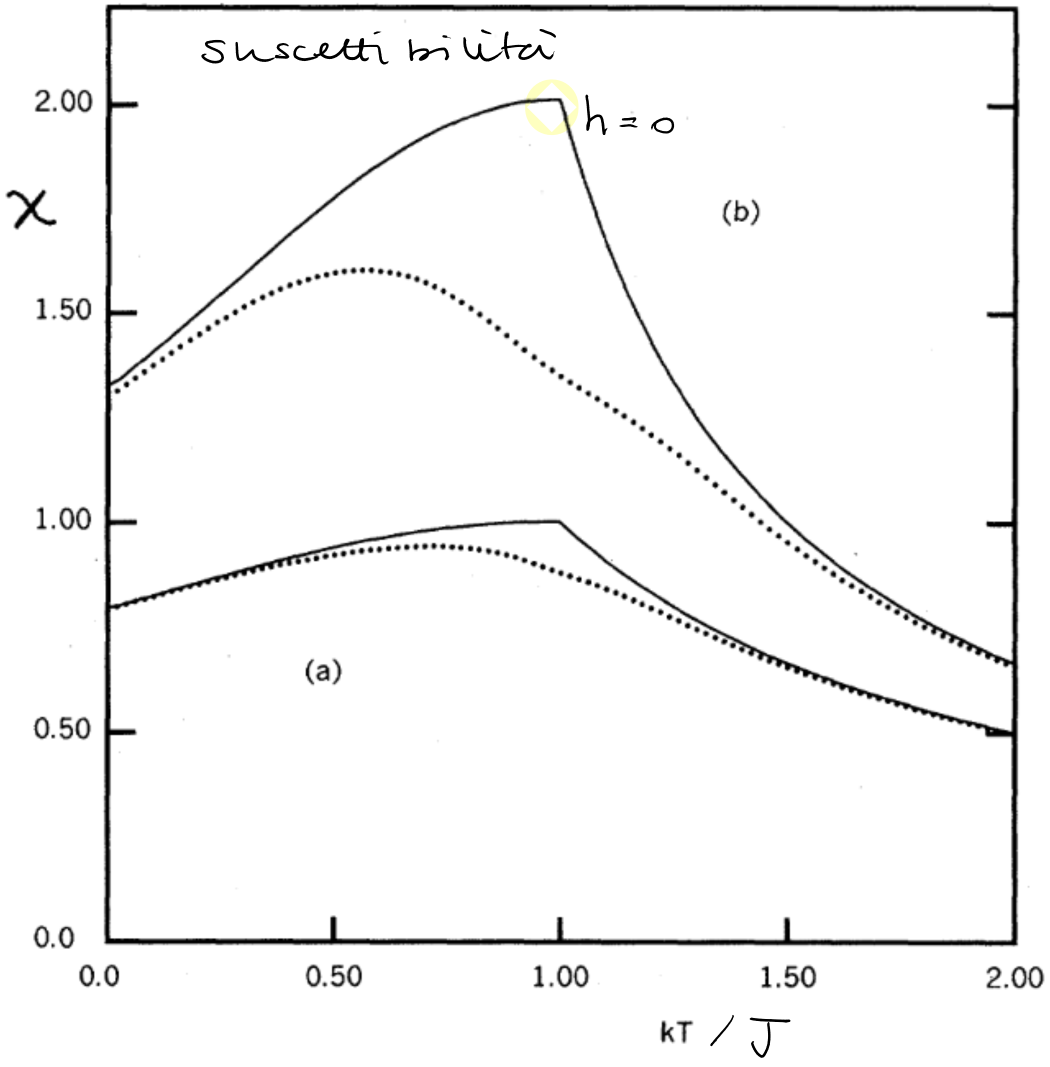
② $\lim_{N\to\infty}$

Modello di Sherrington- Kirkpatrick (1975)

$$H = - \sum_{i=1}^{N} \sum_{j>i}^{N} J_{ij} \sigma_i \sigma_j - h \sum_{i=1}^{N} \sigma_i \qquad \sigma_i = \pm 1 \qquad p(J_{ij}) = \frac{1}{\sqrt{2\pi J^2/N}} \exp\left(- \frac{(J_{ij}-J_0)^2}{2 J^2/N}\right)$$



$J_0 = 0$
$h = 0$



(1) replica trick : $f = - \lim_{N\to\infty} \lim_{n\to 0} \dfrac{\overline{Z^n} - 1}{nN\beta}$

(2) integrali gaussiani : $\sqrt{\dfrac{\lambda}{2\pi}} \displaystyle\int_{-\infty}^{\infty} dx \exp\left(-\frac{\lambda x^2}{2} + \lambda x a\right) = e^{\lambda a^2/2}$

(3) approx punto sella : $\displaystyle\int_{-\infty}^{\infty} dx\, e^{Nf(x)} \approx \sqrt{\frac{2\pi}{N|f''(x^*)|}} \exp\left(Nf(x^*)\right) \qquad N \gg 1$

Metodo delle repliche : $\mathrm{Tr}[\ldots] = \mathrm{Tr}_{\sigma_1^N} \ldots \mathrm{Tr}_{\sigma_n^N} [\ldots]$

$$\overline{Z^n} = \int_{-\infty}^{\infty} \prod_{i=1}^{N} \prod_{j>i} dJ_{ij}\, p(J_{ij})\, \mathrm{Tr}\left[\exp\left(\beta \sum_{a=1}^{n} \sum_{i=1}^{N} \sum_{j>i}^{N} J_{ij} \sigma_{ia} \sigma_{ja}\right)\right]$$

$$= \mathrm{Tr}\left[\int_{-\infty}^{\infty} \prod_i \prod_{j>i} dJ_i \frac{1}{\sqrt{2\pi J^2/N}} \exp\left(-\frac{J_{ij}^2}{2J^2/N} + \beta \sum_i \sum_{j>i} J_{ij} \underbrace{\left(\sum_{a=1}^{n} \sigma_{ia}\sigma_{ja}\right)}_{\phi_{ij}}\right)\right]$$

Ne prendo uno

$$\int_{-\infty}^{\infty} dJ_{ij} \frac{1}{\sqrt{2\pi J^2/N}} \exp\left(-\frac{J_{ij}^2}{2J^2/N} + \beta \phi_{ij} J_{ij}\right) = \exp\left(\frac{\beta^2 J^2 \phi_{ij}^2}{2N}\right)$$

$$\sqrt{\frac{\lambda}{2\pi}} \int_{-\infty}^{\infty} dx \, \exp\left(-\frac{\lambda x^2}{2} + \lambda x a\right) = e^{\lambda a^2/2} \qquad x = J_{ij} \qquad a = (J^2/N)\beta\phi_{ij} \qquad \lambda = \frac{1}{J^2/N}$$

$$\overline{Z^n} = \mathrm{Tr}\left[\prod_i \prod_{j>i} \exp\left(\frac{\beta^2 J^2}{2N} \sum_a \sum_b \sigma_{ia}\sigma_{ja}\sigma_{ib}\sigma_{jb}\right)\right]$$

$$\mathrm{Tr}\left[\exp\left(\frac{\beta^2 J^2}{2N} \sum_i \sum_{j>i} \sum_a \sum_b \underbrace{\sigma_{ia}\sigma_{ja}\sigma_{ib}\sigma_{jb}}_{\cdots}\right)\right]$$

$$\sum_i \sum_{j>i} \sum_a \sum_b \cdots = \left(\frac{1}{2}\sum_i \sum_j \sum_a \sum_b \cdots - \sum_i^N \sum_a \sum_b \sigma_{ia}^2 \sigma_{ib}^2\right)$$

$$= \sum_a \sum_b \left(\frac{1}{2}\sum_i \sum_j \cdots - \frac{N}{2}\right)$$

$$= \sum_a \sum_b{}' \left(\underbrace{\frac{1}{2}\sum_i \sum_j \cdots}_{\sim N^2 \checkmark} - \underbrace{\frac{N}{2}}_{\sim N \times}\right) + \sum_a \left(\underbrace{\frac{1}{2}\sum_i \sum_j \sigma_{ia}^2 \sigma_{ja}^2}_{\sim N^2 \checkmark} - \underbrace{\frac{N}{2}}_{\sim N \times}\right)$$

$$= \frac{1}{2}\sum_a \sum_b{}' \cdots + \frac{1}{2} n N^2$$

$$\overline{Z^n} = \exp\left(\frac{Nn\beta^2J^2}{4}\right) \mathrm{Tr}\left[\exp\left(\frac{\beta^2J^2}{4N}\sum_a \sum_b{}' \underbrace{\sum_i \sum_j \sigma_{ia}\sigma_{ja}\sigma_{ib}\sigma_{jb}}_{\left(\sum_i \sigma_{ia}\sigma_{ib}\right)^2}\right)\right]$$

$$= \exp\left(\frac{Nn\beta^2J^2}{4}\right) \mathrm{Tr}\left[\exp\left(\frac{\beta^2J^2}{2N}\sum_a \sum_{b>a} \left(\sum_i \sigma_{ia}\sigma_{ib}\right)^2\right)\right]$$

"Linearizzo" l'integrale. Prendo un termine

$$\exp\left(\frac{\beta^2J^2}{2N}\left(\sum_i \sigma_{ia}\sigma_{ib}\right)^2\right) = \sqrt{\frac{\beta^2J^2}{2\pi N}} \int_{-\infty}^{\infty} dq_{ab} \exp\left(-\frac{\beta^2J^2}{2N}q_{ab}^2 + \frac{\beta^2J^2}{N} q_{ab} \sum_i \sigma_{ia}\sigma_{ib}\right)$$

$$e^{\lambda A^2/2} = \sqrt{\frac{\lambda}{2\pi}} \int_{-\infty}^{\infty} dx \exp\left(-\frac{\lambda x^2}{2} + \lambda x A\right) \qquad x = q_{ab} \qquad A = \sum_{i=1}^{N} \sigma_{ia}\sigma_{ib} \qquad \lambda = \frac{\beta^2J^2}{N}$$

$$\overline{Z^n} = \int_{-\infty}^{\infty} \prod_a \prod_{b>a}\left(dq_{ab}\sqrt{\frac{\beta^2J^2}{2\pi N}}\right) \cdot \exp\left(\frac{Nn\beta^2J^2}{4}\right) \cdot \exp\left(-\frac{\beta^2J^2}{2N}\sum_a\sum_{b>a} q_{ab}^2\right) \cdot$$

$$\cdot\ \mathrm{Tr}\left[\exp\left(\frac{\beta^2J^2}{N}\sum_a\sum_{b>a} q_{ab}\left(\sum_i \sigma_{ia}\sigma_{ib}\right)\right)\right]$$

Cambio variabile: $q_{ab} = N\tilde{q}_{ab} \to \tilde{q}_{ab} \to q_{ab}$

$$\overline{Z^n} = \int_{-\infty}^{\infty} \prod_a \prod_{b>a} \left(dq_{ab} \sqrt{\frac{\beta^2 J^2 N}{2\pi}} \right) \cdot \exp\left(\frac{Nn\beta^2 J^2}{4} \right) \cdot \exp\left(-\frac{N\beta^2 J^2}{2} \sum_a \sum_{b>a} q_{ab}^2 \right) \cdot$$

$$\cdot \underbrace{\mathrm{Tr}\left[\exp\left(\beta^2 J^2 \sum_a \sum_{b>a} q_{ab} \left(\sum_i \sigma_{ia} \sigma_{ib} \right) \right) \right]}_{\left\{ \mathrm{Tr}_{\sigma_a} \left[\exp\left(\beta^2 J^2 \sum_a \sum_{b>a} q_{ab} \sigma_a \sigma_b \right) \right] \right\}^N}$$

$$\mathrm{Tr}[\ldots] = \exp\left(N \log\left(\mathrm{Tr}_{\sigma_a} \left[\exp\left(\underbrace{\beta^2 J^2 \sum_a \sum_{b>a} q_{ab} \sigma_a \sigma_b}_{L} \right) \right] \right) \right)$$

$$\overline{Z^n} = \int_{-\infty}^{\infty} \prod_a \prod_{b>a} \left(dq_{ab} \sqrt{\frac{\beta^2 J^2 N}{2\pi}} \right) \cdot$$

$$\exp\left(\frac{Nn\beta^2 J^2}{4} - \frac{N\beta^2 J^2}{2} \sum_a \sum_{b>a} q_{ab}^2 + N \log \mathrm{Tr}_{\sigma_a} e^L \right)$$

$$\overline{Z^n} = \int_{-\infty}^{\infty} \prod_a \prod_{b>a} \left(dq_{ab} \sqrt{\frac{\beta^2 J^2 N}{2\pi}} \right) \exp\left(-Nn\beta f(\{q_{ab}\})\right)$$

$$f(\{q_{ab}\}) = \frac{\beta J^2}{4} - \frac{\beta J^2}{2n} \sum_a \sum_{b>a} q_{ab}^2 + \frac{1}{n\beta} \log \operatorname{Tr}_{\sigma_a} e^L$$

approx punto sella per $N \gg 1$

$$\begin{cases} \overline{Z^n} \approx \exp\left(-Nn\beta f(\{q^*_{ab}\})\right) \\ \dfrac{\partial f}{\partial q_{ab}}(\{q^*_{ab}\}) = 0 \end{cases}$$

Linearizzo l'esponenziale per $n \to 0$

$$\overline{Z^n} \approx 1 - Nn\beta f(\{q^*_{ab}\}) \qquad f = -\lim_{N\to\infty} \lim_{n\to 0} \frac{\overline{Z^n} - 1}{Nn\beta} = \lim_{n\to 0} f(\{q^*_{ab}\})$$

$$\frac{\partial}{\partial q_{ab}} \left(\frac{\beta^2 J^2}{2} \sum_a \sum_{b>a} q_{ab}^2 \right) = \frac{\partial}{\partial q_{ab}} \left(\log \operatorname{Tr}_{\sigma_a} \exp\left(\beta^2 J^2 \sum_a \sum_{b>a} q_{ab} \sigma_a \sigma_b \right)\right)$$

$$\beta^2 J^2 q_{ab} = \frac{\beta^2 J^2 \operatorname{Tr}_{\sigma_a}[\sigma_a \sigma_b e^L]}{\operatorname{Tr}_{\sigma_a}[e^L]} \qquad q^*_{ab} = \frac{\operatorname{Tr}_{\sigma_a}[\sigma_a \sigma_b e^L]}{\operatorname{Tr}_{\sigma_a}[e^L]}$$

RS : simmetria tra le repliche $\qquad q_{ab} = q$

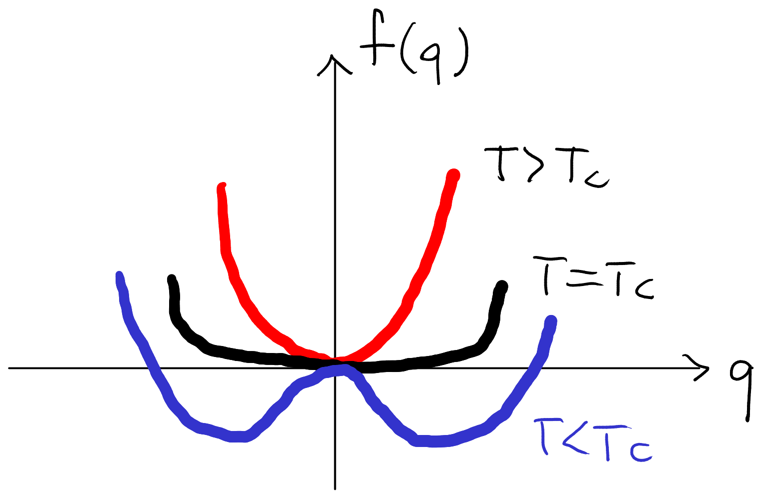
$$\begin{cases} -\beta f = \dfrac{\beta^2 J^2}{2}(1-q)^2 + \displaystyle\int_{-\infty}^{\infty} dz \, \frac{\exp(-z^2/2)}{\sqrt{2\pi}} \log\left[2\cosh(\beta J \sqrt{q}\, z)\right] \\ \dfrac{\beta^2 J^2}{2}(q-1) + \displaystyle\int_{-\infty}^{\infty} dz \, \frac{\exp(-z^2/2)}{\sqrt{2\pi}} \tanh(\beta J \sqrt{q}\, z) \frac{\beta J}{2\sqrt{q}} z = 0 \end{cases} \qquad (\text{es.})$$

$$\Rightarrow \quad q = \int_{-\infty}^{\infty} dz \, \frac{\exp(-z^2/2)}{\sqrt{2\pi}} \tanh^2(\beta J \sqrt{q}\, z)$$

sviluppo in serie di Taylor per piccoli q $\quad$ (es.)

$$\beta f = -\frac{1}{4}\beta^2 J^2 - \log 2 - \underbrace{\frac{\beta^2 J^2}{4}(1-\beta^2 J^2)\, q^2}_{\sim \sum_a \sum_{b>a} q_{ab}^2} + O(q^3)$$

$$\beta_c^2 J^2 = 1 \qquad T_c = J$$



$$-\beta f = \lim_{N\to\infty} \lim_{n\to 0} \frac{\text{cost}}{nN} \log \int_{-\infty}^{\infty} \prod_{a=1}^{n} \prod_{b>a}^{n} dq_{ab} \exp(-\underbrace{Nn\beta f(q_{ab})}_{S(q_{ab})})$$

The second point we have to pay attention to is what we actually mean by the 'minimum' of S. The problem here is that the number of independent elements of Q_{ab} is $n(n-1)/2$, which becomes negative in the limit $n \to 0$. It is hard to say what is the minimum of a function with a negative number of variables! There is however a criterion we can use to select the correct saddle point: the corrections to the saddle point result are given by the Gaussian integration around the saddle point itself. This integration gives as a result the square root of the determinant of the second-derivative matrix of S, and thus, in order to have a sensible result, we must have all the eigenvalues of this matrix positive. Summarizing, we have to select saddle points with a positive-definite second derivative of S [12].

Castellani, Cavagna

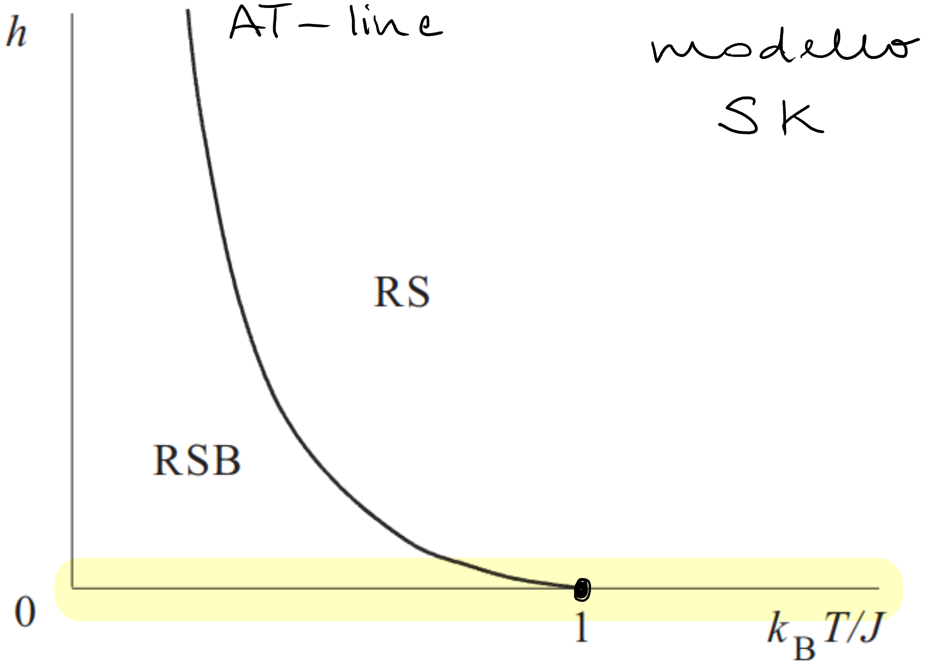
$$\lim_{N\to\infty} \int_{-\infty}^{\infty} dx \exp(Nf(x)) = \sqrt{\frac{(2\pi)^d}{-N|f''(x_0)|}} \exp(Nf(x_0))$$

determinante

Rottura di simmetria delle repliche

1975 : Sherrington Kirkpatrick
1978 : de Almeida - Thouless
1979 - 1980 : Parisi RSB
2006 : Talagrand

G. Parisi
Nobel 2021



modello
SK

Connessione tra le repliche e la fisica

$$q^*_{ab} = \frac{\mathrm{Tr}_{\sigma_a} [\sigma_a \sigma_b e^L]}{\mathrm{Tr}_{\sigma_a} [e^L]} \qquad n \to 0$$

$$q_{EA} = \frac{1}{N} \sum_{i=1}^{N} \overline{\langle \sigma_i \rangle^2} = \frac{1}{N} \sum_{i=1}^{N} \overline{\langle \sigma_i \rangle \langle \sigma_i \rangle}$$

Metodo delle repliche

$$\overline{\langle A \rangle} = \overline{\frac{1}{Z} \mathrm{Tr}_{\sigma_i} \left[A(\{\sigma_i\}) e^{-\beta H[\{\sigma_i\}; J]} \right]} = \overline{\frac{Z^{n-1}}{Z^n} \mathrm{Tr}_{\sigma_i} [A e^{-\beta H}]}$$

$$= \lim_{n \to 0} \overline{Z^{n-1} \mathrm{Tr}_{\sigma_i} [A e^{-\beta H}]} = \lim_{n \to 0} \overline{\mathrm{Tr} \left[A(\{\sigma_{ai}\}) e^{-\beta H^{(n)}} \right]} \qquad H^{(n)} = \sum_{a=1}^{n} H_a$$

$\uparrow$ non deve dipendere da a

$$= \lim_{n \to 0} \mathrm{Tr} \left[A(\{\sigma_{ai}\}) \overline{e^{-\beta H^{(n)}}} \right]$$

$$q_{EA} = \lim_{n\to 0} \mathrm{Tr}\left[\frac{1}{N} \sum_{i=1}^{N} \underbrace{\sigma_{ia}\sigma_{ib}}_{} \, \overline{e^{-\beta H^{(n)}}} \right]$$

non deve dipendere da a e b

$$\approx \dots = \lim_{n\to 0} \frac{\mathrm{Tr}_{\sigma_a} \left[\sigma_a \sigma_b \, e^{L} \right]}{\mathrm{Tr}_{\sigma_a} \left[e^{L} \right]} = \lim_{n\to 0} q^{*}_{ab}$$

Se a e b sono equivalenti (RS) $\qquad [\, q^{*}_{ab} = q_{ab} \,]$

$$\lim_{n\to 0} q_{ab} = \frac{1}{N} \sum_{i=1}^{N} \overline{\langle \sigma_i \rangle^2}$$

Se a e b sono distinguibili

$$\lim_{n\to 0} \frac{2}{n(n-1)} \sum_{a} \sum_{b>a} q_{ab} = \frac{1}{N} \sum_{i=1}^{N} \overline{\langle \sigma_i \rangle^2}$$

Parametro d'ordine EA $\qquad \langle \dots \rangle = \sum_\alpha w_\alpha \langle \dots \rangle_\alpha$

$$\frac{1}{N}\sum_{i=1}^{N} \overline{\langle \sigma_i \rangle^2} = \frac{1}{N}\sum_{i=1}^{N} \overline{\Big(\sum_\alpha w_\alpha \langle \sigma_i \rangle_\alpha\Big)^2} = \frac{1}{N}\sum_{i=1}^{N}\sum_\alpha \sum_\beta \overline{w_\alpha w_\beta \langle \sigma_i \rangle_\alpha \langle \sigma_i \rangle_\beta}$$

$$= \sum_\alpha \sum_\beta \overline{w_\alpha w_\beta \underbrace{\frac{1}{N}\sum_{i=1}^{N} \langle \sigma_i \rangle_\alpha \langle \sigma_i \rangle_\beta}_{q_{\alpha\beta}}} = \sum_\alpha \sum_\beta \overline{w_\alpha w_\beta \, q_{\alpha\beta}}$$

$$= \int_{-\infty}^{\infty} dq \, \overline{\underbrace{\sum_\alpha \sum_\beta w_\alpha w_\beta \, \delta(q - q_{\alpha\beta})}_{p(q)} \, q} = \int_{-\infty}^{\infty} dq \, q \, \overline{p(q)}$$

$$q^{(1)} = \int_{-\infty}^{\infty} dq \, q \, \overline{p(q)} = \frac{1}{N}\sum_{i=1}^{N} \overline{\langle \sigma_i \rangle^2}$$

$$q^{(2)} = \int_{-\infty}^{\infty} dq \, q^2 \, \overline{p(q)} = \frac{1}{N^2}\sum_{i=1}^{N} \overline{\langle \sigma_i \sigma_j \rangle^2}$$

$$q^{(k)} = \int_{-\infty}^{\infty} dq \, q^k \, \overline{p(q)} = \frac{1}{N^k}\sum_{i=1}^{N} \overline{\langle \sigma_{i_1} \cdots \sigma_{i_k} \rangle^2}$$

stati $\longleftrightarrow$ repliche

$$q^{(1)} = \int_{-\infty}^{\infty} dq\, q\, \overline{p(q)} = \lim_{n\to 0} \frac{2}{n(n-1)} \sum_a \sum_{b>a} q_{ab}$$

$$\int_{-\infty}^{\infty} dq\, f(q)\, \overline{p(q)} = \lim_{n\to 0} \frac{2}{n(n-1)} \sum_a \sum_{b>a} f(q_{ab})$$

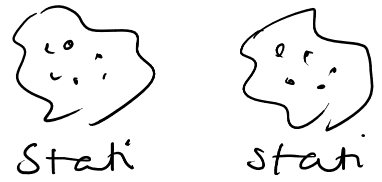
$$\overline{p(q')} = \lim_{n\to 0} \frac{2}{n(n-1)} \sum_a \sum_{b>a} \delta(q'-q_{ab})$$

$f(q) = \delta(q-q')$

$$\boxed{\overline{p(q)} = \lim_{n\to 0} \frac{2}{n(n-1)} \sum_a \sum_{b>a} \delta(q-q_{ab})}$$

distribuzione dell'overlap tra stati $p(q)$ mediata sul disordine = frazione di repliche con overlap pari a q

Parametrizzazione della matrice q_{ab}



$a \neq b$ $\quad q_{ab}$ $\quad$ somiglianza

$a = b$ $\quad q_{aa}$ $\quad$ taglia

configs

$a \neq b$ $\quad q_{ab}$ $\quad \Sigma_a \Sigma_{b>a}$

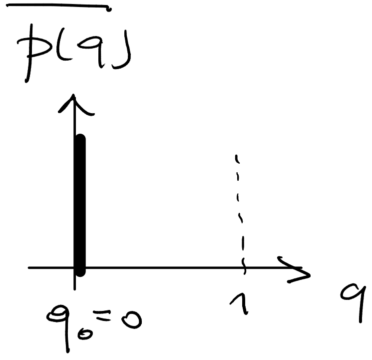
$a = a$ $\quad q_{aa} = 1$

RS : simmetria tra repliche $\qquad q_{ab} = q_0$

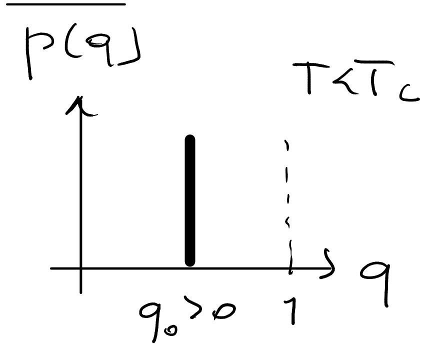
$$
\begin{pmatrix}
0 & & & & & \\
 & 0 & & q_0 & & \\
 & & 0 & & & \\
 & & & 0 & & \\
 & q_0 & & & 0 & \\
 & & & & & 0
\end{pmatrix}
$$

$\overline{p(q)} = \delta(q - q_0)$

$q_0 =$ self-overlap dello stato



para



ferro

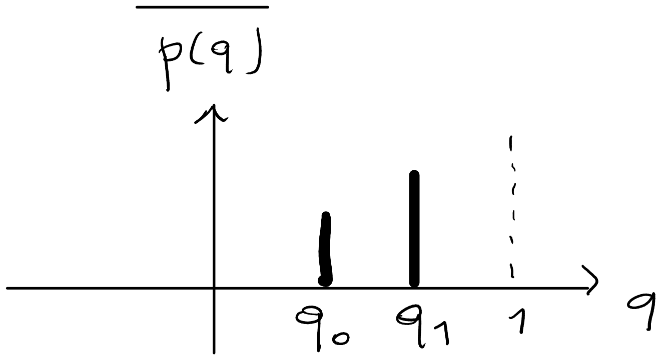
1-RSB : rottura di simmetria a 1 step

$$\begin{array}{c c} m & \left(\begin{array}{ccc|ccc} 0 & q_1 & q_1 & & & \\ q_1 & 0 & q_1 & & q_0 & \\ q_1 & q_1 & 0 & & & \\ \hline & & & 0 & q_1 & q_1 \\ & q_0 & & q_1 & 0 & q_1 \\ & & & q_1 & q_1 & 0 \end{array}\right). \end{array}$$

q_1 self-overlap degli stati

q_0 overlap tra gli stati

$0 \le q_0 \le q_1 \le 1 \qquad 1 \le m \le n$



$$\begin{aligned} \overline{p(q)} &= \lim_{n \to 0} \frac{2}{n(n-1)} \sum_a \sum_{b > a} \delta(q - q_{ab}) \\ &= \frac{m-1}{n-1} \delta(q - q_1) + \frac{n-m}{n-1} \delta(q - q_0) \\ &= (1-m)\, \delta(q - q_1) + m\, \delta(q - q_0) \end{aligned}$$

$$\begin{cases} q_{ab} = q_{ab}(q_0, q_1, m) \\ 0 \le q_0 \le q_1 \le 1 \\ 0 \le m \le 1 \end{cases}$$

Soluzione per SK è instabile

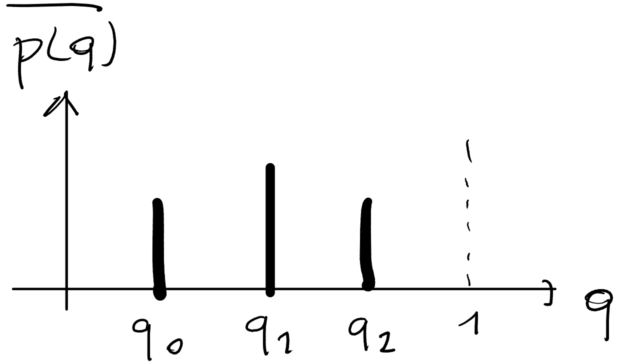
We clearly see that while the elements q_1, q_0 fix the positions of the peaks, the parameter m fixes their heights, and therefore the probabilities of the overlaps. We have now to take the limit $n \to 0$. Here lies possibly the weirdest twist of the replica method. Relation (72) seems to resist strenuously our efforts to send n to zero. However, in this limit it is clear that m must also be promoted to being a real number, rather than an integer. To see how

2 - RSB

$$
\begin{array}{c}
m_1 \\[2em] m_2 \\[2em] \\ \\ \\
\end{array}
\left(
\begin{array}{ccc|ccc|ccc|ccc}
0 & q_2 & q_2 & & & & & & & & & \\
q_2 & 0 & q_2 & & q_1 & & & & & & & \\
q_2 & q_2 & 0 & & & & & & & & & \\
\cline{1-6}
 & & & 0 & q_2 & q_2 & & & q_0 & & & \\
 & q_1 & & q_2 & 0 & q_2 & & & & & & \\
 & & & q_2 & q_2 & 0 & & & & & & \\
\hline
 & & & & & & 0 & q_2 & q_2 & & & \\
 & & & & & & q_2 & 0 & q_2 & & q_1 & \\
 & & q_0 & & & & q_2 & q_2 & 0 & & & \\
\cline{7-12}
 & & & & & & & & & 0 & q_2 & q_2 \\
 & & & & & & & q_1 & & q_2 & 0 & q_2 \\
 & & & & & & & & & q_2 & q_2 & 0
\end{array}
\right)
$$

$$0 \leq q_0 \leq q_1 \leq q_2 \leq 1$$

$$q_{ab} = q_{ab}(q_0, q_1, q_2, m_1, m_2)$$



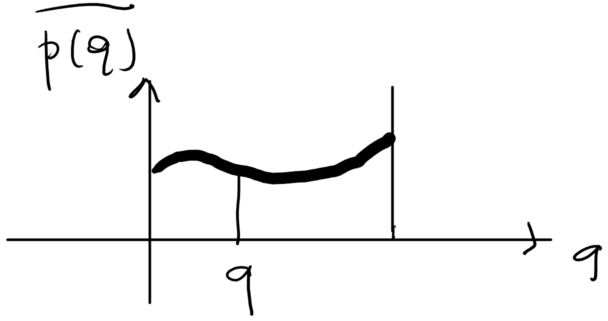
K - RSB

full - RSB

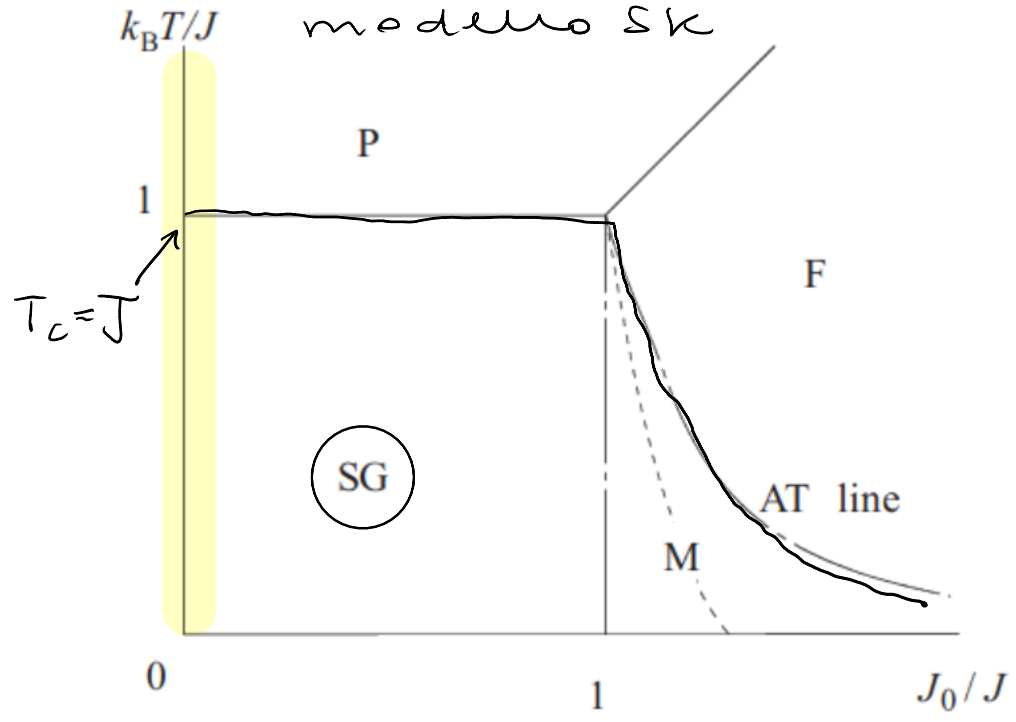
$K \to \infty \qquad q(x) \qquad x \in [0,1]$

$$\frac{1}{n(n-1)} \sum_a \sum_b{}' q_{ab} \to -\int_0^1 dx\, q(x)$$

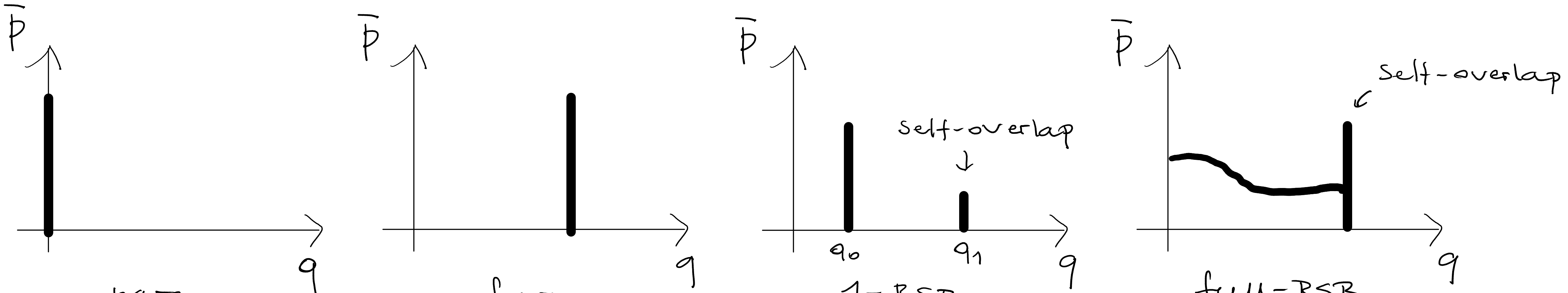
ok per SK!



modello SK



RSB



para | ferro | 1-RSB | full-RSB

1-RSB ↓ p-spin ⇝ vetri

full-RSB ↓ SK ⇝ vetro di spin

$$\bar{p}(q) = (1-m)\,\delta(q-q_1) + m\,\delta(q-q_0)$$

Modelli p-spin sferici

$\sigma_i \in \mathbb{R}$ vincolo sferico : $\sum_{i=1}^{N} \sigma_i^2 = N$ $\quad q_{aa} = 1$

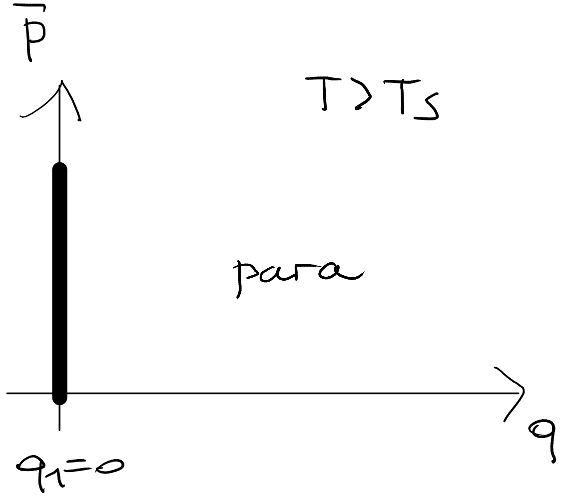
$$H = -\sum_{i_1=1}^{N} \sum_{i_2>i_1}^{N} \cdots \sum_{i_p>i_{p-1}}^{N} J_{i_1 \dots i_p}\, \sigma_{i_1} \cdots \sigma_{i_p} \qquad p \geq 3$$

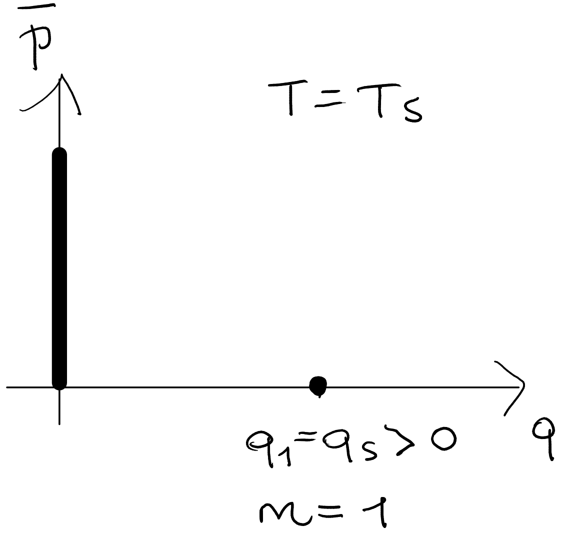
$J_{i_1 \dots i_p}$ gelate gaussiane $\nearrow$ J^2/N^{p-1}

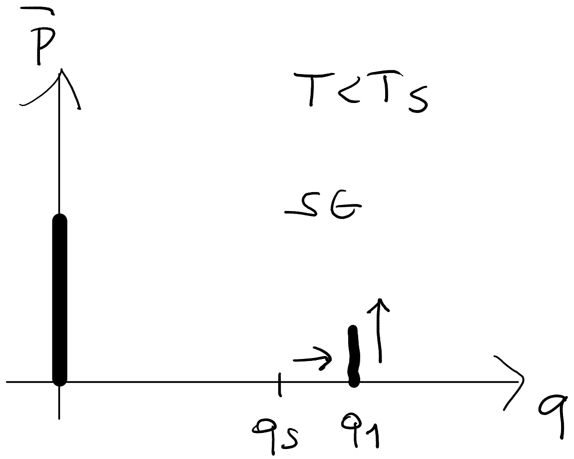
Statica : 1-RSB

$q_{ab}(q_0, q_1, m) \to f(q_0, q_1, m)$

$$\begin{cases} \dfrac{\partial f}{\partial q_0} = 0 \Rightarrow q_0 = 0 \quad \perp \\ \dfrac{\partial f}{\partial q_1} = 0 \quad < \begin{matrix} \text{alta } T \\ \text{bassa } T \end{matrix} \\ \dfrac{\partial f}{\partial m} = 0 \end{cases}$$







Dinamica

Langevin sovrasmorzata

$$\frac{\partial \sigma_i}{\partial t} = -\frac{\partial H}{\partial \sigma_i} + \xi_i(t) - \mu_i(t)\,\sigma_i(t) \qquad \langle \xi_i(t)\,\xi_i(0)\rangle = 2T\,\delta(t-t')$$

$$p=3: \quad \frac{\partial H}{\partial \sigma_i} \sim \sum_j \sum_k J_{ijk}\,\sigma_j\,\sigma_k$$

Eq. moto per σ (singolo sito) senza disordine + memoria

$$\frac{\partial \sigma}{\partial t} = -\mu(t)\,\sigma(t) + \frac{1}{2}p(p-1)\int_{-\infty}^{t} ds\; R(t,s)\, C^{p-2}(t,s)\,\sigma(s) + \tilde{\xi}(t)$$

funzione di risposta

f. correlazione $C(t,s) = \langle \sigma(t)\,\sigma(s)\rangle$

stazionario: $C(t)$

teor. FD: $R(t) = -\beta \frac{dC}{dt}$

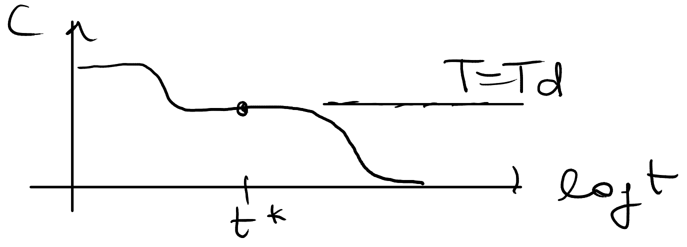
$$\frac{dC}{dt} + T\,C(t) + \frac{p}{2T}\int_0^t ds\; C^{p-1}(t-s)\,\frac{dC}{ds}(s) = 0$$

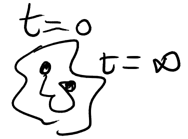
$$\frac{dC}{dt} + T\,C(t) + \frac{3}{2T}\int_0^t ds\; C^{2}(t-s)\,\frac{dC}{ds}(s) = 0 \;\longrightarrow\; \text{MCT schematica}$$

∃ transizione ideale $T = T_d$

$T \to T_d$: $\quad \tau \sim \dfrac{1}{|T-T_c|^{\gamma}}$

$$T_d = \left[\frac{p(p-2)}{2(p-1)^{p-1}}\right]^{1/2} > T_s$$

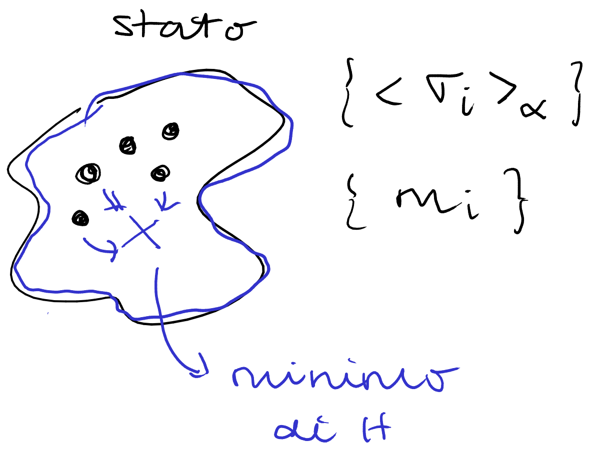




$\langle \sigma(t^*)\, \sigma(0) \rangle$

[Kirkpatrick & Thirumalai PRB 1987]

Connessione con l'energy landscape



$f(\{m_i\})$ stati = minimi di f

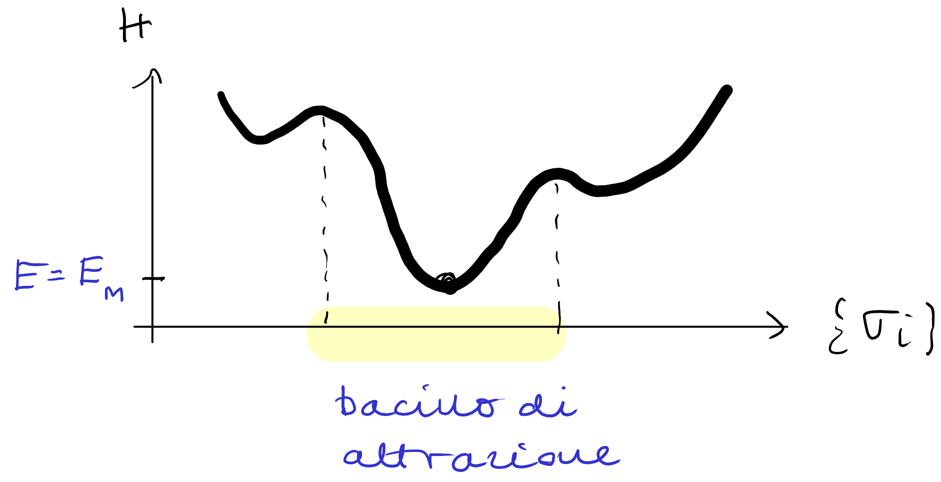
$E(\{\sigma_i\}) \rightarrow$ minimo di f

p-spin

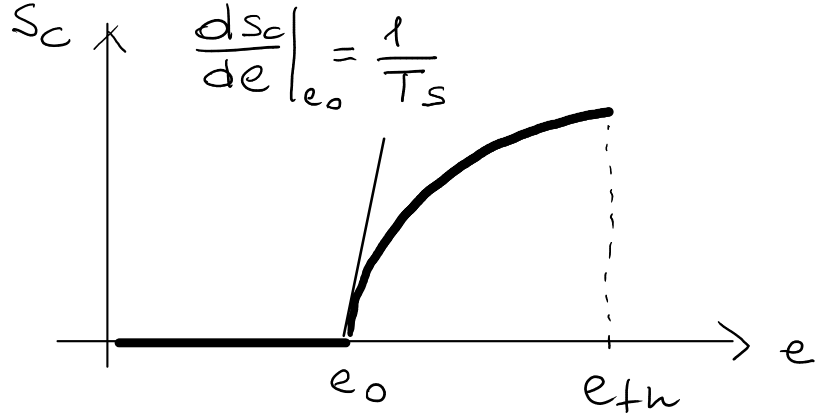
$e = \frac{E}{N}$

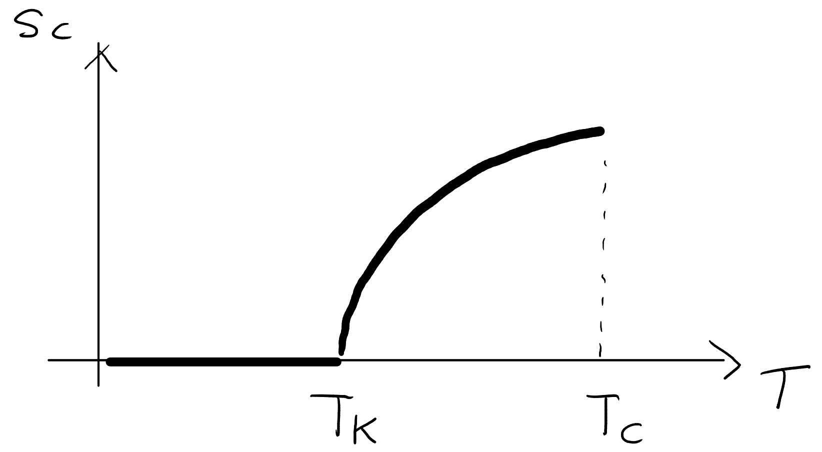
Entropia configurazionale

$$s_c = \frac{K_B}{N} \overline{\log \Omega(e)} \underset{\substack{\uparrow \\ \text{p-spin}}}{=} \frac{K_B}{N} \log \overline{\Omega(e)}$$



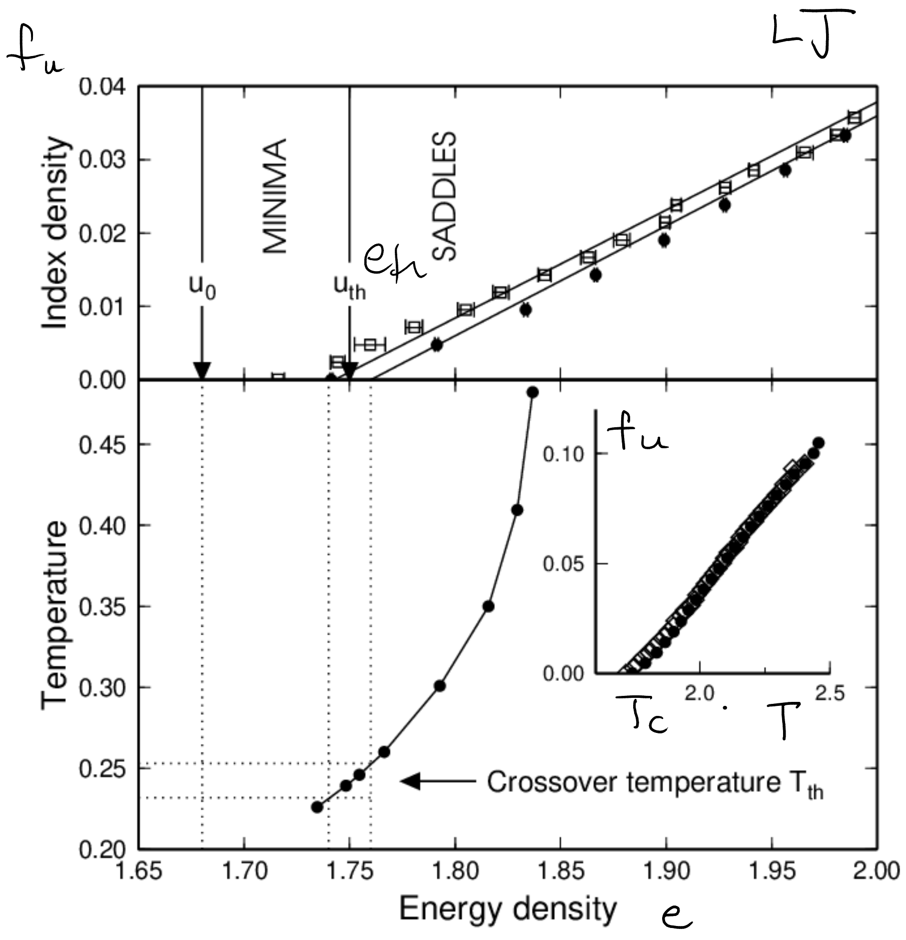
$$\left.\begin{array}{lll} \overline{\Omega}(e) \sim \exp(s_c N) & \Rightarrow & s_c > 0 \\ \overline{\Omega}(e) \sim N^a & \Rightarrow & s_c = 0 \end{array}\right\} N \to \infty$$





$e = e_0$	$s_c = 0$	@ T_s
$e = e_{th}$	s_c indefinita	@ T_d
$e > e_{th}$	punti sella	

p-spin		vetri
T_d	$\longleftrightarrow$	T_c
T_s		T_K

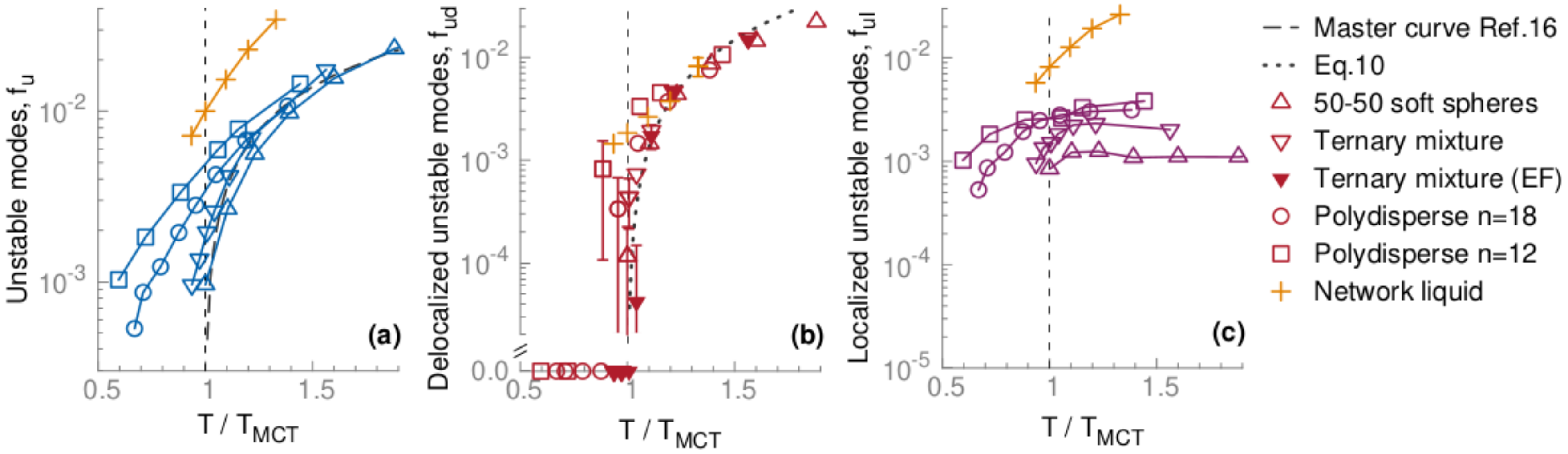


$f_u = \frac{n_u}{3N}$

Grigera, Cavagna, Giardina, Parisi
PRL 2002

(iii) Presence of saddles below T_c. Despite claims based on numerical results that T_c marks a real change in the topology of the landscape [33–35], there are strong indications that this is at best only a crossover [36,37], and at worst a biased interpretation of numerical data [38]. For instance, careful numerical studies have shown that the saddle index $n_s(T)$ remains positive even below T_c [36]. Moreover, Ref. [38] argues convincingly that the data for $n_s(T)$ can be described by an Arrhenius law, $n_s(T) \sim \exp(-E/T)$, with E an energy scale, which means that T_c does not mark any particular change in the saddle index.

Berthier & Garrahan JCP 2003



Coslovich, Ninarello, Berthier Scipost 2019