Lezione 37 - 2/12/2025

Formula di Taylor

f derivabile fino all'ordine n in I int., $x_0 \in I$

$$f(x) = \underbrace{f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\frac{f^{(3)}(x_0)}{3!}(x-x_0)^3+\dots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n}_{\text{polinomio di Taylor}} + \underbrace{R_n(x,x_0)}_{\text{resto}}$$

polinomio di Taylor (di punto iniziale x_0 fino all'ordine n)

con $\lim_{x\to x_0} \frac{R_n(x,x_0)}{(x-x_0)^n}=0$

resto nella forma di PEANO

f derivabile fino all'ordine $n+1$ in I intervallo $x_0\in I$

$\forall x \in I\setminus\{x_0\}$, $\exists \xi \in]x_0,x[$ (oppure $]x,x_0[$)

$$f(x) = \underbrace{f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\frac{f'''(x_0)}{3!}(x-x_0)^3+\dots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n}_{\text{polinomio di Taylor (uguale a sopra)}}+\underbrace{\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}}_{\text{resto nella forma di LAGRANGE}}$$

Esempio: utilizzo la f. di Taylor con resto di Lagrange per calcolare il valore di e con errore inferiore a 10^{-3}

(per il momento sappiamo che $e=\lim_n \left(1+\frac1n\right)^n$ e $2\le e\le 3$)

Scrivo la f. di Taylor per e^x con $x_0=0$ e n qualunque con il resto di Lagrange

$f(x)=e^x$ $\qquad x_0=0$ $\qquad f(0)=e^0=1$

$f'(x)=e^x$ $\qquad f'(0)=e^0=1$

$\vdots$

$f^{(n)}(x)=e^x$ $\qquad f^{(n)}(0)=e^0=1$

$f^{(n+1)}(x)=e^x$ $\qquad f^{(n+1)}(\xi)=e^\xi$

$$e^x = f(x) = f(0)+f'(0)(x-0)^1+\frac{f''(0)}{2!}(x-0)^2+\dots+\frac{f^{(n)}(0)}{n!}(x-0)^n+\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-0)^{n+1}$$
con $\xi\in]0,x[$

$$\boxed{e^x = 1+1\cdot x+\frac{1}{2!}x^2+\dots+\frac{1}{n!}x^n+\frac{e^\xi}{(n+1)!}x^{n+1}}$$ con $0<\xi<x$

$$e^x = 1+x+\frac{x^2}{2}+\frac{x^3}{6}+\dots+\frac{x^n}{n!}+\frac{e^\xi}{(n+1)!}x^{n+1}$$ con $0<\xi<x$

$x=1$

$$e = 1+1+\frac12+\frac16+\dots+\frac1{n!}+\frac{e^\xi}{(n+1)!}$$ con $0<\xi<1$

Ho che una approssimazione di e è

$1+1+\frac12+\frac16+\dots+\frac1{n!}$ con errore $\left|\frac{e^\xi}{(n+1)!}\right|$

¿ quanto piccolo è $\left|\frac{e^\xi}{(n+1)!}\right|$ se $\xi\in]0,1[$?

$2\le e\le 3$, $0\le\xi\le1$, $|e^\xi|\le 3$

$\left|\frac{e^\xi}{(n+1)!}\right|\le\frac{3}{(n+1)!}$ scelgo n in modo che $\frac{3}{(n+1)!}<10^{-3}$

$\left|\frac{e^{\xi}}{(n+1)!}\right| \le \frac{3}{(n+1)!}$ scelgo n in modo che $\frac{3}{(n+1)!} < 10^{-3}$

$n=4$ $(n+1)! = 5! = 120$ è $\frac{3}{120} \cancel{<} 1/1000$? no

$n=5$ $(n+1)! = 6! = 720$ è $\frac{3}{720} \cancel{<} \frac{1}{1000}$? no

$n=6$ $(n+1)! = 7! = 5040$ è $\frac{3}{5040} \le \frac{1}{1000}$? sì

$n=7$ $(n+1)! = 8! \sim 40000$ $\frac{3}{40000} \le 10^{-4}$

e è approssimato da

$1+1+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}+\frac{1}{120}+\frac{1}{720}$ con errore inferiore a 10^{-3}

$\uparrow$ $6!$

$= 2 + \frac{360+120+30+6+1}{720}$

$= 2 + \frac{517}{720}$

360
120
30
6
1
517

$517 : 72 = 7180$
130
580
-40

$517 : 720 = 0{,}7180\ldots$

$\boxed{e = 2{,}7180 \text{ con errore inferiore a } 10^{-3}}$

$e = 1+1+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}+\frac{1}{120}+\frac{1}{720}+\frac{1}{5040}$ con errore inferiore a 10^{-4}

$\sqrt{e} = e^{\frac{1}{2}} = f(0)+f'(0)x+\frac{f''(0)}{2}x^2+\ldots+\frac{f^{(n)}(0)}{n!}x^n+\frac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1}$

$x=\frac{1}{2}$

$= 1+\frac{1}{2}+\frac{1}{2!}\left(\frac{1}{2}\right)^2+\frac{1}{3!}\left(\frac{1}{2}\right)^3+\ldots+\frac{1}{n!}\left(\frac{1}{2}\right)^n+\frac{e^{\xi}}{(n+1)!}\left(\frac{1}{2}\right)^{n+1}$

Es. calcolare $\lim_{x\to 0} \frac{1-\sqrt{1-x}}{x}$ $\quad \frac{0}{0}$

1) $\frac{1-\sqrt{1-x}}{x} = \frac{1-\sqrt{1-x}}{x}\cdot\frac{1+\sqrt{1-x}}{1+\sqrt{1-x}} = \frac{1-(1-x)}{x(1+\sqrt{1-x})}$

$= \frac{\cancel{x}}{\cancel{x}(1+\sqrt{1-x})} = \frac{1}{1+\sqrt{1-x}}$

$\lim_{x\to 0}\frac{1-\sqrt{1-x}}{x} = \lim_{x\to 0}\frac{1}{1+\sqrt{1-x}} = \frac{1}{2}$

$\left((1-x)^{\frac{1}{2}}\right)'$

2) applico Hôpital $\quad \frac{1}{2}(1-x)^{-\frac{1}{2}}\cdot(-1)$

$\lim_{x\to 0}\frac{1-\sqrt{1-x}}{x} \overset{H}{\Leftarrow} \lim_{x\to 0}\frac{-\left(\frac{1}{2}(1-x)^{-\frac{1}{2}}(-1)\right)}{1}$

$\lim_{x\to 0}\frac{1}{2}\frac{1}{\sqrt{1-x}} = \frac{1}{2}$

3) formula di Taylor con resto di Peano per $\sqrt{1-x}$ con $x_0=0$ n qualunque

$f(x) = (1-x)^{\frac{1}{2}}$

$f'(x) = \frac{1}{2}(1-x)^{-\frac{1}{2}}\cdot(-1) = -\frac{1}{2}(1-x)^{-\frac{1}{2}}$

$f''(x) = -\frac{1}{2}\left(-\frac{1}{2}\right)(1-x)^{-\frac{3}{2}}(-1) = -\frac{1}{4}(1-x)^{-\frac{3}{2}}$

$f^{(3)}(x) = -\frac{1}{4}\left(-\frac{3}{2}(1-x)^{-\frac{5}{2}}(-1)\right) = -\frac{3}{8}(1-x)^{-\frac{5}{2}}$

$f(0) = 1$
$f'(0) = -\frac{1}{2}$
$f''(0) = -\frac{1}{4}$
$f'''(0) = -\frac{3}{8}$

$\sqrt{1-x} = 1-\frac{1}{2}x-\frac{1}{4}\cdot\frac{1}{2!}x^2-\frac{3}{8}\cdot\frac{1}{3!}x^3+R_3(x)$ con $\lim_{x\to 0}\frac{R_3(x)}{x^3}=0$

$\sqrt{1-x} = 1 - \frac{1}{2}x - \frac{1}{4}\cdot\frac{1}{2!}x^2 - \frac{3}{8}\cdot\frac{1}{3!}x^3 + r_3(x)$ con $\lim_{x\to 0}\frac{r_3(x)}{x^3}=0$

$$\sqrt{1-x} = 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} + r_3(x) \quad \text{con} \quad \lim_{x\to 0}\frac{r_3(x)}{x^3} = 0$$

$$1-\sqrt{1-x} = 1 - \left(1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} + r_3(x)\right)$$

$$1-\sqrt{1-x} = \frac{x}{2} + \frac{x^2}{8} + \frac{x^3}{16} + r_3(x)$$

$$\frac{1-\sqrt{1-x}}{x} = \frac{1}{2} + \underbrace{\frac{x}{8}}_{\to 0} + \underbrace{\frac{x^2}{16}}_{\to 0} + \underbrace{\frac{r_3(x)}{x^3}x^2}_{\to 0}$$

$$\lim_{x\to 0}\frac{1-\frac{1}{2}x-\sqrt{1-x}}{x^2} = \qquad \sqrt{1-x} = 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} + r_3(x)$$

$$= \lim_{x\to 0}\frac{1-\frac{1}{2}x-\left(1-\frac{1}{2}x-\frac{x^2}{8}-\frac{x^3}{16}+r_3(x)\right)}{x^2}$$

$$= \lim_{x\to 0}\frac{\frac{x^2}{8}+\frac{x^3}{16}-r_3(x)}{x^2}$$

$$= \lim_{x\to 0}\frac{1}{8}+\frac{x}{16}+\frac{r_3(x)}{x^2} = \frac{1}{8}$$

$$\lim_{x\to 0}\frac{1-\frac{1}{2}x-\frac{1}{8}x^2-\sqrt{1-x}}{x^3} = \frac{1}{16}$$

Es. Scrivere la formula di Taylor con il resto di Peano per $f(x) = \sqrt{\cos x}$ con $x_0 = 0$, $n = 2$

$f(x) = (\cos x)^{\frac{1}{2}}$

$f'(x) = \frac{1}{2}(\cos x)^{-\frac{1}{2}}(-\operatorname{sen} x) = -\frac{1}{2}(\cos x)^{-\frac{1}{2}}\operatorname{sen} x$

$f''(x) = -\frac{1}{2}\left(-\frac{1}{2}(\cos x)^{-\frac{3}{2}}(-\operatorname{sen} x)\cdot \operatorname{sen} x + (\cos x)^{+\frac{1}{2}}\right)$

$- \frac{1}{2}\left(-\frac{1}{2}\cdot 1\cdot 0\cdot 0 + (1)\right)$

$f(0) = 1$
$f'(0) = 0$
$f''(0) = -\frac{1}{2}$

$$\sqrt{\cos x} = 1 + 0\cdot x + \frac{1}{2}\left(-\frac{1}{2}\right)x^2 + r_2(x)$$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + r_2(x)$$

$$\boxed{\sqrt{\cos x} = 1 - \frac{1}{4}x^2 + r_2(x) \quad \text{con} \quad \lim_{x\to 0}\frac{r_2(x)}{x^2} = 0}$$

Es. Scrivere la f. di Taylor con resto di Lagrange per $f(x) = x\operatorname{sen} x$, $x_0 = 0$, $n = 2$.

$f(x) = x\operatorname{sen} x$

$f'(x) = \operatorname{sen} x + x\cos x$

$f''(x) = \cos x + \cos x - x\operatorname{sen} x = 2\cos x - x\operatorname{sen} x$

$f'''(x) = -2\operatorname{sen} x - \operatorname{sen} x - x\cos x = -3\operatorname{sen} x - x\cos x$

$f(0) = 0$
$f'(0) = 0$
$f''(0) = 2$

Es. Scrivere la f. di Taylor con resto di Lagrange per $f(x) = x \operatorname{sen} x$, $x_0 = 0$, $n = 2$.

$f(x) = x \operatorname{sen} x$

$f'(x) = \operatorname{sen} x + x \cos x$

$f''(x) = \cos x + \cos x - x \operatorname{sen} x = 2\cos x - x \operatorname{sen} x$

$f'''(x) = -2 \operatorname{sen} x - \operatorname{sen} x - x\cos x = -3 \operatorname{sen} x - x \cos x$

$f(0) = 0$

$f'(0)\ 0$

$f''(0) = 2$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(\xi)}{3!}x^3$$

$$x \operatorname{sen} x = 0 + 0 \cdot x + \frac{2}{2}x^2 + \left(\frac{-3\operatorname{sen}\xi - \xi\cos\xi}{6}\right)x^3$$

$$x \operatorname{sen} x = x^2 + \frac{(-3\operatorname{sen}\xi - \xi\cos\xi)}{6}x^3$$

FUNZIONI CONVESSE.

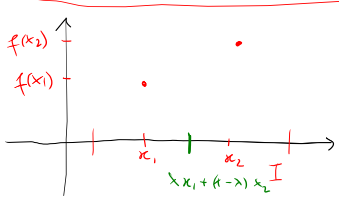
def. Sia $f: I \longrightarrow \mathbb{R}$, I intervallo

f si dice **convessa** in I se

$\forall x_1, x_2 \in I$ con $x_1 < x_2$

$\forall \lambda \in [0,1]$

$$f(\lambda x_1 + (1-\lambda)x_2) \le \lambda f(x_1) + (1-\lambda) f(x_2)$$



$\lambda \in [0,1]$

$\lambda x_1 + (1-\lambda) x_2$

$x_1 < x_2$

$\lambda x_1 < \lambda x_2$

$\lambda \in [0,1]$

$$\underbrace{\lambda x_1 + (1-\lambda)x_1}_{= x_1} \le \lambda x_1 + (1-\lambda)x_2 \le \underbrace{\lambda x_2 + (1-\lambda)x_2}_{= x_2}$$

conclusione

$$x_1 \le \lambda x_1 + (1-\lambda)x_2 \le x_2$$

cioè ogni valore y che sta fra x_1 e x_2

lo posso scrivere $y = \lambda x_1 + (1-\lambda)x_2$

per un opportuno λ

$y = \lambda(x_1 - x_2) + x_2$

$y - x_2 = \lambda(x_1 - x_2)$

$x_2 - y = \lambda(x_2 - x_1)$

(basta prendere $\lambda = \dfrac{x_2 - y}{x_2 - x_1}$)

equazione della retta per i punti $(x_1, f(x_1))$ e $(x_2, f(x_2))$

$$y = \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1) + f(x_1)$$

se $x = \lambda x_1 + (1-\lambda)x_2$ quanto vale y?

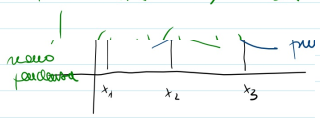
$$\frac{f(x_2) - f(x_1)}{x_2 - x_1}(\lambda x_1 + (1-\lambda)x_2 - x_1) + f(x_1)$$

$$\lambda x_1 + x_2 - \lambda x_2 - x_1 = x_1(\lambda - 1) + x_2(1-\lambda)$$

$$\overset{!}{=} (1-\lambda)(x_2 - x_1)$$

$$\frac{f(x_2)-f(x_1)}{x_2-x_1}\left(\lambda x_1+(1-\lambda)x_2-x_1\right)+f(x_1)$$

$$\lambda x_1 + x_2 - \lambda x_2 - x_1 = x_1(\lambda-1)+x_2(1-\lambda)$$



$$= \frac{f(x_2)-f(x_1)}{\cancel{x_2-x_1}}\cdot(1-\lambda)\cancel{(x_2-x_1)}+f(x_1)$$

$$= \left(f(x_2)-f(x_1)\right)(1-\lambda)+f(x_1)$$

$$= f(x_2)(1-\lambda)+\lambda f(x_1)-\cancel{f(x_1)}+\cancel{f(x_1)}$$

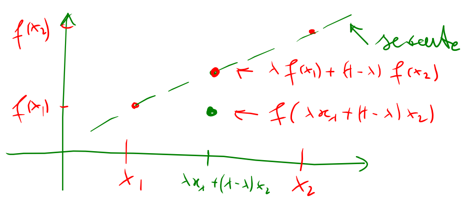
$$= \lambda f(x_1)+(1-\lambda)f(x_2)$$

riassunto

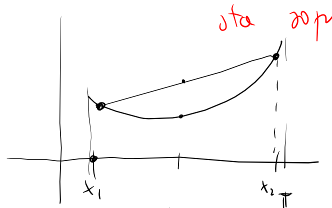
la condizione

$$f(\lambda x_1+(1-\lambda)x_2)\le \lambda f(x_1)+(1-\lambda)f(x_2)$$

è



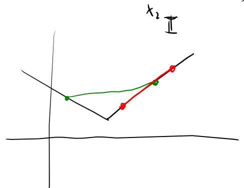
f convessa $\iff$ presi 2 punti sul grafico il segmento che li congiunge sta sopra il grafico

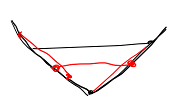


$\forall x_1, x_2 \in I$ con $x_1<x_2$

$\forall \lambda\in[0,1]$

$f(\lambda x_1+(1-\lambda)x_2)\le \lambda f(x_1)+(1-\lambda)f(x_2)$





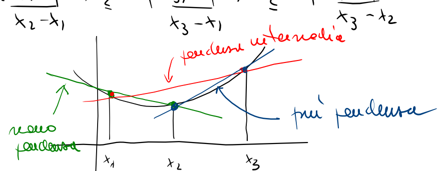
Teorema sia $f: I \to \mathbb{R}$, I intervallo

sono equivalenti le seguenti 3 condizioni

1) f è convessa $\left(\forall x_1,x_2\in I \text{ con } x_1<x_2,\ \forall\lambda\in[0,1]\ \ f(\lambda x_1+(1-\lambda)x_2)\le\lambda f(x_1)+(1-\lambda)f(x_2)\right)$

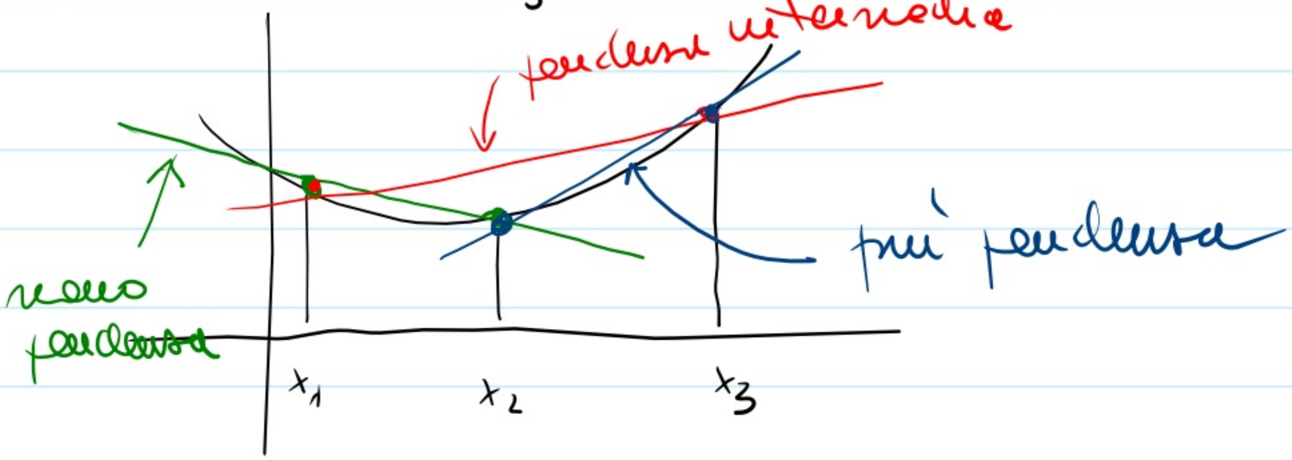
2) $\forall x_1,x_2,x_3\in I$ con $x_1<x_2<x_3$

$$\frac{f(x_2)-f(x_1)}{x_2-x_1}\le\frac{f(x_3)-f(x_1)}{x_3-x_1}\le\frac{f(x_3)-f(x_2)}{x_3-x_2}$$



2) $\forall x_1, x_2, x_3 \in I$ con $x_1 < x_2 < x_3$

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} \leq \frac{f(x_3)-f(x_1)}{x_3-x_1} \leq \frac{f(x_3)-f(x_2)}{x_3-x_2}$$



3) $\forall x_1, x_2, x_3 \in I$, con $x_1 < x_2 < x_3$

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} \leq \frac{f(x_3)-f(x_2)}{x_3-x_2}$$